A K-BEST ORTHOGONAL MATCHING PURSUIT FOR COMPRESSIVE SENSING

Pu-Hsuan Lin[1] , *Shang-Ho Tsai*[1], *and Gene C.-H. Chuang*[2]

Department of Electrical Engineering National Chiao Tung University, Hsinchu, Taiwan[1]
Industrial Technology Research Institute, Hsinchu, Taiwan[2]

ABSTRACT

This paper proposes an orthogonal matching pursuit (OMP-) based recovering algorithm for compressive sensing problems. This algorithm can significantly improve recovering performance while it can still maintain reasonable computational complexity. Complexity analysis and simulation results are provided for the proposed algorithm and compared with other popular recovering schemes. We observe that the proposed algorithm can significantly improve the exact recovering performance compared to the OMP scheme. Moreover, in the cases with high compressed ratio, the proposed algorithm can even outperform the benchmark performance achieved by the subspace programming and linear programming.

Index Terms— Compressed sensing, orthogonal matching pursuit, K-best.

1. INTRODUCTION

Compressive sensing (CS) attracts extensive research attentions in recent years. This may be because most of the data resources can be thrown away without perceptual distortion in practical systems. An overview of CS is given in [1]. The authors in [2] showed that exact reconstruction is guaranteed when a sensing matrix satisfies the restricted isometry property (RIP) with a constant parameter. A sparse approximation problem is demonstrated in [2] and [4] to model the recovering problem. An l_0 minimization problem used to reconstruct sparse signals is provided in [2]. Unfortunately, the l_0 optimization problem is non-convex, and is NP-hard so that such reconstruction is difficult to be applied. To overcome this problem, a relaxed convex l_1 minimization was proposed in [3], and the reconstruction problems can be solved by linear programming (LP). However, the complexity of the LP may not be feasible in many applications.

To fast decode the compressed data and still keep the complexity as low as possible, a family of iterative greedy algorithms have been proposed. A basic iterative algorithm called the orthogonal matching pursuit (OMP) was proposed and analyzed in [6]. Although OMP is much simpler than the LP, it has however been shown that the OMP has weak guarantees of exact recovery. To improve the the recovering ability, several modified schemes based on the OMP have been proposed. One popular iterative algorithm called subspace pursuit (SP), which updates the subspace of column set iteratively instead of picking only one column greedily, was proposed in [5]. However, SP needs to estimate the signals via least square solution with double-size of sparsity and has poor performance with high compressed ratio [7]. Focusing on high compressed ratio situation, the look ahead OMP (LAOMP) algorithm, which uses multi-path OMP procedure and has better performance than SP, is proposed in [7]. The computational complexity of LAOMP is still high for large sparsity. In addition, this algorithm may early discard the correct path and result in serious error if the chosen element is not the correct one. The above discussion motivates us to investigate methods which not only can reduce complexity but also can avoid discarding correct path.

In this paper, we proposed a new recovering algorithm based on the OMP. The proposed algorithm extends and preserves multiple search paths simultaneously so that the probability of finding the correct locations of non-zero elements can be much improved, compared to the LAOMP. The complexity of the proposed scheme is only K times higher than that of the OMP. Simulation results compare two types of recovering performance for the proposed algorithm and various popular recovering methods including the SP, LP, OMP and LAOMP for Gaussian and zero-one signals. The results demonstrate that the proposed algorithm can achieve a good trade-off between computational complexity and recovering performance.

2. REVIEW OF COMPRESSIVE SENSING AND OMP

Compressive sensing is the process of acquiring and reconstructing a signal that is supposed to be sparse or compressible. The transformation between high dimension and low dimension representations can be modeled as the following linear under-determined equation:

$$\mathbf{y} = \mathbf{\Phi x}, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^N$ is a S-sparse signal (S-sparse means there are at most S non-zero elements in $\mathbf{x}$), $\mathbf{y} \in \mathbf{R}^m$ represents a vector of compressed signal from $\mathbf{x}$, and $\mathbf{\Phi}$ is an $m \times N$ sensing matrix which transfers $\mathbf{x}$ to $\mathbf{y}$ and its i-th column is ϕ_i. Since the OMP algorithm will be used to describe the proposed reconstruction method, we list the procedure of OMP in Algorithm 1 for convenience.

