LATTICE STRUCTURES FOR 2-D NON-SEPARABLE OVERSAMPLED LAPPED TRANSFORMS

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ABSTRACT

This paper proposes lattice structures for two-dimensional (2-D) non-separable (NS) oversampled (OS) lapped transforms. The proposed systems consist of the 2-D separable discrete cosine transform and NS support extension processes, and allow us to simply realize rational redundancy as well as the overlapping, paraunitary (PU), symmetric, real-valued and compact support property. The lattice structures have two aspects. One is an extention of OS linear-phase (LP) perfect reconstruction (PR) filter banks (FBs) to the 2-D NS case. The other is a generalization of 2-D NS LPPR FBs to the OS case. The significance is verified by showing some design examples and image restoration results with the iterative shrinkage/thresholding algorithm (ISTA).

Index Terms— Tight frame, LPPUFB, directional transform, lattice structure, image restoration

1. INTRODUCTION

Filter banks, or transforms, are essential components of signal processing and have found a wide variety of applications so far, such as compression, communication, denoising, restoration, feature extraction and so forth [1–4]. For image processing, filter banks are also utilized as indispensable tools to decompose and reconstruct a given image. In the classical approach, one-dimensional (1-D) filter banks are applied to the vertical and horizontal direction independently. This separable manner is quite simple, but does not exploit the full potential of 2-D systems and shows weakness in representation of diagonal edges and textures. Demands on high-quality image processing make us introduce NS filter banks [5–9].

Fig 1 shows a parallel structure of P-channel 2-D non-separable filter banks. The system consists of an analysis and synthesis bank, where $\mathbf{z} \in \mathbb{C}^2$ denotes a 2×1 complex variable vector $(z_\mathrm{y}, z_\mathrm{x})^T$ in the 2-D z-transform domain, $H_p(\mathbf{z})$ and $F_p(\mathbf{z})$ are the transfer functions of the p-th analysis and synthesis filter, respectively, and $\downarrow \mathbf{M}_p$ and $\uparrow \mathbf{M}_p$ are the downsampler and upsampler with factor $\mathbf{M}_p \in \mathbb{Z}^{2\times 2}$, respectively [1, 8]. The sampling ratio of the p-th channel, M_p, is given by $M_p = |\det(\mathbf{M}_p)|$. The total sum of the reciprocals of $\{M_p\}_{p=0}^{P-1}$, i.e. $\mathcal{R} = \sum_{p=0}^{P-1} \frac{1}{M}_p$, is referred to as redundancy. When $\mathcal{R} = 1$, the system is referred to as a critically-sampled (CS) filter bank. On the other hand, a system with $\mathcal{R} > 1$ is called an oversampled (OS) filter bank. Note that $\mathcal{R} \geq 1$ is required for perfect reconstruction (PR).

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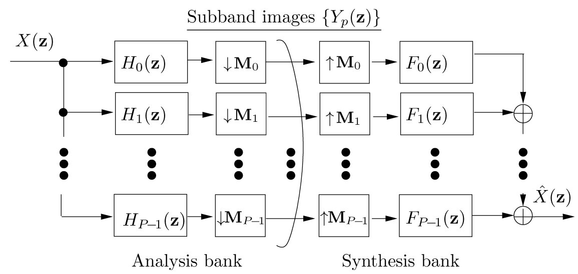

Fig. 1. Parallel structure of a P-channel filter bank.

Regardless of the number of dimensions, superiority of an OS filter bank to CS one is its high degree of design freedom [6, 10]. There is infinite combination of analysis and synthesis banks in the OS case to meet PR, while a CS system has a unique combination. This fact leads an optimal design problem of synthesis (analysis) bank for a given analysis (synthesis) bank [11, 12]. Finding an optimal dual bank can be interpreted as a problem of finding optimal subband images $\{Y_p(\mathbf{z})\}_{p=0}^{P-1}$ for a given image $X(\mathbf{z})$ and a fixed synthesis bank. Since there is infinite number of subband image candidates, we can make an optimal choice for a certain criteria. OS filter banks have close relation to the frame theory of the vector space [10, 13, 14]. With this rigid theory and recent development of optimization techniques, the significance has been verified through the application to image restoration such as deblurring, super-resolution and inpainting, as well as compressive sensing [6, 7, 15–17].

Preferable properties of filter banks for image processing include the linear-phase (LP), PR and compact-support ones. Thus, several LP PR FIR filter banks have been proposed so far [4, 8, 18, 19]. Table 1 summerizes some of such filter banks. The NS and OS properties are also of interest. Two simple ways are known to construct OS filter banks from CS ones. One is a mixture construction of multiple CS systems [7, 20], and the other is a non-subsampled, or shift-invariant, construction, which is realized by removing the downsamplers and upsamplers from a CS system. If the original CS systems are NS, the derived OS systems inherit the NS property. It, however, can hardly exploit the design freedom of OS systems with these two approaches. The redundancy $\mathcal{R}$ is restricted to be integer. Especially, the latter case tends to have large redundancy. Contourlet proposed by Do *et al.* satisfies all of the NS, LP and FIR property and exploits the design freedom [9]. There, however, is a restriction for simultaneously realizing both of the LP and PU property due to the

Table 1. Properties of several LPPR FIR filter banks, where mixture and nonsubsampled constructions are not taken into account. 'NS,' 'OS,' 'LP,' 'PR,' 'PU' and 'FB' denote 'non-separable,' 'oversampled,' 'linear phase,' 'perfect reconstruction,' 'paraunitary,' and 'filter bank,' respectively. As well, 'P' means 'possible.'

