CONVERGENCE AND TRACKING ANALYSIS OF THE $\epsilon-$NSRLMF ALGORITHM

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ABSTRACT

In this work, the convergence and tracking behavior of the $\epsilon-$normalized sign regressor least mean fourth (NSRLMF) algorithm are analyzed in the presence of white and correlated Gaussian data. Furthermore, the stability bound on the step-size of the $\epsilon-$NSRLMF algorithm to ensure convergence in the mean, which also leads us to the mean convergence of the $\epsilon-$normalized sign regressor least mean mixed-norm (NSRLMMN) algorithm is derived. Finally, simulation results are conducted to confirm the validity and performance of the proposed adaptive algorithm for both white and correlated Gaussian regressors.

Index Terms— LMF, NLMF, SRLMF, NSRLMF, Convergence, Tracking.

1. INTRODUCTION

The normalized least mean fourth (NLMF) algorithm was introduced for two reasons [1]–[2]. First, to get better convergence rate as compared to the traditional least mean fourth (LMF) algorithm [3]. Second, to overcome the convergence dependency of the LMF algorithm on the input data correlation statistics.

On the other hand, the sign regressor least mean fourth (SRLMF) algorithm, which is based on clipping of the input data, was introduced in order to reduce the complexity of the LMF algorithm [4]. Then, it was also observed that the LMF and SRLMF algorithms converge at an almost identical rate for the case of real-valued data. However, the convergence behavior of both of these algorithms depends on the input data correlation statistics [3]–[4].

Motivated by the advantages of sign adaptive filters and NLMF algorithm as mentioned above we introduced the normalized version of the SRLMF algorithm and performed its steady-state analysis in [5]. In the present paper, the convergence and tracking behavior of the $\epsilon-$NSRLMF algorithm is analyzed and very well supported by simulations.

The remainder of the paper is organized as follows. In Section 2, a brief description of the $\epsilon-$NSRLMF algorithm is provided, while in Section 3, the tracking analysis of the $\epsilon-$NSRLMF algorithm is derived. Section 4 deals with the convergence analysis of the proposed algorithm. Simulation results are reported in Section 5 to validate the theoretical findings. Finally, Section 6 concludes the paper.

2. THE $\epsilon-$NSRLMF ALGORITHM

The weight update recursion of the $\epsilon-$NSRLMF algorithm is given by the following expression:

$$\mathbf{w}_i = \mathbf{w}_{i-1} + \frac{\mu}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\,\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} e_i^3, \quad i \geq 0, \tag{1}$$

where $\mathbf{w}_i$ is the updated weight vector, μ is the step-size, $\mathbf{u}_i$ is the regressor vector, ϵ is a small positive constant to avoid division by zero when the regressor is zero, e_i is the estimation error, $||\mathbf{u}_i||^2_{\mathrm{H}} = \mathbf{u}_i \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$, $\mathrm{H}[\mathbf{u}_i]$ is some positive-definite Hermitian matrix-valued function of $\mathbf{u}_i$ defined by

$$\mathrm{H}[\mathbf{u}_i] = \mathrm{diag}\left\{\frac{1}{|\mathbf{u}_{i_1}|}, \frac{1}{|\mathbf{u}_{i_2}|}, \ldots, \frac{1}{|\mathbf{u}_{i_{\mathrm{M}}}|}\right\}, \tag{2}$$

M is the filter length and $\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}} = \mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}$.

3. TRACKING ANALYSIS

Tracking analysis of the $\epsilon-$NSRLMF algorithm can be extended in a straightforward way using its mean-square analysis presented in [5] as there are only slight differences. We will therefore be brief in this section.

Here, let us assume that the data $\{d_i, \mathbf{u}_i\}$ satisfy the following conditions of the nonstationary data model [6]:

A.1 There exists an optimal weight vector $\mathbf{w}_i^o$ such that $d_i = \mathbf{u}_i\mathbf{w}_i^o + v_i$, where d_i is the desired sequence and v_i is the noise sequence with variance σ_v^2.

A.2 The weight vector varies according to the random-walk model $\mathbf{w}_i^o = \mathbf{w}_{i-1}^o + \mathbf{q}_i$, and the sequence $\mathbf{q}_i$ is independent and identically distributed (i.i.d.) with covariance matrix $\mathbf{Q}$. Moreover, $\mathbf{q}_i$ is independent of $\{v_j, \mathbf{u}_j\}$ for all i, j.