Algorithm 1: The OMP Algorithm [6].

1: **Input:** A received signal $\mathbf{y}$ and a sensing matrix $\mathbf{\Phi} = [\phi_1, \cdots, \phi_N]$.
2: **Initialization:** Let $\Omega = \emptyset$, and the residual vector $\mathbf{y}_{\text{res}} = \mathbf{y}$.
3: **for** $s = 1$ to S **do**
4: **Identify:** $r = \underset{k \in \{1, \cdots, N\}, k \notin \Omega}{\arg\max} |\langle \mathbf{y}_{\text{res}}, \phi_k \rangle|$, where $\langle \cdot, \cdot \rangle$ is the inner product. $\Omega = \Omega \cup \phi_r$.
5: **Estimate:** $\hat{\mathbf{x}} = \underset{\mathbf{u} \in \mathbb{R}^s}{\arg\min} ||\mathbf{y} - \mathbf{\Phi}_\Omega \mathbf{u}||_2$, where $\mathbf{\Phi}_\Omega$ represents a submatrix of $\mathbf{\Phi}$ whose columns are chosen from the indices set Ω.
6: **Update:** $\mathbf{y}_{\text{res}} = \mathbf{y} - \mathbf{\Phi}_\Omega \hat{\mathbf{x}}$.
7: **end for**
8: **Output:** Return the vector $\mathbf{x}$ with components $x_j = \hat{x}_j$ for $j \in \Omega$ and $x_j = 0$ otherwise.

The main procedure of the OMP are Step 4 to Step 6. Step 4 is for identification. In this step, one column of a sensing matrix $\mathbf{\Phi}$, which is most corrected with the current residuals is selected, where the residual is the remaining part of vector $\mathbf{y}$ after $\mathbf{y}$ deducts the effect of the chosen columns and estimated signal. Then this selected column index are added to the set of the selected column indices $\mathbf{\Phi}_\Omega$. Step 5 calculates the estimated signal $\hat{\mathbf{x}}$, which is the least square solution of the submatrix $\mathbf{\Phi}_\Omega$. For the updating procedure in Step 6, the residual vector $\mathbf{y}_{\text{res}}$ is updated by projecting the compressed signal $\mathbf{y}$ onto the subspace spanned by the selected columns. The steps are repeated until the number of indices in Ω is equal to S.

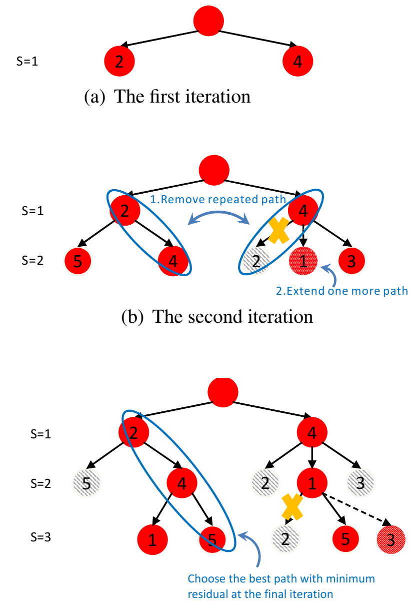
Although the procedure of the OMP is simple, this algorithm only selects one column in each iteration so that it does not have a high exact recovering rate for large sparsity. Thus the OMP has weaker guarantee and more restrict RIP condition than the SP and LP. To improve of the exact recovering rate of the OMP, we propose a modified OMP-based algorithm called the K-best OMP, which preserves the best K elements in each iteration and each one extends K more best paths, in next section.

