THE GENERALIZED SLIDING-WINDOW RECURSIVE LEAST-SQUARES LATTICE FILTER

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ABSTRACT

The *Generalized Sliding-Window Fast Transversal Filter (GSWFTF)* is an extension of the exponentially weighted fast RLS algorithm, which inherits its rapid convergence while featuring the desired robustness of affine projection algorithms. Motivated by the numerical difficulties typical from such adaptive transversal realizations, we develop the lattice version of the GSWRLS. Our simulations show significant improvement on stability compared to the GSWFTF and the classical, sliding-window RLS lattice recursions. The new order-recursive filter exhibits the best of both robust and fast converging algorithms, without showing divergence over millions of iterations.

Index Terms— Adaptive, fast, RLS

1. INTRODUCTION

The *Generalized Sliding-Window Recursive Least Squares(GSWRLS)* algorithm was proposed in [1] as a means of improving the tracking ability of the exponentially weighted RLS (EWRLS) algorithm, and the numerical impairments pertaining to the traditional (rectangular) sliding window RLS (SWRLS) recursions. Whereas the SWRLS completely forgets data beyond a length L of past input samples, the GSWRLS algorithm makes use of only partial downdate recursions, thus preserving the exponential window decay towards infinity. The resulting window exhibits a characteristic "tail" beyond L samples, which helps regularizing the least-squares (LS) problem, therefore contributing to better conditioning and stability of sliding-window adaptive filters (see Fig. 1). Compared to the SWRLS and the *Affine Projection Algorithm* (APA) [2], the GSWRLS is able to retain both desired robustness and fast convergence features within a single algorithm. Note that the APA solves an underdetermined system of equations due to a short window length, and yet both SWRLS and the APA are subject to ill-conditioning and noise enhancement [3].

Now, a fast $\mathcal{O}(Q)$ transversal realization of the GSWRLS algorithm was further developed in [3] for tapped-delay-line models of length Q, along with an error feedback stabilizing mechanism similar to the one employed in the well known fast transversal filter (FTF) recursions [4]. Unfortunately, despite of its efficiency, depending on the choice of the feedback tuning parameters, the GSWFTF can still suffer from the numerical problems inherent to such *nonminimal* transversal realizations [5],[6], which eventually become unstable.

Fast *order-recursive* implementations of the EWRLS are, on the other hand, naturally stable, and require no stabilization mechanism [7],[8]. This fact motivates us to pursue the *lattice* version of the GSWRLS algorithm, which we refer to as the *Generalized Sliding Window RLS Lattice* (GSWRLSL) adaptive filter. We develop new *a posteriori* error based recursions for the additional lattice that results from solving the order-recursive, partial downdate problems, and show how the defining variables of the new arising reflection coefficients relate to the ones in the updating solution.

We have verified, for *Composite Source Signal* (CSS) inputs, that while the GSWFTF and the rectangular SW-lattice algorithms diverge in a few iterations, the GSWRLSL remains stable even after a long observation period. The mean-squared-error (MSE) and the convergence rate of the GSWRLSL are also illustrated in comparison with the ones of the APA, NLMS, SWRLS, and EWRLS algorithms. We observed that the GSWRLSL offers excellent tradeoff, with a mix of fast convergence and quick tracking behavior, given sudden changes in the impulse response of a random walk channel model.

2. REGULARIZED GSWRLS PROBLEM

Given a column vector $y_N \in \mathbb{C}^{N+1}$, a $(N+1) \times M$ complex data matrix $H_{M,N}$, and a positive-definite regularization Π_M, we seek the vector $w_M \in \mathbb{C}^M$ that solves the following minimization problem:

$$\min_{w_M} \left[\lambda^{N+1} w_M^* \Pi_M^{-1} w_M \;\; + \;\; \|y_N - H_{M,N} w_M\|^2_{W_N} \right] \tag{1}$$

where we denote by $(\ )^*$, the complex conjugate transposition, and $W_N \triangleq W_{N,L} = \text{diag}\left((1-\eta_0)\lambda^N, \cdots, (1-\eta_0)\lambda^L, \lambda^{L-1}, \cdots, 1\right)$ is a diagonal matrix defined in terms of a forgetting factor λ satisfying $0 \ll \lambda \leq 1$, and a partial downdate factor η_0, for $0 < \eta_0 \leq 1$. Figure 1 depicts the generalized window:

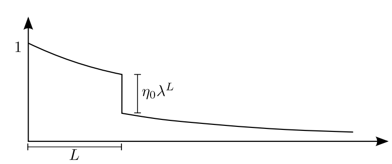

Fig. 1. Generalized Sliding-Window.