	NS	OS	LP	PU
DCT / Haar DWT [4]	No	No	Yes	Yes
5-3 DWT / 9-7 DWT [22]	No	No	Yes	No
LPPRFB [18]	No	Yes	Yes	P
NSLPPRFB [19]	Yes	No	Yes	P
OSLPPRFB [21]	No	Yes	Yes	P
Contourlet [9]	Yes	Yes	Restricted	
NSOLT (Proposal)	Yes	Yes	Yes	P

double filter bank structure. The PU property is another preferable characteristic for filter banks since it preserves the energy between images in the spatial and subband domain and is sometimes helpful for reducing design parameters and simplifying computations.

In this work, we propose a novel filter bank construction to obtain 2-D NS OS LPPR FIR systems. The proposed construction is real-valued and based on lattice structures, where rational redundancy is available and the PU property is simply realized. The structures are regarded as an extention of OSLPPRFBs to the 2-D NS case [21], and a generalization of the NSLPPRFBs to the OS case [8, 19, 20]. We refer to the proposed filter banks as non-separable oversampled lapped transforms and abbreviate them NSOLTs. The significance is verified by showing some design examples with rational redundancy and results of image restoration with ISTA.

2. 2-D NON-SEPARABLE OVERSAMPLED LPPRFB

Let us review the LP and PU condition in terms of polyphase matrices and describe important properties on the relation among the polyphase order, filter symmetry and the number of channels.

2.1. Parallel Structure and Polyphase Representation

Suppose that the sampling factors are all the same in Fig. 1, i.e. $\mathbf{M}_p = \mathbf{M}$ and $M_p = M = |\det(\mathbf{M})|$ for $p \in \{0, 1, 2, \cdots, P-1\}$. Then, the analysis bank $\{H_p(\mathbf{z})\}_{p=0}^{P-1}$ and synthesis bank $\{F_p(\mathbf{z})\}_{p=0}^{P-1}$ are respectively represented by

$$\mathbf{h}(\mathbf{z}) = \left(H_0(\mathbf{z})\ H_1(\mathbf{z})\ \cdots\ H_{P-1}(\mathbf{z}) \right)^T = \mathbf{E}(\mathbf{z}^{\mathbf{M}})\mathbf{d}(\mathbf{z}),$$
$$\mathbf{f}^T(\mathbf{z}) = \left(F_0(\mathbf{z})\ F_1(\mathbf{z})\ \cdots\ F_{P-1}(\mathbf{z}) \right) = \mathbf{d}^T(\mathbf{z}^{-\mathbf{I}})\mathbf{R}(\mathbf{z}^{\mathbf{M}})$$

and $\mathbf{R}(\mathbf{z})$ [1, 8, 20], where the superscript 'T' denotes the transposition and $\mathbf{d}(\mathbf{z})$ shows an $M \times 1$ vector which corresponds to a 2-D delay chain. $\mathbf{E}(\mathbf{z})$ and $\mathbf{R}(\mathbf{z})$ are 2-D polyphase matrices and of size $P \times M$ and $M \times P$, respectively. The redundancy $\mathcal{R}$ is given by $\mathcal{R} = P/M$. When $\mathbf{M}$ is diagonal and the elements are M_y and M_x, we have $\mathbf{z}^{\mathbf{M}} = (z_\mathrm{y}^{M_\mathrm{y}}, z_\mathrm{x}^{M_\mathrm{x}})^T$ and $M = M_\mathrm{y} M\mathrm{x}$. Each element of the delay $\mathbf{d}(\mathbf{z})$ is defined by $[\mathbf{d}(\mathbf{z})]_\ell = z_\mathrm{y}^{-((\ell))_{M_\mathrm{y}}} \cdot z_\mathrm{x}^{-\lfloor \ell/M_\mathrm{y} \rfloor}$ for $\ell \in \{0, 1, \cdots, M-1\}$, where $((x))_m$ denotes the modulo x of m.

2.2. Linear-Phase Condition

Suppose that all filters $\{H_p(\mathbf{z})\}_{p=0}^{P-1}$ and $\{F_p(\mathbf{z})\}_{p=0}^{P-1}$ are of size $L_\mathrm{y} \times L_\mathrm{x} = (N_\mathrm{y}+1)M_\mathrm{y} \times (N_\mathrm{x}+1)M_\mathrm{x}$. The polyphase order $\bar{\mathbf{n}}$ is given by $\bar{\mathbf{n}} = (N_\mathrm{y}, N_\mathrm{x})^T$, and the LP conditions for the analysis and synthesis bank are represented as

$$\mathbf{E}(\mathbf{z}) = \mathbf{z}^{-\bar{\mathbf{n}}}\mathbf{\Gamma}_P\mathbf{E}(\mathbf{z}^{-\mathbf{I}})\mathbf{J}_M, \quad (1)$$
$$\mathbf{R}(\mathbf{z}) = \mathbf{z}^{-\bar{\mathbf{n}}}\mathbf{J}_M\mathbf{R}(\mathbf{z}^{-\mathbf{I}})\mathbf{\Gamma}_P, \quad (2)$$

respectively, where $\mathbf{J}_M$ is the $M \times M$ reversal matrix, and $\mathbf{\Gamma}_P$ is a $P \times P$ diagonal matrix with diagonal elements '1' or '-1,' which correspond to symmetric and antisymmetry filters, respectively.

