A VARIATION OF EMPIRICAL MODE DECOMPOSITION WITH INTELLIGENT PEAK SELECTION IN SHORT TIME WINDOWS

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ABSTRACT

This paper describes analysis of the behaviour, and establishment of different decomposition properties, of a previously presented modification of the empirical mode decomposition algorithm using fractional Gaussian noise. Importantly, the modified algorithm, called empirical mode decomposition-modified peak selection (EMD-MPS), is used to explain certain aspects of the decomposition behaviour of EMD, providing novel insight into the domain. Finally, the utility of EMD-MPS is demonstrated by using it for a novel time-scale based de-trending of signals, using real-world financial time-series as an example.

***Index Terms*—** empirical mode decomposition, modified peak selection, filter-bank behavior, time-scale decomposition, de-trending

1. INTRODUCTION

Empirical mode decomposition (EMD) is a technique for decomposition of non-linear and non-stationary signals into amplitude and frequency modulated waveforms called intrinsic mode functions (IMFs), which are obtained by adaptive extraction of all the oscillatory modes present in a signal. EMD is defined by an algorithm [1], and the IMFs are extracted through a process called sifting. The sifting process considers the oscillatory modes in the signal at the very local time-scale, and the EMD algorithm defines steps to extract these modes. The most local time-scale is defined by two consecutive extrema, hence identification of all the extrema (peaks) in the signal is an important part of the sifting process. This also means that a change in the choice of extrema will result in limiting the time-scale over which the sifting process allows an oscillatory mode in the signal to pass un-decomposed. Exploiting the idea of selective extrema selection, we previously proposed a modification to the EMD algorithm [2], which we now call empirical mode decomposition-modified peak selection (EMD-MPS).

In the EMD-MPS method, the sifting process uses intelligent peak selection in short-time windows of length τ. Based on different values of τ, different decompositions of a signal into what we term as τ-functions are possible. Therefore the short-time window acts as an operator which allows separation of different frequency components in a signal into τ-functions, as determined by the length τ of the short-time window. We establish and empirically verify a relation between the frequency components decomposed and the value of τ.

In this paper, we analyze the behaviour of EMD-MPS by applying it to decompose fractional Gaussian noise (fGn) with different values of the Hurst exponent, and use spectral analysis of the obtained τ-functions to establish decomposition properties of EMD-MPS, as has been done previously in the case of EMD [3, 4]. Importantly, we point out the relationship between EMD and EMD-MPS, and show how EMD-MPS provides novel insight into EMD-based decomposition. As a demonstration of the utility of the method, EMD-MPS is used for a novel time-scale based de-trending of signals, using non-linear financial time-series in the form of S&P index data as an example signal.

2. EMPIRICAL MODE DECOMPOSITION-MODIFIED PEAK SELECTION

The sifting process in EMD-MPS uses a criterion for choosing the extrema based on short-time windows of length τ, instead of using a time-scale based on successive extrema, as is done in the case of EMD. Let us define an operator $W_i^\tau(\cdot),\ i = 1...k, i \in \mathbb{Z},\ 0 < \tau < L, L \in \mathbb{R}$, which, given a signal $x[n]$ of length L, produces the i-th τ-function T_i, such that $\mathrm{T}_i[n] = W_i^\tau(x[n])$. This can be explained as choosing a short-time window length, τ, for a given signal $x[n]$, and selecting the highest/lowest from among the maxima/minima within τ, from each interval τ over the whole signal length. The maxima/minima thus identified (one maxima and minima each per τ) are connected using cubic splines to form the upper and lower envelopes, and the mean of the envelopes $E_{n(mean)}$ is calculated. The signal $x[n]$ is updated by subtracting the mean from it $x[n] \leftarrow x[n] - E_{n(mean)}$. These steps, which are similar to EMD-based sifting, are continued till a stopping criterion is met, at which point $x[n]$ is reduced to a τ-function. This τ-function is subtracted from $x[n]$ to get a residue, which is then taken as the starting point instead of $x[n]$, and previous steps of the algorithm are repeated to find all the τ-functions T_i in the signal.