A.3 The initial conditions $\{\mathbf{w}_{-1}, \mathbf{w}_{-1}^o\}$ are independent of the zero mean random variables $\{d_i, \mathbf{u}_i, v_i, \mathbf{q}_i\}$.

For the adaptive filter of the form in (1), and for any data $\{d_i, \mathbf{u}_i\}$, assuming filter operation in steady-state, the following variance relation holds [6]:

$$\mu\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}\mathrm{g}^2[e_i]\right] + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = 2\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right], \quad \text{as } i \to \infty, \tag{3}$$

where $\mathrm{g}[e_i]$ denotes some function of e_i and for the $\epsilon-$NSRLMF algorithm $\mathrm{g}[e_i]$ is readily given by

$$\begin{aligned}\mathrm{g}[e_i] &= \frac{e_i^3}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}},\\ &= \frac{1}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\{e^3_{a_i} + v^3_i + 3e^2_{a_i}v_i + 3e_{a_i}v^2_i\},\end{aligned} \tag{4}$$

$e_{a_i} = \mathbf{u}_i(\mathbf{w}^o_i - \mathbf{w}_{i-1})$ is the a priori estimation error, the estimation error is

$$e_i = e_{a_i} + v_i, \tag{5}$$

and finally

$$\mathrm{E}[||\mathbf{u}_i||^2_{\mathrm{H}}] = \mathrm{E}[\mathbf{u}_i\mathrm{H}[\mathbf{u}_i]\mathbf{u}_i^{\mathrm{T}}]. \tag{6}$$

In [5], the following expressions for the terms $\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right]$ and $\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}\mathrm{g}^2[e_i]\right]$ were derived:

$$\mathrm{E}\left[e_{a_i}\mathrm{g}[e_i]\right] = \mathrm{E}\left[\frac{e^4_{a_i}}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\right] + 3\sigma^2_v\mathrm{E}\left[\frac{e^2_{a_i}}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\right], \tag{7}$$

$$\begin{aligned}\mathrm{E}\left[||\mathbf{u}_i||^2_{\mathrm{H}}\mathrm{g}^2[e_i]\right] &\approx 6\mathrm{E}\left[\tfrac{||\mathbf{u}_i||^2_{\mathrm{H}}e_{a_i}}{(\epsilon+||\mathbf{u}_i||^2_{\mathrm{H}})^2}\right]\mathrm{E}[v^5_i]\\ &+15\mathrm{E}\left[\tfrac{||\mathbf{u}_i||^2_{\mathrm{H}}e^2_{a_i}}{(\epsilon+||\mathbf{u}_i||^2_{\mathrm{H}})^2}\right]\xi^4_v + \mathrm{E}\left[\tfrac{||\mathbf{u}_i||^2_{\mathrm{H}}}{(\epsilon+||\mathbf{u}_i||^2_{\mathrm{H}})^2}\right]\xi^6_v,\end{aligned} \tag{8}$$

where $\xi^4_v = \mathrm{E}[v^4_i]$ and $\xi^6_v = \mathrm{E}[v^6_i]$ denote the fourth- and sixth-order moments of v_i, respectively. Finally, substituting expressions (7) and (8) into (3) we get

$$\mu\mathcal{Z}_1\xi^6_v + \mu^{-1}\mathrm{Tr}(\mathbf{Q}) = (6\sigma^2_v\mathcal{Z}_2 - 15\mu\mathcal{Z}_1\xi^4_v)\mathrm{E}[e^2_{a_i}], \tag{9}$$

where $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are defined, respectively, as

$$\mathcal{Z}_1 \triangleq \mathrm{E}\left[\frac{||\mathbf{u}_i||^2_{\mathrm{H}}}{(\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}})^2}\right], \tag{10}$$

$$\mathcal{Z}_2 \triangleq \mathrm{E}\left[\frac{1}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\right]. \tag{11}$$

Therefore, the expression for the tracking excess-mean-square error (EMSE) $\zeta = \mathrm{E}[e^2_{a_i}]$ for the $\epsilon-$NSRLMF algorithm is given by

$$\zeta = \frac{\mu\mathcal{Z}_1\xi^6_v + \mu^{-1}\mathrm{Tr}(\mathbf{Q})}{(6\sigma^2_v\mathcal{Z}_2 - 15\mu\mathcal{Z}_1\xi^4_v)}. \tag{12}$$