2.3. Perfect Reconstruction and Paraunitary Condition

The PR condition is sufficiently represented by $\mathbf{R}(\mathbf{z})\mathbf{E}(\mathbf{z}) = \mathbf{z}^{-\bar{\mathbf{n}}}\mathbf{I}_M$ in terms of the polyphase matrices, where $\mathbf{I}_M$ is the $M \times M$ identity matrix. If an analysis and synthesis bank satisfy

$$\mathbf{E}^T(\mathbf{z}^{-\mathbf{I}})\mathbf{E}(\mathbf{z}) = \mathbf{I}_M, \quad (3)$$
$$\mathbf{R}(\mathbf{z})\mathbf{R}^T(\mathbf{z}^{-\mathbf{I}}) = \mathbf{I}_M, \quad (4)$$

respectively, such systems are said to be PU [13, 14, 21]. From the frame-theoretic point of view, a PU system corresponds to a tight frame [14]. If we have a PU analysis bank, then we can obtain a PU synthesis bank by the para-conjugation of $\mathbf{E}(\mathbf{z})$:

$$\mathbf{R}(\mathbf{z}) = \mathbf{z}^{-\bar{\mathbf{n}}}\mathbf{E}^T(\mathbf{z}^{-\mathbf{I}}). \quad (5)$$

It is verified that this pair of banks constitute a PR system together. In the OS case, however, there is infinite number of PR combination of analysis and synthesis banks [11, 12]. (5) illustrates one choice of such combinations and optimal in the sense that the energy, i.e. squared standard norm, of subband images $\{Y_p(\mathbf{z})\}_{p=0}^{P-1}$ is minimized. For a given analysis bank, it is also possible to find a different synthesis bank with another criteria, and vice versa. When a synthesis filter bank is fixed, regardless it is PU or not, a direct search of subband images $\{Y_p(\mathbf{z})\}_{p=0}^{P-1}$ is also available.

2.4. Relation between Polyphase Order and Symmetry

In the article [21], Gan *et al.* discuss theorems on the relation among the polyphase order, sampling ratio, the number of channels and filter symmetry for 1-D OS LPPRFBs. Let us extend the theorems to the 2-D NS case. Note that the result shown here is novel and the derivation is not trivial since there are two variables for the polyphase order, although we omit to show the details of the proof.

Theorem 1. *For a P-channel NS OS LPPR filter bank with sampling ratio $M = M_\mathrm{y}M_\mathrm{x}$, suppose all of the analysis and synthesis filters are of the same size $L_\mathrm{y} \times L_\mathrm{x} = (N_\mathrm{y}+1)M_\mathrm{y} \times (N_\mathrm{x}+1)M_\mathrm{x}$. The number of symmetric filters p_s and antisymmetric filters $p_\mathrm{a} = P - p_\mathrm{s}$ must satisfy the following necessary conditions.*

1. *When M is even and $\bar{\mathbf{n}} = (N_\mathrm{y}, N_\mathrm{x})^T$ is arbitrary, $M/2 \leq p_\mathrm{s} \leq P - M/2$ and $M/2 \leq p_\mathrm{a} \leq P - M/2$.*
2. *When M is odd and both of N_y and N_x are even, $(M+1)/2 \leq p_\mathrm{s} \leq P - (M-1)/2$ and $(M-1)/2 \leq p_\mathrm{a} \leq P - (M+1)/2$.*
3. *When M is odd and either of N_y or N_x is odd, $(M+1)/2 \leq p_\mathrm{s} \leq P - (M+1)/2$ and $(M+1)/2 \leq p_\mathrm{a} \leq P - (M+1)/2$.*

Proof . Follow the proof of [21, Theorem 1]. □

2.5. Condition on Propagation Matrix $\mathbf{G}_{n_d}^{\{d\}}(z_d)$

A lattice structure for a P-channel LPPR FIR filter bank is given by a cascade representation of polyphase matrix $\mathbf{E}(\mathbf{z})$ in terms of propagation matrices $\{\mathbf{G}_{n_d}^{\{d\}}(z_d)\}$:

$$\mathbf{E}(\mathbf{z}) = \mathbf{G}_{K_\mathrm{y}}^{\{\mathrm{y}\}}(z_\mathrm{y})\mathbf{G}_{K_\mathrm{y}-1}^{\{\mathrm{y}\}}(z_\mathrm{y})\cdots\mathbf{G}_{1}^{\{\mathrm{y}\}}(z_\mathrm{y}) \times \mathbf{G}_{K_\mathrm{x}}^{\{\mathrm{x}\}}(z_\mathrm{x})\cdots\mathbf{G}_{2}^{\{\mathrm{x}\}}(z_\mathrm{x})\mathbf{G}_{1}^{\{\mathrm{x}\}}(z_\mathrm{x})\mathbf{E}_0(\mathbf{z}), \quad (6)$$