Unlike IMFs extracted by the EMD algorithm, the coarse-grained τ-functions may contain different coexisting modes of oscillation, each superimposed on the other. This happens since the short-time window τ sets an upper limit on the periods of the oscillations that can be included in any given τ-function obtained using the EMD-MPS method. This limit is determined by :

$$F = \frac{F_s}{\tau} \tag{1}$$

where F_s represents the sampling frequency. As an example for this relation, a value of $\tau = 25$ (in samples) corresponds to a frequency value $F = 40$ samples/second for $F_s = 1000$ samples/second. Using this value of τ, only one peak (maxima and minima each) in each 25 sample interval will be used in the envelope formation, and the sifting process should then decompose all $F \leq 40$ samples/second oscillatory components, and let all components with $F > 40$ samples/second pass through un-decomposed in one τ-function. In practice, the value of τ is qualified by a scaling constant k, such that $\hat{\tau} = k\tau$, and $0 < k \leq 1$. The relation in Eq. 1 and the scaling constant k are empirically validated in Sections 3.2 and 3.3.

Assuming Nyquist sampling, we can also see that the minimum value of τ is given by $\tau_{min} = \frac{F_s}{F_{max}}$, where $F_{max} = \frac{F_s}{2}$. This gives the minimum value of τ as $\tau_{min} = 2$. At this value of τ,

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all the maxima and minima present in the signal are selected, and EMD-MPS is equivalent to the EMD algorithm. This means that all frequency components present in the signal are decomposed, and the τ-functions are the same as IMFs. This relation of EMD-MPS with EMD allows us to shed new light on the decomposition behaviour of EMD in terms of EMD-MPS. Similarly, τ_{max} represents the maximum possible value of τ, at which value the signal remains un-decomposed. Therefore the range of possible values of τ can be written as:

$$2 \leq \tau < F_s/F_{min} \tag{2}$$

where F_{min} is the lowest frequency present in the signal to be decomposed. Practically, however, the maximum value of τ will be limited by the length of the signal (see also Sec. 3.1.2 for relation to EMD algorithm termination).

3. DECOMPOSITION OF FRACTIONAL GAUSSIAN NOISE USING EMD-MPS

Fractional Gaussian noise (fGn) represents a versatile model for a full-spectrum process which is not dominated by any particular frequency band, and has been previously used in studies to establish properties of EMD [4]. The fGn of parameter H, which is the Hurst exponent, can be defined as the zero-mean stationary Gaussian process with autocorrelation sequence $r_H[k] := \mathrm{E}\{x_H[n]x_H[n+k]\}$, $r_H[k] = \frac{\sigma^2}{2}(\mid k-1\mid^{2H} - 2\mid k\mid^{2H} + \mid k+1\mid^{2H})$. For $H = 0.5$, fGn reduces to discrete white noise, whereas other values of H correspond to non-zero correlations (negative for: $0 < H < 0.5$; positive for: $0.5 < H < 1$).

For this work, extensive experiments were carried out with decomposition of fGn processes using EMD-MPS with different values of τ in the operator $W_i^\tau(\cdot)$. For each of the different values of H used (0.2,0.5.0.8, and also 0.1 and 0.9, but for fewer values of τ), 2500 independent samples paths of fGn were generated, each of length 2048 samples. EMD-MPS was applied to each of the 2500 sample paths, using the following values of τ and i in the operator $W_i^\tau(\cdot)$: $i = 1,2,3,4,5$, and $\tau = 1-50$. This means that for each value of τ for each sample path, a fixed five τ-functions (T_1 to T_5) were obtained.

3.1. Filter bank behavior of EMD-MPS

Filter banks represent a collection of bandpass filters which can isolate different frequency bands in the input signal. Spectral analysis of IMFs obtained by application of EMD on white noise or fGn has shown frequency responses similar to that of a dyadic filter bank [3, 4]. In order to test the behaviour of τ-functions T_i obtained by decomposing fGn with EMD-MPS, the power spectrum of each τ-function (T_1 to T_5, obtained for different values of τ) was estimated by computing its autocorrelation function for each fGn sequence, which was then ensemble averaged over all sequences and then Fourier transformed. Figure 1 shows the spectra of the τ-functions for 2 different values of τ for fGn sequences with H=0.5.