The entries of y_N are given by $\{d(i)\}$, while the rows of $H_{M,N}$ are defined by the input regressors $\{u_{M,i}\}$. We further denote by $\{x_{m,N}\}$ the m-th column of $H_{M,N}$. The solution to (1) is given by $w_{M,N}$, which we explicitly specify in terms of its length M at time N. This will be the case of several LS variables defined in this paper.

Due to the discontinuity in the window shape of Fig. 1, the GSWRLS algorithm computes $w_{M,N}$ iteratively based on the solution to a partial downdate problem, $w^d_{M,N}$, which is obtained by replacing W_N in (1) with $W^d_N \triangleq W_{N,L-1}$. In this case, the *Kalman* gain and likelihood variables $\{g^d_{M,N}, \gamma^d_M(N)\}$ corresponding to the downdating solution are analogously defined as in the update RLS equations. Table 1 lists the GSWRLS recursions of [1].

Note: Since we shall later exploit the shift structure nature of the input regressors $u_{M,i} = [\, u_0(i)\ u_1(i)\ \cdots\ u_{M-1}(i) \,]$, we will assume that the regularization matrix Π_M is of the special diagonal form

$\Pi_M^{-1} \triangleq \mu \cdot \mathrm{diag}(\lambda^{Q-1}, \lambda^{Q-2}, \cdots, \lambda^{Q-M})$, for a small positive μ, and where Q is the order of the model we wish to estimate.

Initialization		
Set:	η_0 satisfying $0 < \eta_0 \leq 1$	$w_{M,-1} = 0$
	λ satisfying $0 \ll \lambda \leq 1$	$P_{M,-1} = \Pi_M$
For $N \geq 0$, repeat:		
$\gamma_M^d(N-1) = (-\eta_0^{-1}\lambda^{1-L} + u_{M,N-L}P_{M,N-1}u_{M,N-L}^*)^{-1}$		
$g_{M,N-1}^d = P_{M,N-1}u_{M,N-L}^*\gamma_M^d(N-1)$		
$P_{M,N-1}^d = P_{M,N-1} - g_{M,N-1}^d g_{M,N-1}^{d*}\gamma_M^{-d}(N-1)$		
$w_{M,N-1}^d = w_{M,N-1} + g_{M,N-1}^d[d(N-L) - u_{M,N-L}w_{M,N-1}]$		
$\gamma_M(N) = (1 + \lambda^{-1}u_{M,N}P_{M,N-1}^d u_{M,N}^*)^{-1}$		
$g_{M,N} = \lambda^{-1}P_{M,N-1}^d u_{M,N}^*\gamma_M(N)$		
$P_{M,N} = \lambda^{-1}P_{M,N-1}^d - g_{M,N}g_{M,N}^*\gamma_M^{-1}(N)$		
$w_{M,N} = w_{M,N-1}^d + g_{M,N}[d(N) - u_{M,N}w_{M,N-1}^d]$		

Table 1. *Generalized Sliding Window RLS (GSWRLS) algorithm.*

3. GSWRLS LATTICE RECURSIONS

Let us define the *a posteriori* estimation error vectors:

$$e_{M,N} \triangleq y_N - H_{M,N}w_{M,N}, \tag{2}$$

$$e_{M,N}^d \triangleq y_N - H_{M,N}w_{M,N}^d. \tag{3}$$

According to our notation, scalars are indicated with parenthesis, e.g., the last element of $e_{M,N}$ is given by $e_{M,N}(N)$, while the $(N-L)$-th entry of $e_{M,N-1}^d$ is denoted by $e_{M,N-1}^d(N-L)$.

Now suppose one more column is appended to $H_{M,N}$:

$$H_{M+1,N} = \begin{bmatrix} H_{M,N} & x_{M,N} \end{bmatrix}. \tag{4}$$

The joint process estimation order-updates $e_{M,N}$ and $e_{M,N}^d$ to $e_{M+1,N}$ and $e_{M+1,N}^d$, respectively. This is accomplished by writing the updates for $P_{M+1,N}^{-1}$ and $P_{M+1,N}^{-d}$ using (4), where

$$P_{M,N}^{-1} \triangleq \lambda^{N+1}\Pi_M^{-1} + H_{M,N}^* W_N H_{M,N}, \tag{5}$$

$$P_{M,N}^{-d} \triangleq \lambda^{N+1}\Pi_M^{-1} + H_{M,N}^* W_N^d H_{M,N}. \tag{6}$$