where $\mathbf{E}_0(\mathbf{z})$ is a $P \times M$ initial matrix of polyphase order $\bar{\mathbf{n}}_0 = (N_{\mathrm{y}0}, N_{\mathrm{x}0})^T$, and $\mathbf{G}_{k_\mathrm{y}}^{\{\mathrm{y}\}}(z_\mathrm{y})$ and $\mathbf{G}_{k_\mathrm{x}}^{\{\mathrm{x}\}}(z_\mathrm{x})$ are $P \times P$ propagation matrices of polyphase order $\bar{\mathbf{n}}_{\mathrm{y}1} = (N_{\mathrm{y}1}, 0)^T$ and $\bar{\mathbf{n}}_{\mathrm{x}1} = (0, N_{\mathrm{x}1})^T$, respectively. The condition on $\mathbf{G}_{k_d}^{\{d\}}(z_d)$ is shown as follows:

Theorem 2. *In* (6)*, if each $\mathbf{G}_{k_d}^{\{d\}}(z_d)$ is FIR invertible, i.e. the determinant is a monomial in z_d, and satisfies*

$$\mathbf{G}_{k_d}^{\{d\}}(z_d) = \mathbf{z}_d^{-N_{d1}}\mathbf{\Gamma}_P\mathbf{G}_{k_d}^{\{d\}}(z_d^{-1})\mathbf{\Gamma}_P,\ d \in \{\mathrm{y}, \mathrm{x}\}, \quad (7)$$

then $\mathbf{E}(\mathbf{z})$ corresponds to an NS OS LPPRFB with all filters of size $L_\mathrm{y} \times L_\mathrm{x} = (K_\mathrm{y}N_{\mathrm{y}1} + N_{\mathrm{y}0} + 1)M_\mathrm{y} \times (K_\mathrm{x}N_{\mathrm{x}1} + N_{\mathrm{x}0} + 1)M_\mathrm{x}$ each.

Proof. Follow the proof of [21, Theorem 2]. □

Theorem 3. *If $\mathbf{G}_{k_d}^{\{d\}}(z_d),\ d \in \{\mathrm{y}, \mathrm{x}\}$ is invertible and satisfied* (7) *with $N_{d1} = 1$, then $p_\mathrm{s} = p_\mathrm{a}$.*

Proof. Follow the proof of [21, Theorem 3]. □

Similar to the 1-D case, from Theorem 3, 2-D NS OS LPPR filter banks are categorized into the following two Types:

1. Type-I: $p_\mathrm{s} = p_\mathrm{a}$, $N_{d1} = 1$, $d \in \{\mathrm{y}, \mathrm{x}\}$
2. Type-II: $p_\mathrm{s} \neq p_\mathrm{a}$, $N_{d1} \neq 1$, $d \in \{\mathrm{y}, \mathrm{x}\}$

3. LATTICE STRUCTURES FOR NSOLT

Let us propose a P-channel NS OS LPPR filter banks with diagonal sampling factor $\mathbf{M} = \mathrm{diag}(M_\mathrm{y}, M_\mathrm{x})$. As a preliminary, we define a $P \times P$ butterfly matrix $\mathbf{B}_P^{(m)}$ as

$$\mathbf{B}_P^{(m)} = \begin{cases} \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{I}_m & \mathbf{I}_m \\ \mathbf{I}_m & -\mathbf{I}_m \end{pmatrix}, & m = \frac{P}{2} \\ \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{I}_m & \mathbf{o} & \mathbf{I}_m \\ \mathbf{o}^T & \sqrt{2}\mathbf{I}_{P-2m} & \mathbf{o}^T \\ \mathbf{I}_m & \mathbf{o} & -\mathbf{I}_m \end{pmatrix}, & m < \frac{P}{2} \end{cases},$$

where $\lceil M/2 \rceil \leq m \leq \lfloor P/2 \rfloor$ and $\mathbf{o}$ is a zero column vector.

3.1. Type-I NSOLT

When the number of channels P is even, it is possible to set $p_\mathrm{s} = p_\mathrm{a} = P/2$. From Theorem 3, a Type-I lattice can be constructed with propagation matrices $\{\mathbf{G}_{k_d}^{\{d\}}(z_d)\}$ of order $N_{d1} = 1$. The corresponding polyphase matrix $\mathbf{E}(\mathbf{z})$ is represented by

$$\mathbf{E}(\mathbf{z}) = \prod_{n_\mathrm{y}=1}^{N_\mathrm{y}}\left\{\mathbf{R}_{n_\mathrm{y}}^{\{\mathrm{y}\}}\mathbf{Q}(z_\mathrm{y})\right\} \cdot \prod_{n_\mathrm{x}=1}^{N_\mathrm{x}}\left\{\mathbf{R}_{n_\mathrm{x}}^{\{\mathrm{x}\}}\mathbf{Q}(z_\mathrm{x})\right\} \cdot \mathbf{R}_0\mathbf{E}_0, \quad (8)$$