Consequently, the optimum step-size of the $\epsilon-$NSRLMF algorithm can be obtained by minimizing (12) with respect to μ and can be shown to be

$$\mu_{\mathrm{opt}} = \sqrt{\frac{\mathrm{Tr}(\mathbf{Q})}{\xi^6_v}\left[\frac{25(\xi^4_v)^2\mathrm{Tr}(\mathbf{Q})}{4\sigma^4_v(\mathcal{Z}_2)^2\xi^6_v} + \frac{1}{\mathcal{Z}_1}\right]} - \frac{5\xi^4_v\mathrm{Tr}(\mathbf{Q})}{2\sigma^2_v\mathcal{Z}_2\xi^6_v}. \tag{13}$$

Finally, the corresponding minimum value of the tracking mean-square error (MSE) of the $\epsilon-$NSRLMF algorithm is derived derived straight forward from (12) and is given by

$$\mathrm{E}\left[e^2_i\right] = \frac{\mu_{\mathrm{opt}}\mathcal{Z}_1\xi^6_v + \mu^{-1}_{\mathrm{opt}}\mathrm{Tr}(\mathbf{Q})}{(6\sigma^2_v\mathcal{Z}_2 - 15\mu_{\mathrm{opt}}\mathcal{Z}_1\xi^4_v)} + \sigma^2_v. \tag{14}$$

4. CONVERGENCE ANALYSIS

To carry out the convergence analysis of the $\epsilon-$NSRLMF algorithm we rely on the following assumptions [2]:

A.4 The noise sequence v_i is independent of $\mathbf{u}_j$ for all i, j and both sequences have zero mean.

A.5 The weight error vector $\widetilde{\mathbf{w}}_i$ (defined below) is independent of the input $\mathbf{u}_j$ for all i, j.

Subtracting both sides of (1) from $\mathbf{w}^o_i$ we get

$$\widetilde{\mathbf{w}}_i = \widetilde{\mathbf{w}}_{i-1} - \frac{\mu}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\,\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}e^3_i, \tag{15}$$

where the weight error vector $\widetilde{\mathbf{w}}_i$ is given by

$$\widetilde{\mathbf{w}}_i = \mathbf{w}^o_i - \mathbf{w}_i. \tag{16}$$

We know that, the desired sequence d_i is given by

$$d_i = \mathbf{u}_i\mathbf{w}^o_i + v_i, \tag{17}$$

and the estimation error e_i is given by

$$e_i = d_i - \mathbf{u}_i\mathbf{w}_{i-1}. \tag{18}$$

Then substituting (16) and (17) into (18) and expanding the term e^3_i, we get

$$e^3_i = (\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1})^3 + v^3_i + 3(\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1})^2v_i + 3\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1}v^2_i. \tag{19}$$

At convergence [2], the following holds:

$$(\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1})^3 \le \mathbf{u}_i\widetilde{\mathbf{w}}_{i-1}. \tag{20}$$

Since $(\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1})^3$ is a convex function for $\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1} \ge 0$, the above inequality is always true as long as $\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1} \le 1$. Therefore, (19) can be approximated by

$$e^3_i \approx \mathbf{u}_i\widetilde{\mathbf{w}}_{i-1} + v^3_i + 3(\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1})^2v_i + 3\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1}v^2_i. \tag{21}$$

Substituting (21) into (15) we get

$$\begin{aligned}\widetilde{\mathbf{w}}_i &= \widetilde{\mathbf{w}}_{i-1} - \frac{\mu}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\,\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}\Big[\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1} + v_i^3 \\ &\quad +3(\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1})^2 v_i + 3\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1}v_i^2\Big], \\ &= \left[\mathbf{I} - \frac{\mu}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}\,\mathbf{u}_i\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}(1+3v_i^2)\right]\widetilde{\mathbf{w}}_{i-1} \\ &\quad - \frac{\mu\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}}{\epsilon + ||\mathbf{u}_i||^2_{\mathrm{H}}}[v_i^3 + 3(\mathbf{u}_i\widetilde{\mathbf{w}}_{i-1})^2 v_i]. \end{aligned} \tag{22}$$

Taking the expectation of both sides of (22) under the above-mentioned assumptions and by ignoring ϵ as it is very small, we obtain

$$\mathrm{E}[\widetilde{\mathbf{w}}_i] = \left[\mathbf{I} - \mu\mathrm{E}\left[\frac{\mathbf{u}_i\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}}{||\mathbf{u}_i||^2_{\mathrm{H}}}\right](1+3\sigma_v^2)\right]\mathrm{E}[\widetilde{\mathbf{w}}_{i-1}]. \tag{23}$$