Let the solutions to the update and downdate backward prediction problems be given by $w_{M,N}^b \triangleq P_{M,N}H_{M,N}^* W_N x_{M,N}$ and $w_{M,N}^{bd} \triangleq P_{M,N}^d H_{M,N}^* W_N^d x_{M,N}$, respectively. It follows that the minimum costs $\zeta_M^b(N)$ and $\zeta_M^{bd}(N)$ from these projections are

$$\zeta_M^b(N) \triangleq \mu\lambda^{N+Q-M} + x_{M,N}^* W_N b_{M,N} \tag{7}$$

$$\zeta_M^{bd}(N) \triangleq \mu\lambda^{N+Q-M} + x_{M,N}^* W_N^d b_{M,N}^d, \tag{8}$$

where the *backward* prediction error vectors are defined as

$$b_{M,N} \triangleq x_{M,N} - H_{M,N}w_{M,N}^b, \tag{9}$$

$$b_{M,N}^d \triangleq x_{M,N} - H_{M,N}w_{M,N}^{bd}. \tag{10}$$

The so-called reflection coefficients for the update and downdate solutions are thus written as

$$\begin{cases} \kappa_M(N) \triangleq \dfrac{b_{M,N}^* W_N y_N}{\zeta_M^b(N)} \triangleq \dfrac{\rho_M^*(N)}{\zeta_M^b(N)}, & (11) \\[2ex] \kappa_M^d(N) \triangleq \dfrac{b_{M,N}^{d*} W_N^d y_N}{\zeta_M^{bd}(N)} \triangleq \dfrac{\rho_M^{d*}(N)}{\zeta_M^{bd}(N)}. & (12) \end{cases}$$

For convenience, we denote the inner products in (7) and (8) by

$$\xi_M^b(N) \triangleq x_{M,N}^* W_N b_{M,N}, \quad \xi_M^{bd}(N) \triangleq x_{M,N}^* W_N^d b_{M,N}^d. \tag{13}$$

Applying the matrix inversion lemma to $P_{M+1,N}^{-1}$ and $P_{M+1,N}^{-d}$, and substituting the results in (2) and (3), respectively, we obtain order-updates for the *a posteriori* estimation error vectors:

$$e_{M+1,N}(N) = e_{M,N}(N) - \kappa_M(N)b_{M,N}(N) \tag{14}$$

$$e_{M+1,N-1}^d(N-L) = e_{M,N-1}^d(N-L) - \kappa_M^d(N-1)b_{M,N-1}^d(N-L) \tag{15}$$

3.1. Forward and Backward Order-Updates

In order to update the *a posteriori* errors, we shall consider the following decompositions:

$$H_{M+1,N} = \begin{bmatrix} x_{0,N} & \bar{H}_{M,N} \end{bmatrix} \tag{16}$$

$$\bar{H}_{M+1,N} = \begin{bmatrix} \bar{H}_{M,N} & x_{M+1,N} \end{bmatrix} \tag{17}$$

We define the LS solutions for the forward update and downdate problems as $w_{M,N}^f \triangleq \bar{P}_{M,N}\bar{H}_{M,N}^* W_N x_{0,N}$, and $w_{M,N}^{fd} \triangleq \bar{P}_{M,N}^d \bar{H}_{M,N}^* W_N^d x_{0,N}$, as well as the ones of the backward prediction problems $w_{M,N}^{\bar{b}} \triangleq \bar{P}_{M,N}\bar{H}_{M,N}^* W_N x_{M+1,N}$ and $w_{M,N}^{\bar{b}d} \triangleq \bar{P}_{M,N}^d \bar{H}_{M,N}^* W_N^d x_{M+1,N}$, where, similarly to (5) and (6),

$$\bar{P}_{M,N}^{-1} \triangleq \lambda^N \Pi_M^{-1} + \bar{H}_{M,N}^* W_N \bar{H}_{M,N}, \tag{18}$$

$$\bar{P}_{M,N}^{-d} \triangleq \lambda^N \Pi_M^{-1} + \bar{H}_{M,N}^* W_N^d \bar{H}_{M,N}. \tag{19}$$

In order to write the *prediction* reflection coefficients, we define

$$f_{M,N} \triangleq x_{0,N} - \bar{H}_{M,N}w_{M,N}^f \tag{20}$$

$$f_{M,N}^d \triangleq x_{0,N} - \bar{H}_{M,N}w_{M,N}^{fd} \tag{21}$$

$$\zeta_M^f(N) \triangleq \mu\lambda^{N+Q} + x_{0,N}^* W_N f_{M,N} \tag{22}$$

$$\zeta_M^{fd}(N) \triangleq \mu\lambda^{N+Q} + x_{0,N}^* W_N^d f_{M,N}^d \tag{23}$$

$$\xi_M^f(N) \triangleq x_{0,N}^* W_N f_{M,N}, \quad \xi_M^{fd}(N) \triangleq x_{0,N}^* W_N^d f_{M,N}^d \tag{24}$$