where

$$\mathbf{Q}(z_d) = \mathbf{B}_P^{(\frac{P}{2})}\begin{pmatrix} \mathbf{I}_{p_\mathrm{s}} & \mathbf{O} \\ \mathbf{O} & z_d^{-1}\mathbf{I}_{p_\mathrm{a}} \end{pmatrix}\mathbf{B}_P^{(\frac{P}{2})},\ \mathbf{R}_n^{\{d\}} = \begin{pmatrix} \mathbf{I}_{p_\mathrm{s}} & \mathbf{O} \\ \mathbf{O} & \mathbf{U}_n^{\{d\}} \end{pmatrix},$$

where $\mathbf{O}$ is a zero matrix and $\mathbf{U}_n^{\{d\}} \in \mathbb{R}^{p_\mathrm{a} \times p_\mathrm{a}}$ is an arbitrary invertible matrix. We adopt the initial matrix $\mathbf{E}_0(\mathbf{z})$ defined by the product of the matrix representation of 2-D discrete cosine transform (DCT) $\mathbf{E}_0 \in \mathbb{R}^{M \times M}$ and

$$\mathbf{R}_0 = \begin{pmatrix} \mathbf{W}_0 & \mathbf{O} \\ \mathbf{O} & \mathbf{U}_0 \end{pmatrix}\begin{pmatrix} \mathbf{I}_{\lceil M/2 \rceil} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{\lfloor M/2 \rfloor} \\ \mathbf{O} & \mathbf{O} \end{pmatrix} \in \mathbb{R}^{P \times M}, \quad (9)$$

where $\mathbf{W}_0 \in \mathbb{R}^{p_\mathrm{s} \times p_\mathrm{s}}$ and $\mathbf{U}_0 \in \mathbb{R}^{p_\mathrm{a} \times p_\mathrm{a}}$ are arbitrary invertible matrices. The order of $\mathbf{E}_0(\mathbf{z}) = \mathbf{R}_0\mathbf{E}_0$ is $\bar{\mathbf{n}}_0 = (N_{\mathrm{y}0}, N_{\mathrm{x}0})^T = (0, 0)^T$. It can be verified that $\mathbf{E}(\mathbf{z})$ in (8) satisfies both of (1) and (3) simultaneously. For $P = M$, the lattice structure reduces to that of even-channel CS filter banks [8, 19].

3.2. Type-II NSOLT

For $p_\mathrm{s} \neq p_\mathrm{a}$, even if P is even, propagation matrices $\{\mathbf{G}_{k_d}^{\{d\}}(z_d)\}$ of order $N_{d1} \neq 1$ are required to construct a lattice from Theorem 3. We here consider only the case $p_\mathrm{s} > p_\mathrm{a}$ with even N_y and even N_x. Fig 2 shows an example of such a lattice structure with propagation matrices of order $N_{d1} = 2$. The corresponding polyphase matrix $\mathbf{E}(\mathbf{z})$ is represented by

$$\mathbf{E}(\mathbf{z}) = \prod_{\ell_\mathrm{y}=1}^{N_\mathrm{y}/2}\left\{\mathbf{R}_{\mathrm{E}\ell_\mathrm{y}}^{\{\mathrm{y}\}}\mathbf{Q}_\mathrm{E}(z_\mathrm{y})\mathbf{R}_{\mathrm{O}\ell_\mathrm{y}}^{\{\mathrm{y}\}}\mathbf{Q}_\mathrm{O}(z_\mathrm{y})\right\} \times \prod_{\ell_\mathrm{x}=1}^{N_\mathrm{x}/2}\left\{\mathbf{R}_{\mathrm{E}\ell_\mathrm{x}}^{\{\mathrm{x}\}}\mathbf{Q}_\mathrm{E}(z_\mathrm{x})\mathbf{R}_{\mathrm{O}\ell_\mathrm{x}}^{\{\mathrm{x}\}}\mathbf{Q}_\mathrm{O}(z_\mathrm{x})\right\} \cdot \mathbf{R}_0\mathbf{E}_0, \quad (10)$$

where

$$\mathbf{Q}_\mathrm{E}(z_d) = \mathbf{B}_P^{(p_\mathrm{a})}\begin{pmatrix} \mathbf{I}_{P-p_\mathrm{a}} & \mathbf{O} \\ \mathbf{O} & z_d^{-1}\mathbf{I}_{p_\mathrm{a}} \end{pmatrix}\mathbf{B}_P^{(p_\mathrm{a})},\ \mathbf{R}_{\mathrm{E}\ell}^{\{\mathrm{y}\}} = \begin{pmatrix} \mathbf{W}_\ell^{\{d\}} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{p_\mathrm{a}} \end{pmatrix},$$

$$\mathbf{Q}_\mathrm{O}(z_d) = \mathbf{B}_P^{(p_\mathrm{a})}\begin{pmatrix} \mathbf{I}_{p_\mathrm{a}} & \mathbf{O} \\ \mathbf{O} & z_d^{-1}\mathbf{I}_{P-p_\mathrm{a}} \end{pmatrix}\mathbf{B}_P^{(p_\mathrm{a})},\ \mathbf{R}_{\mathrm{O}\ell}^{\{\mathrm{x}\}} = \begin{pmatrix} \mathbf{I}_{p_\mathrm{s}} & \mathbf{O} \\ \mathbf{O} & \mathbf{U}_\ell^{\{d\}} \end{pmatrix}.$$