3. PROPOSED K-BEST ORTHOGONAL MATCHING PURSUIT

The proposed K-best method can be explained by tree-search structure. A simple example can help describing the proposed scheme easier. Let $N = 5$, $m = 4$, $S = 3$ and $K = 2$. Fig. 1(a) to Fig. 1(c) show the idea of the proposed searching process. In this example, three iterations are needed and two best paths are preserved at each iteration. The first iteration is shown in Fig. 1(a), where we have conducted the inner product operation (Step 4. in Algorithm 1), and then preserve the best $K = 2$ columns with the maximum absolute inner-product values. Here the second and the fourth columns are preserved and the numbers in the circles of the tree nodes represent the preserved column indices of $\mathbf{\Phi}$. At the second iteration, each of the preserved two nodes extends $K = 2$ nodes with the maximum absolute inner-product values. Therefore, there are total $K^2 = 4$ paths. However, some of the K^2 paths are repeated, *e.g.* the paths $\{2, 4\}$ and {4,2} in Fig. 1(b) are repeated, which means these two paths select the same columns but with different order. In this case, we find another new node which has the third maximum absolute inner-product value, and then eliminate the repeated path. In this example, the new path is $\{4, 1\}$. Then the residuals are updating for these K^2 paths, and the K best paths with minimum residuals are chosen from them. Here paths $\{2, 4\}$ and $\{4, 1\}$ are preserved at the second iteration. The same procedure are conducted until the preserved number of columns equals to the number S of sparse. Fig. 1(c) shows the final iteration, where the best path $\{2, 4, 5\}$ is selected to be the solution with the minimum residual. The proposed K-best OMP algorithm is concluded in Algorithm 2.

4. COMPLEXITY ANALYSIS

Since the proposed scheme is based on the OMP scheme, we first analyze the complexity for OMP. For OMP procedure, the dominated computations are Steps 4, 5 and 6 in Algorithm 1. The evaluation in Step 4 needs $N - s + 1$ inner products of $m \times 1$ vectors in the s-th iteration. Thus the complexity is with order of $\mathcal{O}(m(N - s + 1))$. The estimated signal in Step 6 is obtained by computing the least square solution (pseudo-inverse); that is, $\mathbf{\Phi}_\Omega^\dagger = (\mathbf{\Phi}_\Omega^H \mathbf{\Phi}_\Omega)^{-1} \mathbf{\Phi}_\Omega^H$. Hence, in the s-th iteration, the computational complexity is of order $\mathcal{O}(ms^2 + s^3)$. The residual update in Step 7 requires complexity order of $\mathcal{O}(ms)$. Therefore, we can conclude the overall complexity



(a) The first iteration

(b) The second iteration

(c) The final iteration

Fig. 1. An example for the proposed K-beat OMP with $N = 5$, $m = 4$, $S = 3$ and $K = 2$.

Algorithm 2: Proposed K-best OMP.

Input: A received signal $\mathbf{y}$ and a sensing matrix $\mathbf{\Phi}$.
Initialization: For the first iteration, choose the K best columns with the maximum inner-product values and obtain the residuals obtained for viewing K nodes together.
for $s = 1$ to S **do**
  Identify: For each preserved path, extend it to K nodes with the maximum inner-product values.
  Check and Renew: Check whether there is repeated path for all K^2 extending paths. If it is, remove the repeat path and add a new path and, then check again until all K^2 paths contain different column indices.
  Estimate: Estimate the signals for all K^2 preserved paths using the Estimate step in OMP.
  if $s < S$ **then**
    Update and Preserve: Obtain the residuals for all K^2 preserved paths. Compare the l_2 norms for all residuals, and preserve the K paths which has the smallest residuals.
  else if $s = S$ **then**
    Decision: Choose the path with the smallest l_2 norm of residual and whose index are the desired column set.
  end if
end for
Output: Return the estimated signal vector corresponding to desired column set.

is with order of

$$\sum_{s=1}^{S} \mathcal{O}(m(N - s + 1) + ms^2 + s^3 + ms) \approx \mathcal{O}(NmS + \frac{mS^3}{3}). \tag{2}$$