$$\bar{b}_{M,N} \triangleq x_{M+1,N} - \bar{H}_{M,N}w_{M,N}^{\bar{b}} \tag{25}$$

$$\bar{b}_{M,N}^d \triangleq x_{M+1,N} - \bar{H}_{M,N}w_{M,N}^{\bar{b}d} \tag{26}$$

$$\zeta_M^{\bar{b}}(N) \triangleq \mu\lambda^{N+Q-M-1} + x_{M+1,N}^* W_N \bar{b}_{M,N} \tag{27}$$

$$\zeta_M^{\bar{b}d}(N) \triangleq \mu\lambda^{N+Q-M-1} + x_{M+1,N}^* W_N^d \bar{b}_{M,N}^d \tag{28}$$

$$\xi_M^{\bar{b}}(N) \triangleq x_{M+1,N}^* W_N \bar{b}_{M,N}, \; \xi_M^{\bar{b}d}(N) \triangleq x_{M+1,N}^* W_N^d \bar{b}_{M,N}^d \tag{29}$$

so that:

$$\begin{cases} \kappa_M^f(N) \triangleq \dfrac{\bar{b}_{M,N}^* W_N x_{0,N}}{\zeta_M^{\bar{b}}(N)} \triangleq \dfrac{\delta_M^*(N)}{\zeta_M^{\bar{b}}(N)} & (30) \\[2ex] \kappa_M^{fd}(N) \triangleq \dfrac{\bar{b}_{M,N}^{d*} W_N^d x_{0,N}}{\zeta_M^{\bar{b}d}(N)} \triangleq \dfrac{\delta_M^{d*}(N)}{\zeta_M^{\bar{b}d}(N)} & (31) \\[2ex] \kappa_M^b(N) \triangleq \dfrac{f_{M,N}^* W_N x_{M+1,N}}{\zeta_M^f(N)} = \dfrac{\delta_M(N)}{\zeta_M^f(N)} & (32) \\[2ex] \kappa_M^{bd}(N) \triangleq \dfrac{f_{M,N}^{d*} W_N^d x_{M+1,N}}{\zeta_M^{fd}(N)} = \dfrac{\delta_M^d(N)}{\zeta_M^{fd}(N)} & (33) \end{cases}$$

Now, using (16) and (17) to get the order-updates for $P_{M+1,N}$, $P_{M+1,N}^d$, $\bar{P}_{M+1,N}$, $\bar{P}_{M+1,N}^d$, and substituting these expressions into the estimation errors definitions (9), (10), (20), and (21), we obtain

$$b_{M+1,N}(N) = \bar{b}_{M,N}(N) - \kappa_M^b(N)f_{M,N}(N) \tag{34}$$

x_N	$\overline{H}_N$	z_N	Δ^d_{N-1}	$\tilde{x}^d_{N-1}(N-L)$	$\tilde{z}^d_{N-1}(N-L)$	$\overline{\gamma}^d(N-1)$	Δ_N	$\tilde{x}_N(N)$	$\tilde{z}_N(N)$	$\overline{\gamma}(N)$
y_N	$H_{M,N}$	$x_{M,N}$	$\rho^d_M(N-1)$	$e^d_{M,N-1}(N-L)$	$b^d_{M,N-1}(N-L)$	$\gamma^d_M(N-1)$	$\rho_M(N)$	$e_{M,N}(N)$	$b_{M,N}(N)$	$\gamma_M(N)$
$x_{M+1,N}$	$\bar{H}_{M,N}$	$x_{0,N}$	$\delta^{d*}_M(N-1)$	$\bar{b}^d_{M,N-1}(N-L)$	$f^d_{M,N-1}(N-L)$	$\bar{\gamma}^d_M(N-1)$	$\delta^*_M(N)$	$\bar{b}_{M,N}(N)$	$f_{M,N}(N)$	$\bar{\gamma}_M(N)$
$x_{M+1,N}$	$\bar{H}_{M,N}$	$x_{M+1,N}$	$\xi^{\bar{b}d}_M(N-1)$	$\bar{b}^d_{M,N-1}(N-L)$	$\bar{b}^d_{M,N-1}(N-L)$	$\bar{\gamma}^d_M(N-1)$	$\xi^{\bar{b}}_M(N)$	$\bar{b}_{M,N}(N)$	$\bar{b}_{M,N}(N)$	$\bar{\gamma}_M(N)$
$x_{0,N}$	$\bar{H}_{M,N}$	$x_{0,N}$	$\xi^{fd}_M(N-1)$	$f^d_{M,N-1}(N-L)$	$f^d_{M,N-1}(N-L)$	$\bar{\gamma}^d_M(N-1)$	$\xi^f_M(N)$	$f_{M,N}(N)$	$f_{M,N}(N)$	$\bar{\gamma}_M(N)$
$x_{M,N}$	$H_{M,N}$	$x_{M,N}$	$\xi^{bd}_M(N-1)$	$b^d_{M,N-1}(N-L)$	$b^d_{M,N-1}(N-L)$	$\gamma^d_M(N-1)$	$\xi^b_M(N)$	$b_{M,N}(N)$	$b_{M,N}(N)$	$\gamma_M(N)$