It can be observed from Fig.1 that when τ-functions display a band-pass behavior, the pass-bands overlap in a way such that the lower half-band of τ-function T_i represents a frequency range which is roughly covered by the upper half-band of τ-function T_{i+1}, indicating a quasi-dyadic filter-bank structure. In the case of EMD, the dyadic filter-bank structure for IMFs has previously been quantified by establishing an exponentially decreasing relationship between the number of zero-crossings in IMFs (as an indication of the mean frequency in an IMF) and the IMF index number [4, 5]. Since

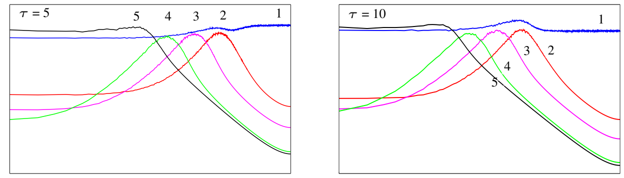

Fig. 1. Mean spectra (power spectral density in dB on y-axis, and $\log_2(frequency)$ on x-axis) of τ-functions T_1 to T_5. Unlabeled numbers in the figures represent the indicies of τ-functions.

τ-functions may contain multiple modes of oscillation, the number of zero-crossings is not a good estimate of the mean frequency. Instead, we used the center-frequency F_i of the τ-functions T_i with band-pass frequency response, and found that F_i is a decreasing exponential of the τ-function index i related by $F_i \propto \beta^{-i}$, where β is very close to 2. It is interesting to note that the value of β remains close to 2 even as τ increases, i.e. the dyadic filter-bank behaviour is demonstrated as long as τ-functions have band-pass like characteristics.

One other aspect of EMD-MPS that may be noted from the filter-bank behaviour is that the spectrum of τ-functions starts changing from band-pass to low-pass as τ increases. For sufficiently large value of τ, all τ-functions T_i , $i > 1$ demonstrate low-pass behaviour. This is to be expected, since as the value of τ increases, most of the frequency components pass un-decomposed in the first τ-function, and only low frequency components present in the fGn sequence are decomposed into τ-functions with index $i > 1$. Similarly, for a large enough increase in τ, τ-function T_1 assumes approximately all-pass characteristics.

3.1.1. Horizontal and vertical behavior

Given that the decomposition of EMD-MPS and EMD is driven by the sifting process, the similarity in behaviour is not surprising. However, according to Eq. 1, we can relate the dyadic behaviour across modes (τ-functions or IMFs) to a change in the value of τ. A change in the value of frequency by half implies an increase in value of τ by a factor of 2. This means that after the extraction of the first mode (whether IMF or τ-function), the sifting process behaves as if the value of τ has been doubled. To test this, the value of τ was increased by a factor of 2 after each τ-function had been extracted, with this change resulting in no difference in the spectra of the τ-functions. However, increasing the value of τ by a factor of 4 after each τ-function extraction resulted in decrease of the spectra to half the previous value.

This suggests that the "horizontal" (i.e. across modes) dyadic filter-bank behaviour also exists "vertically" (i.e. for the same mode, but for specific values of τ) for EMD-MPS, in that the same τ-function T_n^τ displays a dyadic behaviour for different values τ_j, where $\tau_{j+1} = 2\tau_j$. As for the horizontal case, the vertical behaviour can be quantified as $F_n^\tau \propto \beta^{-\tau_j}$, where β is very close to 2. This is shown in Figure 2, which demonstrates the vertical behaviour for values of τ_j starting from 2 (left figure) and 4 (right figure) using F_2^τ. The solid lines in red in both figures are the least-square fits with a slope nearly equal to -1.

