# ON THE DESIGN OF ORTHONORMAL WAVELETS FOR FINITE-LENGTH SIGNALS

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### ABSTRACT

We describe a procedure for designing compactly supported orthonormal wavelets for finite-length wavelet transform. The designed wavelets yield perfect reconstruction with a more general class of signal extensions, including symmetric extension and zero-padding, rather than the common periodic extension. This combines the compact representation from orthonormal wavelets with the good behaviour at the borders from smooth extensions, which enables their use in signal coding. We show few examples that illustrate the improvement for coding finite-length signals.

***Index Terms***— orthonormal wavelets, compact support, finite-length, symmetric extension, optimization.

## 1. INTRODUCTION

The conventional design procedures of orthonormal wavelets, e.g., [1], inherently assume an infinite-length wavelet transform. Nevertheless, many real-life signals, e.g., images and ultrasound signals, have finite length. Migrating common orthonormal wavelets to finite-length requires special treatment at the borders to guarantee perfect reonstruction. There are two common approaches for the wavelet transform of finite-length signls: designing time-varying filters at the boundary, and using special signal extension. Signal extension includes periodic extension, symmetric extension and zero padding [2]. Periodic extension always guarantees perfect reconstruction for orthonormal wavelets whereas symmetric extension and zero padding are mostly used with biorthogonal wavelets. The second approach for finite-length wavelet transform uses time varying boundary filters [3, 4, 5]. These filters can be optimized to have certain desirable features, e.g., orthogonality, while satisfying the perfect reconstruction condition.

In this work we consider the problem of designing orthonormal wavelets of compact support that are suited for finite-length wavelet transform. The perfect reconstruction conditions of the finite-length transform are set as constraints in the wavelet design problem. The kernel wavelet filter is designed using the wavelet parameterization in [6]. Other desirable wavelet features could also be set as constraints in the design problem. We propose pratical examples where the wavelet is designed to match a predefined template while satisfying the finite-length perfect reconstruction conditions. This wavelet design is necessary for signal coding applications when the length of the signal is relatively small, e.g., in compressing ultrasound Doppler signals [7]. In section 5, we demonstrate the effectiveness of the proposed algorithms by few examples from signal and image coding. To our knowledge, this is the first work that addresses the problem of designing compactly supported orthonormal wavelets for finite-length signals.

Throughout the paper, we use bold-faced capital letters for matrices, and bold-faced small letters for column vectors. $\mathbf{A}'$ denotes the transpose of the matrix $\mathbf{A}$ (we assume all matrices and data vectors are real). $\kappa(\mathbf{A})$ denotes the condition number of $\mathbf{A}$ [8]. We assume that the length of the wavelet low pass filter is $N = 2K + 2$ where $K$ is even. The notation $\widetilde{\mathbf{a}}$ of a column vector $\mathbf{a} = [a(0), a(1), a(2), ..., a(2k-1)]'$ is:

$$\widetilde{\mathbf{a}} \triangleq [a(2k-1), -a(2k-2), a(2k-3), \ldots, a(1), -a(0)]' \quad (1)$$

## 2. BACKGROUND

### 2.1. Orthogonal Wavelets with Compact Support

Compactly supported orthonormal wavelets are parameterized by a low-pass filter $h(n)$ that satisfies few existence and orthogonality conditions [1]. By choosing the high-pass wavelet filter as

$$g(n) = (-1)^n h(N-1-n) \quad (2)$$

i.e., $\mathbf{g} = \widetilde{\mathbf{h}}$, then we have in general $K + 1$ degress of freedom in the design of the orthonormal wavelet filter [6]. Therefore, Additional design criteria, e.g., vanishing moments, are usually included in the wavelet design. Let $x_L(n)$ and $x_H(n)$ denote respectively the approximate and detailed coefficients after one stage wavelet decomposition. If the output of the analysis filter bank is organized as:

$$\mathbf{y} = [..., x_L(-1), x_H(-1), x_L(0), x_H(0), x_L(1), x_H(1), ...]' \quad (3)$$

then the analysis filter bank can put in a matrix form [9]

$$\mathbf{y} = \mathbf{H}\mathbf{x} \quad (4)$$

where $\mathbf{H}$ is an infinite-dimensional orthonormal matrix defined as [10]:

$$\mathbf{H} = \begin{pmatrix} \ddots & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\ \ldots & \mathbf{0} & \mathbf{U} & \mathbf{E} & \mathbf{L} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots \\ \ldots & \mathbf{0} & \mathbf{0} & \mathbf{U} & \mathbf{E} & \mathbf{L} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots \\ \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{U} & \mathbf{E} & \mathbf{L} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots \\ \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{U} & \mathbf{E} & \mathbf{L} & \mathbf{0} & \mathbf{0} & \ldots \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \end{pmatrix} \quad (5)$$

where the submatrices $\mathbf{U}$, $\mathbf{E}$, and $\mathbf{L}$ (each of size $K \times K$) are block-Toeplitz matrices defined as

$$\mathbf{U} \triangleq \begin{pmatrix} h(2K+1) & h(2K) & h(2K-1) & \ldots & h(K+2) \\ g(2K+1) & g(2K) & g(2K-1) & \ldots & g(K+2) \\ 0 & 0 & h(2K+1) & \ldots & h(K+4) \\ 0 & 0 & g(2K+1) & \ldots & g(K+4) \\ \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & h(2K) \\ 0 & 0 & 0 & \ldots & g(2K) \end{pmatrix} \quad (6)$$

$$\mathbf{E} \triangleq \begin{pmatrix} h(K+1) & h(K) & \dots & h(3) & h(2) \\ g(K+1) & g(K) & \dots & g(3) & g(2) \\ h(K+3) & h(K+2) & \dots & h(5) & h(4) \\ g(K+3) & g(K+2) & \dots & g(5) & g(4) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h(2K-1) & h(2K-2) & \dots & h(K+1) & h(K) \\ g(2K-1) & g(2K-2) & \dots & g(K+1) & g(K) \end{pmatrix} \quad (7)$$

$$\mathbf{L} \triangleq \begin{pmatrix} h(1) & h(0) & 0 & \dots & 0 & 0 \\ g(1) & g(0) & 0 & \dots & 0 & 0 \\ h(3) & h(2) & h(1) & \dots & 0 & 0 \\ g(3) & g(2) & g(1) & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots & \vdots & \vdots \\ h(K-1) & h(K-2) & h(K-3) & \dots & h(1) & h(0) \\ g(K-1) & g(K-2) & g(K-3) & \dots & g(1) & g(0) \end{pmatrix} \quad (8)$$

The orthogonality of the filter bank implies that $\mathbf{H}'\mathbf{H} = \mathbf{H}\mathbf{H}' = \mathbf{I}$. This could be rewritten as:

$$\mathbf{L}'\mathbf{U} = \mathbf{L}\mathbf{U}' = \mathbf{0} \quad (9)$$

$$\mathbf{L}'\mathbf{E} + \mathbf{E}'\mathbf{U} = \mathbf{L}\mathbf{E}' + \mathbf{E}\mathbf{U}' = \mathbf{0} \quad (10)$$

$$\mathbf{L}'\mathbf{L} + \mathbf{E}'\mathbf{E} + \mathbf{U}'\mathbf{U} = \mathbf{L}\mathbf{L}' + \mathbf{E}\mathbf{E}' + \mathbf{U}\mathbf{U}' = \mathbf{I} \quad (11)$$

It was shown in [10] that the Singular Value Decomposition (SVD) of $\mathbf{L}, \mathbf{U}$ and $\mathbf{E}$ are closely related. If the SVD of $\mathbf{L}$ has the form,

$$\mathbf{L} = \sum_{i=1}^{r} \sigma_i \mathbf{w}_i \mathbf{v}_i' \quad (12)$$

with $\sigma_i \leq 1$ for all $i$; then the SVD's of $\mathbf{U}$ and $\mathbf{E}$ have the form

$$\mathbf{U} \quad = \quad -\sum_{i=1}^{r} \sigma_i \widetilde{\mathbf{w}}_i \widetilde{\mathbf{v}}_i' \quad (13)$$

$$\mathbf{E} \quad = \quad \sum_{i=1}^{r} s_i \sqrt{1-\sigma_i^2}\left(\widetilde{\mathbf{w}}_i \mathbf{v}_i' + \mathbf{w}_i \widetilde{\mathbf{v}}_i'\right) + \sum_{i=2r+1}^{K} \mathbf{w}_i \mathbf{v}_i' \quad (14)$$