Table 2. Description of all quantities involved in the computation of $\{\kappa^f_M(N), \kappa^{fd}_M(N), \kappa^b_M(N), \kappa^{bd}_M(N), \kappa_M(N), \kappa^d_M(N)\}$.

$$b^d_{M+1,N-1}(N-L) = \bar{b}^d_{M,N-1}(N-L) - \kappa^{bd}_M(N-1) f^d_{M,N-1}(N-L) \quad (35)$$

$$f_{M+1,N}(N) = f_{M,N}(N) - \kappa^f_M(N)\bar{b}_{M,N}(N) \quad (36)$$

$$f^d_{M+1,N-1}(N-L) = f^d_{M,N-1}(N-L) - \kappa^{fd}_M(N-1)\bar{b}^d_{M,N-1}(N-L) \quad (37)$$

3.2. Reflection Coefficients

The fact that the updating recursions of the GSWRLS algorithm rely on the downdating variables $\{P^d_{M,N-1}, w^d_{M,N-1}\}$ and vice-versa, suggests that analogous recursions relating the defining quantities of the reflection coefficients (11), (12) and (30)-(33) must too exist. In this section, we develop time-updates for the coefficients $\{\delta_M(N),\ \delta^d_M(N),\ \rho_M(N),\ \rho^d_M(N),\ \xi^b_M(N),\ \xi^{bd}_M(N),\ \xi^f_M(N),\ \xi^{fd}_M(N),\ \xi^{\bar{b}}_M(N),\ \xi^{\bar{b}d}_M(N)\}$ by following the same strategy of [7], i.e., we refer to a generic result for matrices explicitly written as

$$\begin{bmatrix} x_N & \overline{H}_N & z_N \end{bmatrix} \triangleq \begin{bmatrix} x_{N-1} & \overline{H}_{N-1} & z_{N-1} \\ x(N) & h_N & z(N) \end{bmatrix} \quad (38)$$

for an arbitrary structure $\overline{H}_N$, and column vectors x_N and z_N. Our goal is to update the inner product $\Delta_N \triangleq x^*_N W_N \tilde{z}_N$ as well as to obtain its partial downdate counterpart $\Delta^d_N \triangleq x^*_N W^d_N \tilde{z}^d_N$, considering $\tilde{z}_N \triangleq z_N - \overline{H}_N w^z_N$, $\tilde{z}^d_N \triangleq z_N - \overline{H}_N w^{zd}_N$, and the definitions:

$$w^z_N \triangleq \overline{P}_N \overline{H}^*_N W_N z_N, \quad w^{zd}_N \triangleq \overline{P}^d_N \overline{H}^*_N W^d_N z_N \quad (39)$$

$$\overline{P}_N \triangleq (\lambda^N \overline{\Pi}^{-1} + \overline{H}^*_N W_N \overline{H}_N)^{-1} \quad (40)$$

$$\overline{P}^d_N \triangleq (\lambda^N \overline{\Pi}^{-1} + \overline{H}^*_N W^d_N \overline{H}_N)^{-1} \quad (41)$$

$$\tilde{x}_N \triangleq x_N - \overline{H}_N w^x_N, \quad \tilde{x}^d_N \triangleq x_N - \overline{H}_N w^{xd}_N \quad (42)$$

$$w^x_N \triangleq \overline{P}_N \overline{H}^*_N W_N x_N, \quad w^{xd}_N \triangleq \overline{P}^d_N \overline{H}^*_N W^d_N x_N \quad (43)$$

$$\overline{\gamma}^{-1}(N) \triangleq 1 + \lambda^{-1} h_N \overline{P}^d_{N-1} h^*_N \quad (44)$$

$$\overline{\gamma}^{-d}(N) \triangleq -\lambda^{1-L}\eta_0^{-1} + h_{N-L-1}\overline{P}_N h^*_{N-L-1} \quad (45)$$

where $\overline{\Pi}$ is any positive-definite regularization matrix.