For the proposed K-best OMP, since there are K preserved paths from the second to K-th iterations the computations of the inner product are K times higher than that of the OMP scheme. Also since each path extends to K nodes. As a result K^2 times evaluations of pseudo-inversions and residual updates are needed compared to the OMP. From the analysis for OMP, we can conclude that overall complexity is with order of

$$\sum_{s=1}^{S} \mathcal{O}(Km(N-s+1) + K^2(ms^2 + s^3 + ms))$$
$$\approx \mathcal{O}(KNmS + \frac{K^2 mS^3}{3}). \quad (3)$$

To compare the complexity to other schemes conveniently, we summarize the computational complexity of various schemes including the OMP, LAOMP, the proposed OMP, SP and LP in Tab. 1.

5. SIMULATION RESULTS

In the section, simulation results are provided to demonstrate the performance of the proposed algorithms. In all simulation, more than 500 different sensing matrices are used. The elements of the sensing matrix $\mathbf{\Phi}$ are i.i.d Gaussian variables with zero mean, and the columns are normalized to unit-norm. We use S-sparse real signals whose non-zero elements generated from Gaussian source with zero-mean and unit variance. Moreover, from [7], we define average support-cardinality error (ASCE) as $1 - \frac{\mathbb{E}\{\text{\# correct sparse locations}\}}{S}$ and the fraction of measurements (FoM) as $\alpha = \frac{m}{N}$.

Example 1. Frequency of exact reconstruction for Gaussian signal: Let $N = 200$ and $m = 50$. Fig. 2 shows the the exact recovering rate as a function of the sparsity S. For the Gaussian signal, we see that both the SP and LP schemes has the recovering ability that can perfectly reconstruct signals with sparsity (called *critical sparsity* in [5]) up to $S = 8$; the OMP algorithm which has the worst performance can achieve exact reconstruction for $S \leq 4$ due to its weak and nonuniform guarantees of exact recovery. Compared to the OMP, the proposed scheme with $K = 2$ can improve the critical sparsity from 4 to 8 and outperforms SP and LP for $S \geq 8$.

Compared to the LAOMP [7], the proposed algorithm is not only better than the LAOMP under the same K but the complexity of the proposed method is much lower than the LAOMP (see Tab. 1). It is worth to emphasize that, unlike LAOMP whose recover ability is insensitive to the increment of K, the recover rate performance of proposed algorithm can be improved significantly while K is increased. In a case with $K = 5$ and 10, the recovering ability of the proposed method can be up to 12, and the performance gap between the proposed method and the LAOMP becomes more pronounced.

Example 2. Average support-cardinality error for Gaussian: In this example, the distortion of locations for sparse elements is considered. Let $N = 200$ and $S = 10$. ASCE performance versus α, *i.e.* FoM, for the Gaussian signal is shown in Fig. 3. Observing that for the range of α from 0.1 to 0.3, the SP and LP schemes perform worse than the OMP-based methods. This implies that in high compressed ratio (N/m) scenario, if the sparse signals cannot be exactly reconstructed, the SP and LP may lead to serious error of the non-zero locations. Interestingly, the OMP-based methods have much better correcting rate of non-zero locations than the SP and LP. Furthermore, the multi-path OMP schemes *i.e. LAOMP and the proposed K-best OMP* greatly outperform the conventional OMP scheme. On the other hand, the ASCE of the proposed K-best OMP is slightly worse than that of the LAOMP when $K = 2$. For

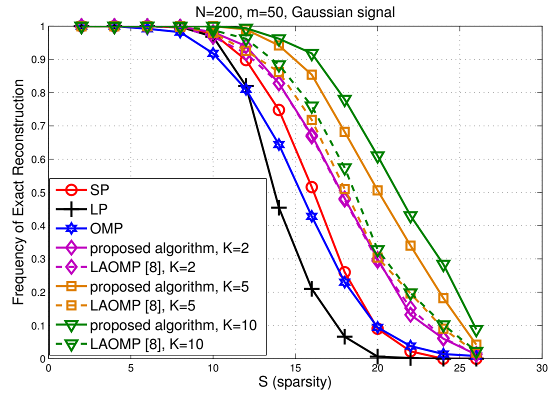


Fig. 2. Recovering rate for Gaussian signal ($N = 200$, $m = 50$).