$\mathbf{W}_\ell^{\{d\}} \in \mathbb{R}^{p_\mathrm{s} \times p_\mathrm{s}}$ and $\mathbf{U}_\ell^{\{d\}} \in \mathbb{R}^{p_\mathrm{a} \times p_\mathrm{a}}$ are arbitrary invertible matrices. We adopt the initial matrix $\mathbf{E}_0(\mathbf{z})$ defined by the product of the 2-D DCT $\mathbf{E}_0 \in \mathbb{R}^{M \times M}$ and $\mathbf{R}_0 \in \mathbb{R}^{P \times M}$ in (9). It can be verified that $\mathbf{E}(\mathbf{z})$ in (10) simultaneously satisfies both of (1) and (3). For $P = M$, the lattice structure reduces to that of odd-channel CS filter banks [8, 19]. Note that this Type-II construction requires for the order of the initial matrix $\mathbf{E}_0(\mathbf{z})$ in the direction d, i.e. N_{d0}, to be one when N_d is odd. In this paper, we omit to show such cases as well as the case $p_\mathrm{s} < p_\mathrm{a}$.

3.3. No-DC-leakage Condition

The no-DC-leakage property is of interest for filter banks applied to image processing [3]. With a similar discussion to the article [8, 20], the conditions for Type-I NSOLT in (8) and Type-II NSOLT in (10) are derived as $\mathbf{W}_0 = \begin{pmatrix} 1 & \mathbf{o}^T \\ \mathbf{o} & \overline{\mathbf{W}} \end{pmatrix}$ and $\prod_{\ell_\mathrm{y}=1}^{N_\mathrm{y}/2}\mathbf{W}_{\ell_\mathrm{y}}^{\{\mathrm{y}\}} \cdot \prod_{\ell_\mathrm{x}=1}^{N_\mathrm{x}/2}\mathbf{W}_{\ell_\mathrm{x}}^{\{\mathrm{x}\}} \cdot \mathbf{W}_0 = \begin{pmatrix} 1 & \mathbf{o}^T \\ \mathbf{o} & \overline{\mathbf{W}} \end{pmatrix}$, respectively, where $\overline{\mathbf{W}} \in \mathbb{R}^{(p_\mathrm{s}-1) \times (p_\mathrm{s}-1)}$ is an arbitrary invertible matrix.

4. DESIGN EXAMPLES

In this section, we show some examples of NSOLTs in order to verify the significance of the lattice structures.

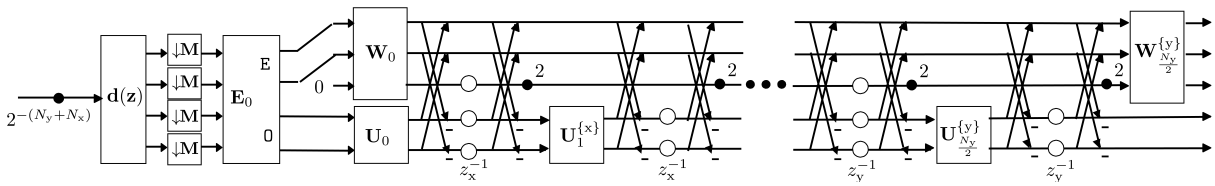

Fig. 2. Type-II lattice structure of analysis bank of a 2-D NS OS lapped transform (NSOLT).

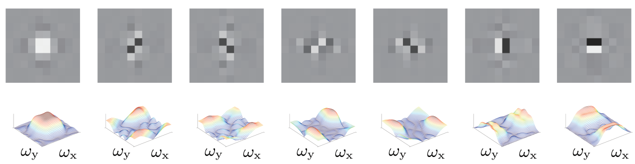

Fig. 3. Design example of Type-II NSOLT. The impulse and frequency amplitude responses in $(\omega_{\mathrm{y}}, \omega_{\mathrm{x}})^T \in [-\pi, \pi)^2$ are shown, where $P = p_{\mathrm{s}} + p_{\mathrm{a}} = 5+2 = 7$, $M_{\mathrm{y}} = M_{\mathrm{x}} = 2$ and $N_{\mathrm{y}} = N_{\mathrm{x}} = 4$.

4.1. Critically-sampled Haar transform

NSOLT in (8) include CS Haar transform as a special case. The transform can be obtained by the following settings.

- Sampling ratio: $M_{\mathrm{y}} = M_{\mathrm{x}} = 2$ $(M = 4)$
- ♯Channels: $P = p_{\mathrm{s}} + p_{\mathrm{a}} = 2 + 2 = 4$ $(\mathcal{R} = P/M = 1)$
- Polyphase order: $N_{\mathrm{y}} = N_{\mathrm{x}} = 0$
- Type-I structure with $\mathbf{W}_0 = \mathbf{I}_2$, $\mathbf{U}_0 = \mathbf{I}_2$

This filter bank is PU and has no DC-leakage.

4.2. Nonsubsampled Haar Transform

NSOLT in (8) include nonsubsampled Haar transform as a special case. The transform can be obtained by the following settings.