By using the LS update and downdates for $\{w^x_N, w^{xd}_N, w^z_N, w^{zd}_N\}$, it can be shown that the following key recursions hold:

$$\boxed{\begin{aligned} \Delta^d_{N-1} &= \Delta_{N-1} + \left(\lambda^{L-1}\eta_0\right)^2 \frac{\tilde{x}^{d*}_{N-1}(N-L)\tilde{z}^d_{N-1}(N-L)}{\overline{\gamma}^d(N-1)} & (46) \\ \Delta_N &= \lambda\Delta^d_{N-1} + \tilde{x}^*_N(N)\tilde{z}_N(N)/\overline{\gamma}(N) & (47) \end{aligned}}$$

where the likelihood variables for these problems are given by

$$\gamma^{-1}_M(N) \triangleq 1 + \lambda^{-1} u_{M,N} P^d_{M,N-1} u^*_{M,N} \quad (48)$$

$$\gamma^{-d}_M(N-1) \triangleq -\eta_0^{-1}\lambda^{1-L} + u_{M,N-L} P_{M,N-1} u^*_{M,N-L} \quad (49)$$

$$\bar{\gamma}^{-1}_M(N) \triangleq 1 + \lambda^{-1} \bar{u}_{M,N} \bar{P}^d_{M,N-1} \bar{u}^*_{M,N} \quad (50)$$

$$\bar{\gamma}^{-d}_M(N-1) \triangleq -\eta_0^{-1}\lambda^{1-L} + \bar{u}_{M,N-L} \bar{P}_{M,N-1} \bar{u}^*_{M,N-L} \quad (51)$$

with $\bar{u}_{M,N-L}$ and $\bar{u}_{M,N}$ given by the $(N-L)$-th and N-th rows of $\bar{H}_{M,N}$, respectively.

Using the general results of (46) and (47), we identify all the required variables for the reflection coefficients updates listed on Table 2.

Remark: It is straightforward to show that similar expressions hold for the modified costs, with ξ replaced by ζ, as, e.g., in (7) and (8).

Furthermore, it can be easily shown that the likelihood variables $\{\gamma_M(N), \gamma^d_M(N-1)\}$ are order-updated as

$$\gamma_{M+1}(N) = \gamma_M(N) - |b_{M,N}(N)|^2/\zeta^b_M(N), \quad (52)$$

$$\gamma^d_{M+1}(N-1) = \gamma^d_M(N-1) - (\lambda^{L-1}\eta_0)^2 \frac{|b^d_{M,N-1}(N-L)|^2}{\zeta^{bd}_M(N-1)}. \quad (53)$$

3.3. Exploiting Data Structure

Given that the data in $H_{M,N}$ possesses a shift structure (i.e., $H_{M,N}$ is Toeplitz-like), it follows that $\bar{\gamma}_M(N) = \gamma_M(N-1)$, $\bar{\gamma}^d_M(N-1) = \gamma^d_M(N-2)$, $\bar{b}_{M,N}(N) = b_{M,N-1}(N-1)$, $\bar{b}^d_{M,N-1}(N-L) = b^d_{M,N-2}(N-1-L)$, $\zeta^{\bar{b}}_M(N) = \zeta^b_M(N-1)$ and $\zeta^{\bar{b}d}_M(N-1) = \zeta^{bd}_M(N-2)$. With $u_0(i) \triangleq u(i)$, this allows us to finalize the flow graph that order-updates all the estimation errors, as shown on Fig. 2.

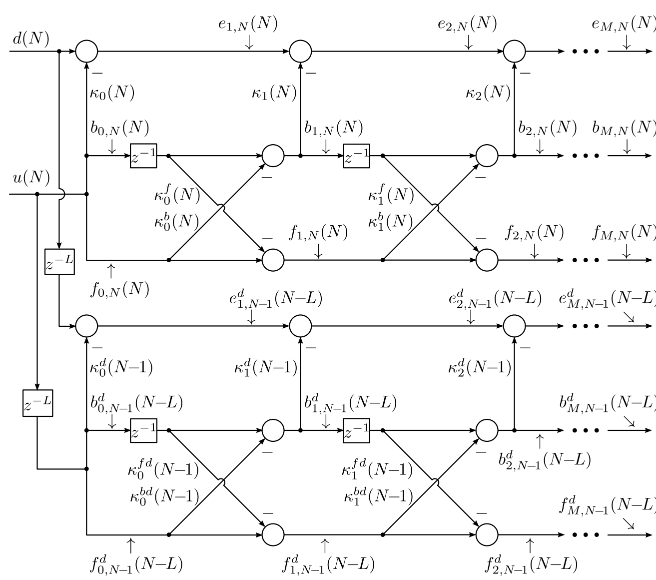

Fig. 2. Flow graph of the GSWRLSL algorithm.

The complete algorithm listing is shown in Table 3. It requires $16M$ complex additions, $10M$ real-complex products, $16M$ complex products and $16M$ complex divisions for each iteration. This represents twice as much as the EWRLSL, and roughly the same used by the GSWFTF recursions (without a stabilizing mechanism).