$K = 5$ and 10, however, the performance of the proposed method exceeds the LAOMP. At extreme α values from 0.1 to 0.14, the proposed algorithm with $K = 5$ even outperforms LAOMP with $K = 10$. Therefore, we conclude when the complexity of the proposed scheme and the LAOMP is comparable, the proposed algorithm should outperform the LAOMP.

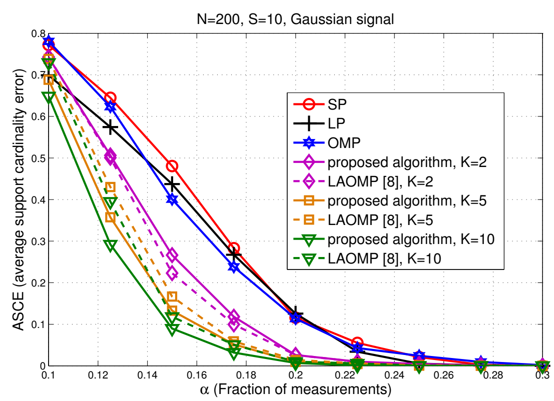


Fig. 3. ASCE performance for Gaussian signal ($N = 200$, $S = 10$).

Example 3. Recovery performance and complexity under fixed $\frac{m}{S}$ values: In this example, we would like to investigate how the compressed ratio affect the recovering performance. Let $N = 200$ and fix the ratio of $\frac{m}{S} = 5$. Fig. 4 shows the exact recovering rate versus m for the Gaussian signal. Observe that when $\alpha < 0.15$, the OMP-based algorithms perform better than the SP and LP. That is, in a high compressed ratio, the SP and LP do not work well, especially the performance of the LP may crash down seriously. On the other hand, for the OMP-based schemes, the proposed K-best OMP and the LAOMP both significantly improve the performance of the conventional OMP. For $K = 2$, the proposed algorithm and the LAOMP can even keep the recovering ratio above 0.9 at $\alpha = 0.1$; for $K = 5$, the recovering ratio can further be increased up to 0.95. Moreover, at this case we can find the proposed algorithm

Table 1. Complexity comparison of various CS recovering algorithms

Algorithm	OMP [6]	LAOMP [7]	The proposed K-best OMP	SP [5]	LP [8]
Complexity order	$O\left(NmS+\frac{mS^3}{3}\right)$	$O\left(\frac{KS}{2}(NmS+\frac{mS^3}{3})\right)$	$O\left(KNmS+\frac{K^2mS^3}{3})\right)$	$O\left(NmS+2mS^3\right)$	$O\left(m^2N^{\frac{3}{2}}\right)$
Example for $N=256, m=64$ $S=16, K=2$	3.495×10^5	5.592×10^6	8.738×10^5	7.864×10^5	1.678×10^7

outperforms the LAOMP slightly with the same K. This observation is exciting because it means that under the cases of high compressed ratio, the proposed algorithm can maintain outstanding recovering rate but with much lower complexity than other schemes. To see this, the corresponding complexity comparison is also provided in Fig. 5. We see that the OMP has the lowest complexity; the SP has slightly higher complexity than the OMP. The LP has the highest complexity because its complexity order is proportional to $N^{3/2}$. When $K = 2$, the complexity order of proposed algorithm is very closed to the SP and the OMP because the dominated term of (3) is $KNmS$, only roughly two times of the OMP. On the other hand, the LAOMP is much complicated than the proposed algorithm because its complexity is proportional to S. When $K = 5$, although the complexity growth of the proposed algorithm becomes fast, it is still much lower than the LAOMP and LP. Moreover, the complexity of the LAOMP is grows faster than the LP for $K = 5$; more specifically when $\alpha > 0.46$, the complexity of the LAOMP exceeds the LP.