Now, let us use the following approximation:

$$\mathrm{E}\left[\frac{\mathbf{u}_i\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}}{||\mathbf{u}_i||^2_{\mathrm{H}}}\right] \approx \frac{\mathrm{E}\left[\mathbf{u}_i\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}\right]}{\mathrm{E}[||\mathbf{u}_i||^2_{\mathrm{H}}]}. \tag{24}$$

From [4], we have

$$\mathrm{E}[||\mathbf{u}_i||^2_{\mathrm{H}}] = \mathrm{E}\left[\mathbf{u}_i\mathrm{sign}[\mathbf{u}_i]^{\mathrm{T}}\right] = \sqrt{\frac{2}{\pi\sigma_u^2}}\mathrm{Tr}(\mathbf{R}). \tag{25}$$

Upon substituting (24) and (25) into (23), we have

$$\mathrm{E}[\widetilde{\mathbf{w}}_i] = \left[1 - \mu(1+3\sigma_v^2)\right]\mathrm{E}[\widetilde{\mathbf{w}}_{i-1}]. \tag{26}$$

From (26), it is easy to show that the mean behavior of the weight error vector, that is $\mathrm{E}[\widetilde{\mathbf{w}}_i]$, converges to the zero vector if the step-size μ is bounded by:

$$0 < \mu < \frac{2}{1+3\sigma_v^2}. \tag{27}$$

Note that the step-size bound of the $\epsilon-$NSRLMF algorithm in (27) is the same as that obtained for the NLMF algorithm in [2]. It is clear from (27) that the upper bound on the step-size of the $\epsilon-$NSRLMF algorithm no longer depends on the maximum eigenvalue, $\lambda_{\max}$, of the input data autocorrelation matrix as was in the case for the SRLMF algorithm [4].

In [7], it was mentioned that the step-size bound of the $\epsilon-$NSRLMMN algorithm can be obtained by combining the step-size bounds of the $\epsilon-$NSRLMS and $\epsilon-$NSRLMF algorithms. This is very clear from the fact that the $\epsilon-$NSRLMMN algorithm reduces to $\epsilon-$NSRLMF and $\epsilon-$NSRLMS algorithms when the mixing parameter, δ, takes the value 0 and 1, respectively. Therefore, by utilizing equation (27), the mean convergence of the $\epsilon-$NSRLMMN algorithm can now be approximated by:

$$0 < \mu_{\epsilon-\mathrm{NSRLMMN}} < 2\delta + \frac{2(1-\delta)}{1+3\sigma_v^2}. \tag{28}$$

5. SIMULATION RESULTS

Several simulation results are conducted to corroborate the theoretical findings in an unknown system identification scenario. For this purpose, $\epsilon = 10^{-6}$ and $\mathrm{M} = 5$ have fixed throughout this study. In Figures 1-2, the variance of the Gaussian noise sequence $\mathbf{q}_i$ in the random-walk model is fixed at $\sigma_q^2 = 10^{-8}$. Moreover, the correlated data can be obtained in the same way as was done in [7].

Figures 1-2 depict the tracking MSE of the $\epsilon-$NSRLMF algorithm using correlated and white Gaussian regressors, respectively. In these figures, the MSE is depicted as a function of the step-size for a signal-to-noise ratio (SNR) of 20 dB under an additive white Gaussian noise (AWGN) environment. It is seen in Figure 1 that the simulation results are in a close match with the analytical results for values of μ up to 0.5. A zoom into the region around $\mu = 0.1$ in Figure 1 shows that the tracking MSE possesses a minimum value of 0.01006151 at $\mu = 0.114$, which are in excellent agreement with the corresponding theoretical values of 0.01005815 and $\mu_{\mathrm{opt}} = 0.1143$ obtained from expressions (14) and (13), respectively. However, the simulation and analytical results are found to be in reasonable agreement for white Gaussian data as depicted in Figure 2.

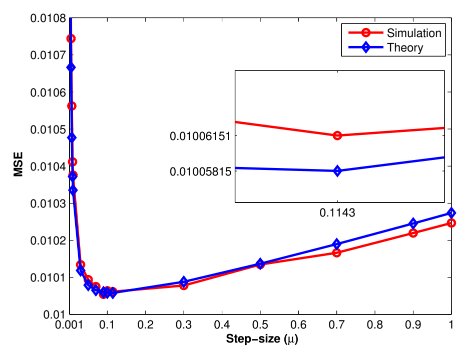

Fig. 1. Theoretical and simulated tracking MSE of the $\epsilon-$NSRLMF algorithm using correlated Gaussian regressors.