4. SIMULATIONS

Figure 3 illustrates a typical behavior of the SW-Lattice, the GSWFTF, and the proposed GSWRLSL algorithm (considering a window of length L for the former), for $\lambda = 0.98$, and $\eta_0 = 0.7$. The input signal is a *Composite Source Signal* (CSS), which was filtered through the impulse response of a typical line echo channel of length 248. The simulations were performed under 53 bits mantissa.

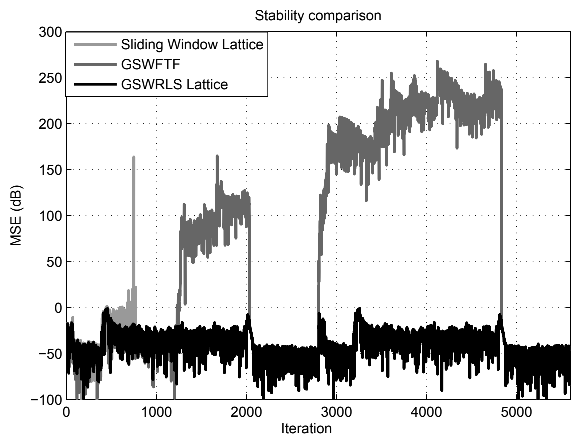

Fig. 3. Learning curves considering a CSS input.

We verified that, while the existing algorithms diverge at several points, the GSWRLSL remains stable for long observation periods. For this sort of input, we noticed that the choice $\lambda < 0.95$ produces unstable behavior for all algorithms (we also considered an autoregressive input with power spectrum function $\sigma_x^2/(1 - 0.9z^{-1})$ under 23 bits mantissa, for which no divergence was observed over millions of iterations, for $\lambda \geq 0.5$).

In order to compare the learning curves of algorithms with different windowing mechanisms, we have adjusted the length L so that the fastest convergence is achieved in the GSWRLS, APA, and the SWRLS filters. We have averaged 10^5 runs of a random walk model, and applied the same AR input as above, under a sudden change in the channel coefficients at $N = 300$, for $Q = 20$. Figure 4 illustrates the MSE decays for $L_{\text{GSWRLS}} = L_{\text{SWRLS}} = 25$, and $L_{\text{APA}} = 10$.

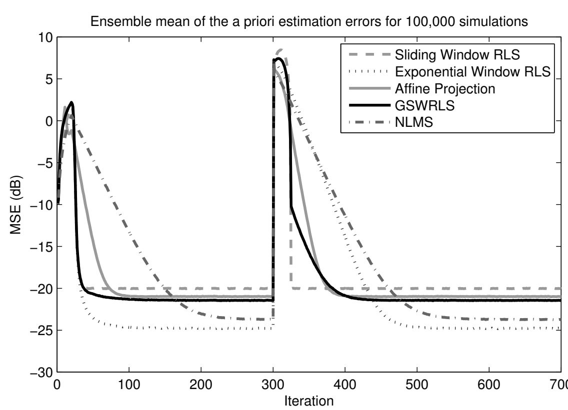

Fig. 4. Convergence rate and minimum MSE performance. We used $\lambda_{\text{GSWRLS}} = \lambda_{\text{EWRLS}} = 0.97$ and $\eta_0 = 0.98$.

Observe that, in this experiment, we assumed that the numerical issues inherent to fast transversal realizations do not arise. In practice, the fast APA and the SWRLS eventually become unstable, so we assumed infinite precision calculations in this example. The GSWRLS shows a mix of fast convergence characteristics, as well as rapid recovery from sudden changes in the channel coefficients. A lower MSE performance level closer to the one of the EWRLS can still be met by properly tuning the GSWRLS window parameters.

Initialization

Set:	*For $M \geq 0$ to $M = Q - 1$, repeat:*
η_0 satisfying $0 < \eta_0 \leq 1$	$\gamma_M(-1) = 1$
λ satisfying $0 \ll \lambda \leq 1$	$\gamma_M^d(-2) = -\eta_0\lambda^{L-1}$
μ small	$\zeta_M^b(-1) = \mu\lambda^{Q-M-1}$
Q the estimated channel order	$\zeta_M^{bd}(-2) = \mu\lambda^{Q-M-2}$
L the size of the window	$\zeta_M^f(-1) = \mu\lambda^{Q-1}$
	$\delta_M(-1) = \rho_M(-1) = 0$
	$b_{M,-1}(-1) = b_{M,-2}^d(-L-1) = 0$

For $N \geq 0$, repeat:

$\gamma_0(N) = 1$
$\gamma_0^d(N-1) = -\eta_0\lambda^{L-1}$
$e_{0,N}(N) = d(N)$
$e_{0,N-1}^d(N-L) = d(N-L)$
$f_{0,N}(N) = b_{0,N}(N) = u(N)$
$f_{0,N-1}^d(N-L) = b_{0,N-1}^d(N-L) = u(N-L)$

For $M \geq 0$ to $M = Q - 1$, repeat:

$\zeta_M^{bd}(N-1) = \zeta_M^b(N-1) + (\lambda^{L-1}\eta_0)^2|b_{M,N-1}^d(N-L)|^2/\gamma_M^d(N-1)$
$\zeta_M^{fd}(N-1) = \zeta_M^f(N-1) + (\lambda^{L-1}\eta_0)^2|f_{M,N-1}^d(N-L)|^2/\gamma_M^d(N-2)$
$\delta_M^d(N-1) = \delta_M(N-1) + (\lambda^{L-1}\eta_0)^2\,\frac{f_{M,N-1}^{d*}(N-L)b_{M,N-2}^d(N-L-1)}{\gamma_M^d(N-2)}$
$\rho_M^d(N-1) = \rho_M(N-1) + (\lambda^{L-1}\eta_0)^2\,\frac{e_{M,N-1}^{d*}(N-L)b_{M,N-1}^d(N-L)}{\gamma_M^d(N-1)}$
$\gamma_{M+1}^d(N-1) = \gamma_M^d(N-1) - (\lambda^{L-1}\eta_0)^2|b_{M,N-1}^d(N-L)|^2/\zeta_M^{bd}(N-1)$
$\kappa_M^d(N-1) = \rho_M^{d*}(N-1)/\zeta_M^{bd}(N-1)$
$\kappa_M^{bd}(N-1) = \delta_M^d(N-1)/\zeta_M^{fd}(N-1)$
$\kappa_M^{fd}(N-1) = \delta_M^{d*}(N-1)/\zeta_M^{bd}(N-2)$
$e_{M+1,N-1}^d(N-L) = e_{M,N-1}^d(N-L) - \kappa_M^d(N-1)b_{M,N-1}^d(N-L)$
$b_{M+1,N-1}^d(N-L) = b_{M,N-2}^d(N-L-1) - \kappa_M^{bd}(N-1)f_{M,N-1}^d(N-L)$
$f_{M+1,N-1}^d(N-L) = f_{M,N-1}^d(N-L) - \kappa_M^{fd}(N-1)b_{M,N-2}^d(N-L-1)$
$\zeta_M^b(N) = \lambda\zeta_M^{bd}(N-1) + |b_{M,N}(N)|^2/\gamma_M(N)$
$\zeta_M^f(N) = \lambda\zeta_M^{fd}(N-1) + |f_{M,N}(N)|^2/\gamma_M(N-1)$
$\delta_M(N) = \lambda\delta_M^d(N-1) + f_{M,N}^*(N)b_{M,N-1}(N-1)/\gamma_M(N-1)$
$\rho_M(N) = \lambda\rho_M^d(N-1) + e_{M,N}^*(N)b_{M,N}(N)/\gamma_M(N)$
$\gamma_{M+1}(N) = \gamma_M(N) - |b_{M,N}(N)|^2/\zeta_M^b(N)$
$\kappa_M(N) = \rho_M^*(N)/\zeta_M^b(N)$
$\kappa_M^b(N) = \delta_M(N)/\zeta_M^f(N)$
$\kappa_M^f(N) = \delta_M^*(N)/\zeta_M^b(N-1)$
$e_{M+1,N}(N) = e_{M,N}(N) - \kappa_M(N)b_{M,N}(N)$
$b_{M+1,N}(N) = b_{M,N-1}(N-1) - \kappa_M^b(N)f_{M,N}(N)$
$f_{M+1,N}(N) = f_{M,N}(N) - \kappa_M^f(N)b_{M,N-1}(N-1)$

Table 3. *Generalized Sliding Window RLS Lattice Algorithm.*

5. CONCLUSIONS

As expected, compared to the well known EWRLS lattice filter, the new algorithm requires an additional lattice in order to realize the partial downdate solution. Just like the covariance and the solutions of the downdate and update problems are obtained from one another in the non-fast recursions, the time-updates of the new variables used in the reflection coefficients of both lattices are analogously obtained from each other. The GSWRLSL filter is fast converging, robust, and far more stable then the GSWFTF, without requiring a stabilizing or rescuing procedure. The main advantage of the proposed filter is its inherent stability when compared to fast transversal realizations. Our next step is to obtain the lattice version of the GSWRLS algorithm for recurrence related input models (not a tapped-delay-line), as a counterpart of [9]. Fast array and normalized lattice variants of the GSWRLSL algorithm will be pursued in a forthcoming publication.

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