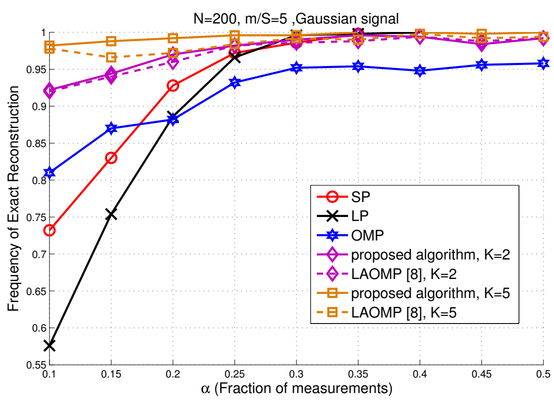


Fig. 4. Recovering rate for Gaussian signal ($N = 200$, $\frac{m}{S} = 5$).

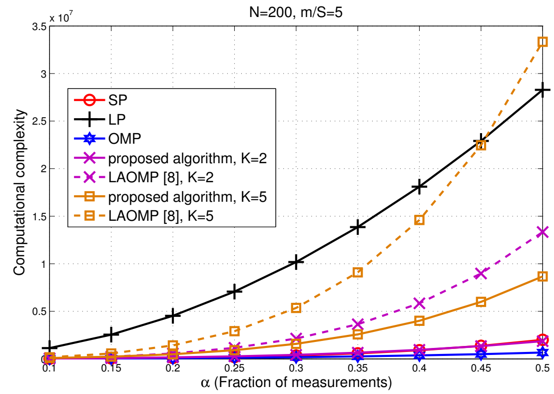


Fig. 5. Complexity order for Gaussian signal ($N = 200$, $\frac{m}{S} = 5$).

6. CONCLUSION

In this paper, a new algorithm, named K-best OMP that finds the best matching projections onto an over-complete dictionary, is proposed. The proposed method preserves the K best paths at each iteration to enlarge the probability of keeping the correct locations of sparse signals. Furthermore, the computational complexity analysis of the proposed scheme and its comparison to other schemes including the OMP, LAOMP, SP and LP were also provided. Finally, simulation results were given to show the advantages of the proposed scheme in terms of complexity and recovering performance.

7. REFERENCES

[1] E. J. Candès and M. B. Wakin, "An introduction to compressive sensing,"*IEEE signal processing magazine*, Mar. 2008.

[2] E. J. Candès, and T. Tao, "Decoding by linear programming," *IEEE Trans. Information Theory*, pp. 4203-4215, Dec. 2005.

[3] S. S. Chen, D. L. Donoho and M. A. Saunders, "Atomic decomposition by basis pursuit," *SIAM J. Sci. Comput.*, 1998.

[4] D. J. Donoho, "Compressed sensing," *IEEE Trans. Information Theory*, vol. 52, pp. 1289-1306, Apr. 2006.

[5] W. Dai and O. Milenkovic, "Subspace pursuit for compressive sensing signal reconstruction," *IEEE Trans. Information Theory*, vol. 55, pp. 2230-2249, May 2009.

[6] S. G. Mallat and Z. Zhifeng "Matching pursuits with time-frequency dictionaries," *IEEE Trans. Signal Processing,*, vol. 41, pp. 3397-3415, Dec. 1993.

[7] S. Chatterjee, D. Sundman and M. Skoglund, "Look ahead orthogonal matching pursuit" in *Proc. ICASSP'2011*, May 2005.

[8] I. E. Nesterov, A. Nemirovskii and Y. Nesterov, *Interior-point polynomial algorithms in convex programming*, 1st ed., Society for Industrial and Applied Mathematics, 1994