Finally, the results in Figures 3-4 compare the convergence behavior of the $\epsilon-$NSRLMF and $\epsilon-$NLMF algorithms in AWGN and uniform noise environments, respectively. In these figures, the convergence curves are plotted for both correlated and white Gaussian data at an SNR of 10 dB. As can be seen from these figures, the $\epsilon-$NLMF algorithm outperforms the $\epsilon-$NSRLMF algorithm with correlated Gaussian input. However, the performance of both algorithms is found to be similar in white Gaussian data.

6. CONCLUSIONS

In this work, expressions are derived for the tracking MSE and optimum step-size of the $\epsilon-$NSRLMF algorithm. A

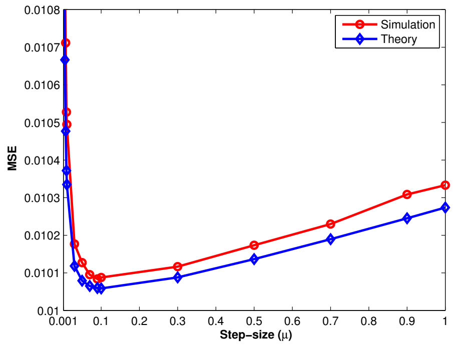

Fig. 2. Theoretical and simulated tracking MSE of the $\epsilon-$NSRLMF algorithm using white Gaussian regressors.

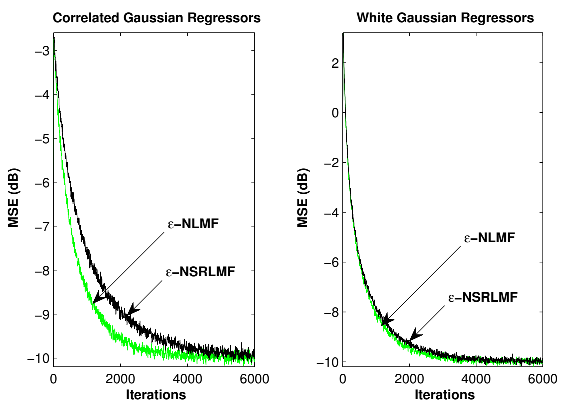

Fig. 3. Comparison of the MSE learning curves of $\epsilon-$NLMF and $\epsilon-$NSRLMF algorithms in AWGN environment.

sufficient condition for the convergence in the mean of the $\epsilon-$NSRLMF algorithm is also derived and is found to be the same as that of the NLMF algorithm. It is also shown that the upper bound on the step-size of the $\epsilon-$NSRLMF algorithm depends on the noise variance only and is independent of the input data correlation statistics. As a by-product of this work, the mean convergence of the $\epsilon-$NSRLMMN algorithm is also obtained. Finally, a close match between analytical and simulation results for correlated Gaussian data than white Gaussian data is obtained. Moreover, the effect of clipping on the performance of the $\epsilon-$NSRLMF algorithm is found to be more evident for correlated Gaussian data than white Gaussian data.

Current work is devised for the recently newly version of the NLMF algorithm [8]. Similarly, as was done in this work, future work is extending the presented idea to that in [8] and eventually compare their results. Due to the nature of the normalization in [8], it is expected that the analytical approach will be very involved.

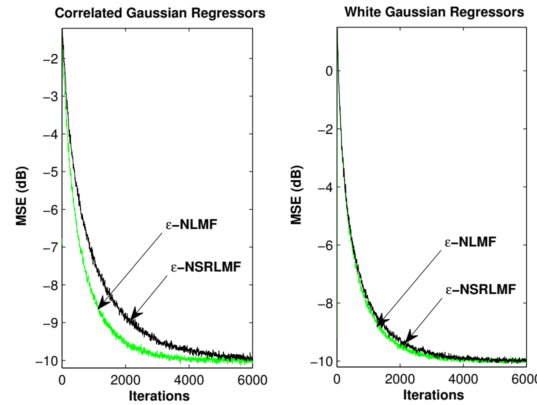

Fig. 4. Comparison of the MSE learning curves of $\epsilon-$NLMF and $\epsilon-$NSRLMF algorithms in a uniform noise environment.

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