At the same time, Figure 2 reveals a very interesting phenomenon for values of τ starting from 2 (left figure), which corresponds to normal EMD behaviour. There is a very clear deviation of the frequency value F_2^2, indicating a lower value than expected. This implies that for EMD decomposition, the oscillatory components in the first mode have higher frequencies than expected by the vertical

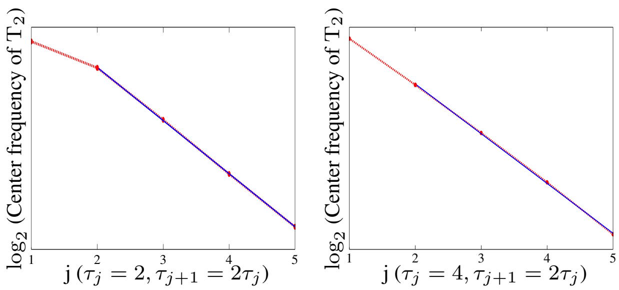

Fig. 2. Center frequency of τ-function T_2 ($\log_2$ values) obtained for different values of τ, plotted against the index j.

model, and going from mode 1 to 2 is not the doubling of τ, but instead amounts to a τ_2 which is almost three times τ_1, $\tau_2 \approx 3\tau_1$, (slight difference in values for different values of H is not discussed here). According to Eq. 1, the frequencies to be separated and the value of τ are inversely proportional, hence for good separation of components with frequencies f_1 and f_2, we need at least $\frac{f_1}{f_2} = \frac{1}{3}$. Interestingly, this matches the separation limit required for two-tone separation established in [6]. Using the analysis presented in this paper, it can be said that this limit is a fundamental property of EMD decomposition, holding beyond the two tone model.

Similarly, the difference in behaviour of the first IMF has been mentioned previously in [7], but formulation of this behaviour in terms of the vertical behaviour presented by EMD-MPS, as done in the last paragraph, can provide better insight into the decomposition properties of EMD, and relate it to previous results as well. Furthermore, the vertical behaviour can be used to demonstrate and explain the evolution of the probability density function of the extracted modes, which has been shown to be multi-modal for the first IMF, and Gaussian-like for other IMFs in the case of EMD [3][8], though this is not done here due to space constraints. Also, the band-pass to low-pass change in IMF behaviour [4] can be explained keeping in view IMF extraction in terms of an increase in τ as mentioned at the end of Section 3.1.

3.1.2. Convergence of EMD algorithm

Interestingly, the convergence of the EMD algorithm may also be explained in terms of an increase in τ. Previous studies have established the limit for the number of IMFs that can result from applying EMD to a signal as $J \leq \log_2 N$, where J is the number of IMFs obtained from a signal of length N [9] . In terms of EMD-MPS behaviour, we can formulate the extraction of an IMF as the doubling of τ. Therefore, for the $\log_2 N$-th IMF, the value of $\tau = N$, which means no further decomposition is possible, since at most one minima and maxima each will be selected, and the algorithm will terminate as no envelope formation is possible. In practice, the number of IMFs can be less than $\log_2 N$, because, depending on the signal, the number of extrema can drop to less than 3 before $\tau = N$, which is when most implementations are programmed to terminate the algorithm.

3.2. Relation between decomposed frequencies and τ

To study the relationship between the decomposed frequencies and τ, the center-frequency F_2 of the second τ-function T_2 is used, since T_2 maintains a band-pass behaviour over a large range of τ for the data length used in our experiments. Fig. 3, which plots F_2 against τ for $H = 0.5$, demonstrates the existence of a non-linear relationship between center-frequency F_i and τ, and can also be seen as characterizing the non-linear nature of the extrema-based decomposition.

It is possible to empirically validate Eq. 1 by using the following relation for a τ-function T_i having band-pass frequency response, with center frequency F, $\forall\, n, m \in \mathbb{Z}, n \neq m$:

$$\frac{\tau_n}{\tau_m} = x(\frac{F_n}{F_m})^{-1} \tag{3}$$

Eq. 1 suggests a values of $x = 1$ (since $F_s = \tau F$), however $x \approx 1$ in Eq. 3 holds only for values of $\tau > 6$ (for all values of H), whereas $0.7 \leq x \leq 1$ for $2 \leq \tau \leq 6$, with x approaching 1 as τ increases to a value of $\tau = 6$. The reason for this behaviour is related to the different vertical behaviour inherent in the decomposition using a value of τ=2, as mentioned in Section 3.1.1.