- Sampling ratio: $M_{\mathrm{y}} = M_{\mathrm{x}} = 1$ $(M = 1)$
- ♯Channels: $P = p_{\mathrm{s}} + p_{\mathrm{a}} = 2 + 2 = 4$ $(\mathcal{R} = P/M = 4)$
- Polyphase order: $N_{\mathrm{y}} = N_{\mathrm{x}} = 1$
- Type-I structure with $\mathbf{W}_0 = \mathbf{U}_1^{\{y\}} = \mathbf{I}_2$, $\mathbf{U}_0 = \mathbf{U}_1^{\{x\}} = \mathbf{J}_2$

This filter bank is PU and has no DC-leakage.

4.3. Type-II NSOLT with Rational Redundancy

Let us show a design example of Type-II NSOLT in (10) with rational redundancy. The followings summerizes the design settings:

- Sampling ratio: $M_{\mathrm{y}} = M_{\mathrm{x}} = 2$ $(M = 4)$
- ♯Channels: $P = p_{\mathrm{s}} + p_{\mathrm{a}} = 5 + 2 = 7$ $(\mathcal{R} = P/M = 7/4)$
- Polyphase order: $N_{\mathrm{y}} = N_{\mathrm{x}} = 4$
- Type-II structure with orthonormal parameter matrices $\mathbf{W}_0$, $\mathbf{U}_0$, $\{\mathbf{W}_{\ell_d}^{\{d\}}\}$ and $\{\mathbf{U}_{\ell_d}^{\{d\}}\}$

Fig 3 shows a design example, where the no-DC-leakage condition is taken into account. This filter bank is also PU.

5. EXPERIMENTAL RESULTS

In order to verify the significance of NSOLT in 4.3, we will show an experimental results of image restoration with ISTA [15, 23]. Let us assume a typical linear degradation model $\mathbf{x} = \mathbf{P}\mathbf{u} + \mathbf{w}$, where $\mathbf{x}$ is an observed image, $\mathbf{u}$ is an unknown original image, $\mathbf{P}$ is a linear degradation process, $\mathbf{w}$ is a zero-mean additive white Gaussian noise (AWGN). An image restoration problem is to find a good candidate $\hat{\mathbf{u}}$ of the unknown image $\mathbf{u}$ from the observed image $\mathbf{x}$. Suppose that $\mathbf{u}$ is represented by a dictionary $\mathbf{D}$ as $\mathbf{u} = \mathbf{D}\mathbf{y}$ and $\mathbf{y}$ is sparse. We use ISTA to solve the optimization problem $\hat{\mathbf{y}} = \arg\min_{\mathbf{y}} \frac{1}{2}\|\mathbf{x} - \mathbf{P}\mathbf{D}\mathbf{y}\|_2^2 + \lambda\|\mathbf{y}\|_1$ and estimate the original $\mathbf{u}$ as $\hat{\mathbf{u}} = \mathbf{D}\hat{\mathbf{y}}$ [23].

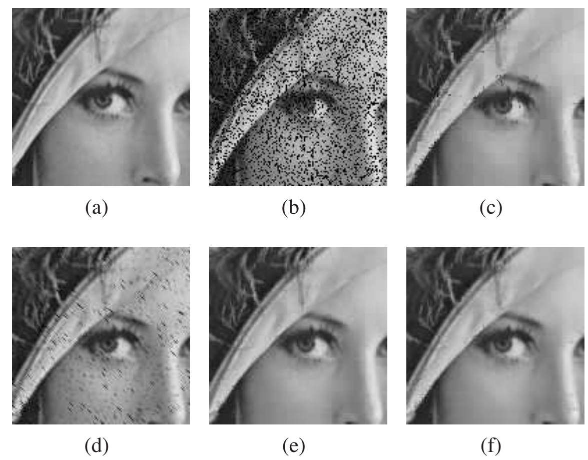

Fig. 4. Inpainting results with ISTA. (a) Original (128 × 128, 8bit gray scale). (b) Observation (pixel loss ratio: 20%), PSNR: 11.93dB. (c) 6-level Haar DWT ($\mathcal{R} = 1$), PSNR: 27.60dB. (d) 2-level nonsubsampled Haar DWT ($\mathcal{R} = 7$), PSNR: 24.21dB. (e) 6-level union of DirSOWTs ($\mathcal{R} = 5$), PSNR: 33.39dB [23]. (f) 6-level Type-II NSOLT in Fig 3 ($\mathcal{R} = 1.999755859375$), PSNR: 32.24dB.

In this experiment, we modeled the degradation $\mathbf{P}$ as random downsampling with 20%-pixel loss. As dictionary $\mathbf{D}$, LPPU synthesis banks of 6-level Haar DWT in 4.1, 2-level nonsubsampled Haar DWT in 4.2, 6-level union of DirSOWTs of order $N_{\mathrm{y}} = N_{\mathrm{x}} = 4$ [20, 23–25] and 6-level NSOLT in Fig. 3 were adopted. The redundancy ratios are $\mathcal{R} = 1, 7, 5$ and 1.999755859375, respectively. The basis termination technique was applied for all cases as a boundary operation [26]. Fig. 4 summerizes the results, where the control parameter of ISTA λ was set to the best value for each transform experimentaly. From Fig. 4, the NSOLT shows better performance than both of NS Haar DWT and comparable performance to UDN4 with lower redundancy.

6. CONCLUSIONS

In this work, we proposed novel lattice structures for 2-D non-separable linear-phase perfect reconstruction FIR filter banks. The proposed structures can realize a real-valued symmetric tight frame. Through experiments of image inpainting with ISTA, we verified the significance of the proposed transforms.