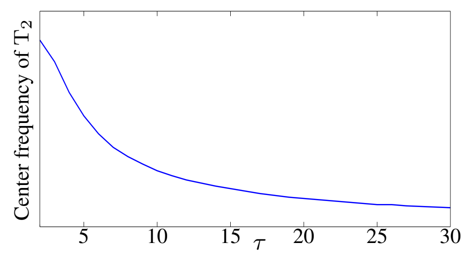

Fig. 3. Relationship between decomposed frequencies and τ.

3.3. Scaling factor for τ

It was previously mentioned in Section 2 that the value of τ in Eq. 1 is qualified by a scaling factor k. The scaling factor k was previously estimated as $0.5 < k < 0.63$ in [2], based on experiments with different test signals and different values of τ. Although this range of k held well for simulated and real-world signals, the work in [2] reported a value lower than $k = 0.5$ giving much better results in terms of expected decomposition into τ-functions. Using fGn analysis as described in this paper, we have a better estimation of the scaling factor for τ, which is now written as (valid for all values of H):

$$0.25 < k < 0.44 \tag{4}$$

The range of the values of k was found by estimating the center frequency of τ-function T_2 resulting from a value of $\hat{\tau}$ used for decomposition, and relating it to the value of τ by $\hat{\tau} = k\tau$, where τ is the value which would have resulted in the same frequency value according to Eq. 1.

Importantly, the value of k is related to the difference in behaviour of τ in the same way as previously mentioned, with the value of $k = 0.44$ valid for values of $\tau > 6$, and $0.25 \leq k < 0.44$ for $2 \leq \tau < 6$, with k increasing exponentially to 0.44 as τ increases to a value of 6. The estimation of the scaling constant k also allows us to relate the change in frequency Δf to the change in τ between values τ_1 and τ_2 as:

$$\Delta f = F_s(\frac{k_1}{\tau_1} - \frac{k_2}{\tau_2}) \tag{5}$$

3.3.1. Validation of the scaling factor k using two-tone separation

The two tone separation problem using EMD has been studied in [6] using a performance measure to quantify the quality of separation of the two tones using a signal model $x(t; f) = \cos 2\pi t + \cos(2\pi f t), t \in \mathbb{R}$. To validate the scaling factor described in the previous section, we use EMD-MPS to test separation of two tones, using the same signal model, and the performance measure given by Eq. 6. This performance measure gives a value close to zero for good separation, and a value approaching 1 in case of poor separation between components [6], as we expect the high frequency tone (frequency f_1) to pass un-decomposed into τ-function T_1, and the low-frequency tone (frequency f_2) to be decomposed into τ-function T_2.

$$c(f) = \frac{||\mathrm{T}_1 - \cos 2\pi t||_2}{||\cos 2\pi f t||_2} \tag{6}$$

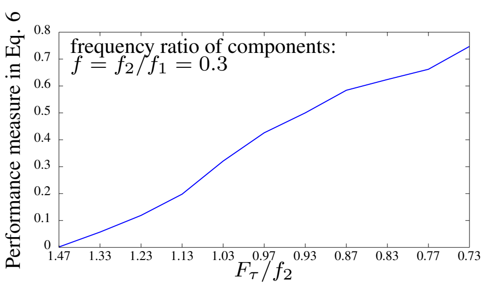

Fig. 4. Value of the performance measure in Eq. 6 plotted against F_τ/f_2.

Fig. 4 shows the results of using EMD-MPS to decompose the signal $x(t)$, where the frequency ratio f of the two components has a value 0.3. At this value of f, the performance measure $c(f)$ should have a value close to zero [6]. The performance measure is plotted against F_τ/f_2, where $F_\tau = F_s \cdot \hat{k}/\tau$, obtained using Eq. 1 and the scaling relation $\hat{\tau} = k\tau$. Here $\hat{\tau}$ represents the actual value of τ used to decompose the signal, and $k = 0.44$. EMD-MPS does not decompose a frequency component with value greater than F_τ, which passes un-decomposed in τ-function T_1.