7. REFERENCES

[1] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice Hall, 1993.

[2] Martin Vetterli and Jelena Kovačević, *Wavelets and Subband Coding*, Prentice Hall, 1995.

[3] Gilbert Strang and Truong Nguyen, *Wavelets and Filter Banks, Second Edition*, Wellesley Cambridge Pr, 1996.

[4] K. Ramamohan Rao and Peter C. L. Yip, Eds., *The Transform and Data Compression Handbook*, Crc Pr I Llc, 2000.

[5] Martin Vetterli, "Wavelets, approximation, and compression," *IEEE Signal Process. Mag.*, pp. 59–73, Sept. 2001.

[6] Stèphane Mallat, *A Wavelet Tour of Signal Processing, Third Edition: The Sparse Way*, Academic Press, 2008.

[7] Jean-Luc Starck, Fionn Murtagh, and Jala M. Fadili, *Sparse Image and Signal Processing: Wavelets, Curvelets, Morphological Diversity*, Cambridge University Press, 2010.

[8] Shogo Muramatsu, Akihiko Yamada, and Hitoshi Kiya, "A design method of multidimensional linear-phase paraunitary filter banks with a lattice structure," *IEEE Trans. Signal Process.*, vol. 47, no. 3, pp. 690–700, Mar. 1999.

[9] Minh N. Do and Martin Vetterli, "The contourlet transform: An efficient directional multiresolution image representation," *IEEE Trans. Image Process.*, vol. 14, no. 12, pp. 2091–2106, Dec. 2005.

[10] Jelena Kovačević and Amina Chebira, "Life beyond bases: The advent of frames (part i)," *IEEE Signal Process. Mag.*, vol. 24, no. 4, pp. 86 –104, 2007.

[11] T. Tanaka, "A direct design of oversampled perfect reconstruction FIR filter banks," *IEEE Trans. Signal Process.*, vol. 54, no. 8, pp. 3011–3022, Aug. 2006.

[12] Li Chai, Jingxin Zhang, and Yuxia Sheng, "Optimal design of oversampled synthesis FBs with lattice structure constraints," vol. 59, no. 8, pp. 3549–3559, Aug. 2011.

[13] Zoran Cvetković and Martin Vetterli, "Oversamped filter banks," *IEEE Trans. Signal Process.*, vol. 46, no. 5, pp. 1245–1255, May 1998.

[14] H. Bolcskei, F. Hlawatsch, and H.G. Feichtinger, "Frame-theoretic analysis of oversampled filter banks," *IEEE Trans. Signal Process.*, vol. 46, no. 12, pp. 3256 –3268, Dec. 1998.

[15] Daniel P. Palomar and Yonina C. Eldar, Eds., *Convex Optimization in Signal Processing and Communications*, Cambridge University Press, 2009.

[16] Michael Elad, *Sparse and Redundant Representations: From Theory to Applications in Signal and Image Processing*, Springer, 2010.

[17] Yonia C. Eldar and Gitta Kutyniok, Eds., *Compressed Sensing, Theory and Applications*, Cambridge University Press, 2012.

[18] T.D. Tran, R.L. de Queiroz, and T.Q. Nguyen, "Linear-phase perfect reconstruction filter bank: lattice structure, design, and application in image coding," *IEEE Trans. Signal Process.*, vol. 48, no. 1, pp. 133–147, Jan. 2000.

[19] Lu Gan and Kai-Kuang Ma, "A simplified lattice factorization for linear-phase perfect reconstruction filter bank," *IEEE Signal Process. Lett.*, vol. 8, no. 7, pp. 207–209, July 2001.

[20] Shogo Muramatsu, Dandan Han, Tomoya Kobayashi, and Hisakazu Kikuchi, "Directional lapped orthogonal transform: Theory and design," *IEEE Trans. Image Process.*, vol. 21, no. 5, pp. 2434–2448, May 2012.

[21] Lu Gan and Kai-Kuang Ma, "Oversampled linear-phase perfect reconstruction filterbanks: theory, lattice structure and parameterization," *IEEE Trans. Signal Process.*, vol. 51, no. 3, pp. 744 – 759, Mar. 2003.

[22] David S. Taubman and Michael W. Marcellin, "JPEG2000, image compression fundamentals, standards and practice," *Kluwer Academic Publishers*, 2002.

[23] Shogo Muramatsu, Natsu Aizawa, and Masahiro Yukawa, "Image restoration with union of directional orthonormal DWTs," in *Proc. of APSIPA ASC*, Dec. 2012.

[24] Shogo Muramatsu and Dandan Han, "Image denoising with union of directional orthonormal DWTs," in *IEEE Proc. of ICASSP*, Mar. 2012, pp. 1089–1092.

[25] Shogo Muramatsu, "SURE-LET image denoising with multiple directional LOTs," in *IEEE Proc. of PCS*, May 2012.

[26] Shogo Muramatsu, Tomoya Kobayashi, Minoru Hiki, and Hisakazu Kikuchi, "Boundary operation of 2-D non-separable linear-phase paraunitary filter banks," *IEEE Trans. Image Process.*, vol. 21, no. 4, pp. 2314–2318, Apr. 2012.