For this experiment, a value of the ratio $F_\tau/f_2 \gg 1$ represents the case where F_τ is greater than f_2, which means the tone with higher frequency f_1 passes un-decomposed in τ-function T_1, whereas the tone with frequency $f_2 \ll F_\tau$ is decomposed into τ-function T_2. However, the ratio F_τ/f_2 decreasing towards 1 and lower represents F_τ becoming close to, and then less than f_2. In this case, a clean separation of the tones is not possible. This is depicted in Fig. 4, where, for values of the ratio F_τ/f_2 close to and less than 1, the value of performance measure increases towards 1, showing an increasingly less effective separation of the tones.

4. TIME-SCALE BASED DE-TRENDING OF SIGNALS

EMD-MPS allows a novel time-scale based de-trending of signals, which does not require estimation of a trend model for model-based de-trending, or knowledge of the statistical properties of IMFs, as is the case for EMD-based de-trending approaches proposed in literature, e.g. [10, 11]. The limitations of the energy-ratio approach for trend extraction presented in [11] have been pointed out in [12]. Also, the work in [12] presents a seasonality checking approach to deal with seasonal time-series to be de-trended, which includes identifying seasonal IMFs based on criteria defined on the extrema. In contrast, EMD-MPS allows de-trending based on time-scales, hence can de-trend seasonal time-series at different required time-scales directly without resorting to any model. In this context, with EMD-MPS, appropriate selection of τ representing a required time-scale allows separation of the faster oscillations from the slowly-varying trend, where "slowly-varying" is controlled by the value of τ, according to Eq. 1. In the context of the vertical behaviour, this is equivalent to combination of lower order IMFs, which pass un-decomposed in one τ-function, and of higher-order IMFs, which combine to form the trend.

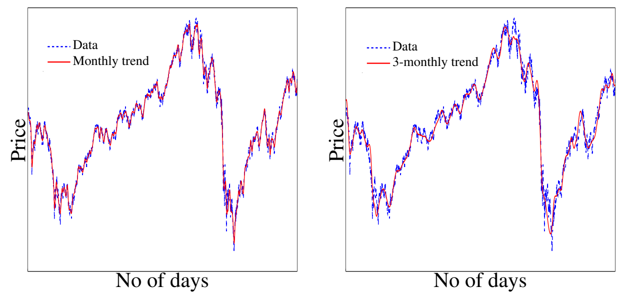

Fig. 5. Time-scale based de-trending of S & P 500 index data.

Time-scale based de-trending using EMD-MPS is illustrated by the use of real-world financial data in the form of S&P 500 daily index from November 6th, 2001 to October 11th, 2011, which consists of 3000 data points. Values of τ representing monthly and 3-monthly cycles were calculated in the following way. Let F_s be equal to N, where N represents the number of days of the daily data being de-trended. For a 3-monthly cycle, let F be equal to 90, and the value of τ is obtained from Eq. 1, which is scaled by k=0.44 to obtain $\hat{\tau}$. This way, the values of $\hat{\tau}$ representing monthly and 3-monthly cycles are calculated, and a different trend is extracted from the S&P data for each of the two time-scales. The trends at different time-scales, superimposed with the original time-series, are shown in Fig. 5, which illustrates the different levels of the slowly-varying detail captured by trends at the different time-scales. It should be mentioned here that an explicit notion of the time-scale associated with the trend is present in the work presented in [13]. However, this time-scale is obtained through the use of instantaneous frequencies after the IMFs have been subjectively combined. This is different from the EMD-MPS approach, where the notion of time-scale is built into the decomposition.

5. CONCLUSION

In this paper EMD-MPS was presented, which is a novel variation of the EMD algorithm, and its properties were studied by application of EMD-MPS to fractional Gaussian noise. The relation between the decomposed frequencies and the value of short-time window τ was also established and validated. Also important is the estimation of the scaling factor relating decomposed frequencies and τ. It was also shown how EMD-MPS allows for a simple methodology for time-scale based de-trending of signals, compared to more involved and subjective approaches based on EMD. Next stages of the work include improving the EMD-MPS algorithm whereby the length of the short-time window τ is not fixed, but is chosen adaptive to the signal.

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