where $s_i \in \{1, -1\}$ and $\{\mathbf{v}_i\}_{i=2r+1}^{K}$ are in the null space of $\mathbf{L} + \mathbf{U}$. It was shown in [10, 6] that the low-pass filter $\mathbf{h} = [h(0) \ \dots \ h(N-1)]'$ is the solution of the linear system of equations:

$$\mathbf{\Gamma}\mathbf{h} = \mathbf{b} \quad (15)$$

where $\mathbf{b} = [0 \ 0 \ \dots \ 0 \ \sqrt{2}]'$; and $\mathbf{\Gamma}$ is an $N \times N$ matrix that is parameterized by a vector $[\mathbf{v}' \ \sigma]'$ of length $N/2$, which is defined as (recall that $N = 2K + 2$):

$$\mathbf{\Gamma}(\mathbf{v}, \sigma) = \left(\widetilde{\gamma}_1 \ \gamma_1 \ \dots \ \widetilde{\gamma}_K \ \gamma_K \ \widetilde{\mathbf{u}}' \ \mathbf{u}'\right)' \quad (16)$$

where $\mathbf{u}$ is an all-ones vector and

$$\widetilde{\mathbf{u}} \triangleq (1 \ -1 \ \dots \ 1 \ -1)' \quad (17)$$

and $\{\boldsymbol{\gamma}_i\}$ are vectors of length $N$ that are parameterized by the parameter vector $(\mathbf{v}, \sigma) = (v_1 \ \dots \ v_K \ \sigma)$. For $1 \leq i \leq K$ we have

$$\boldsymbol{\gamma}_i \triangleq \begin{cases} [\mathbf{0} \ v_K \ \dots \ v_{K+1-2i}]' & \text{if } i \leq K/2 \\ [\mathbf{0} \ v_K \ \dots \ v_1 \ -\sigma v_1 \ \sigma v_2 \ \dots \\ \quad -\sigma v_{2(i-K-1)} \ \sigma v_{2(i-K)}]' & \text{if } i > K/2 \end{cases} \quad (18)$$

where it has $N - 2i$ zeros.The above parameterization describes the wavelet filter coefficients as a continuous function of the unknown decision variables. This enables the deployment of standard optimization search techniques, e.g., gradient descent or Newton search [11], for optimizing regular objective functions.

## 3. FINITE-LENGTH WAVELET TRANSFORM

In case of finite length signals, the filtering matrix (5) has finite dimensions. For example, if the signal length is $M = lK$, where $l$ is an integer, then the truncated filtering matrix would be a square matrix of size $M \times M$:

$$\mathbf{H}_t \triangleq \begin{pmatrix} \mathbf{E} & \mathbf{L} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{U} & \mathbf{E} & \mathbf{L} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U} & \mathbf{E} & \mathbf{L} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & & \vdots & & \vdots & & \vdots & & & \ddots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{U} & \mathbf{E} & \mathbf{L} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{U} & \mathbf{E} \end{pmatrix} \quad (19)$$

Denote $i$-th input block (of size $K$) by $\mathbf{x}_i$, and the $i$-th output block of size $K$ by $\mathbf{y}_i$. The output in this case corresponds in zero-padding of the input signal. Clearly $\mathbf{H}_t$ is no longer orthogonal. Let $\mathbf{H}^{\#}$ (of size $(M - 2K) \times M$) denote the matrix that is composed of the central $M - 2K$ rows of $\mathbf{H}_t'$. It is straightforward to show that [12]

$$\mathbf{H}^{\#}\mathbf{H}_t = \begin{pmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \end{pmatrix} \quad (20)$$

where $\mathbf{I}$ is the identity matrix of size $M - 2K$. Hence, by this direct matrix multiplication we could recover the middle $M - 2K$ samples of the input signal, and we are only left with the first and last $K$ samples, i.e., $\mathbf{x}_1$ and $\mathbf{x}_l$. This result could be straightforwardly generalized for any size $M > 2K + 2$ (i.e., $M$ does not have to be multiple of $K$). In all cases we have,

$$\mathbf{y}_i = \mathbf{U}\mathbf{x}_{i-1} + \mathbf{E}\mathbf{x}_i + \mathbf{L}\mathbf{x}_{i+1} \ \text{ for } 1 < i < l \quad (21)$$

Several approaches have been devised to recover remaining $2K$ samples [12], e.g.,

1. *Periodic Extension*: In this case, we have

$$\mathbf{y}_1 \quad = \quad \mathbf{E}\mathbf{x}_1 + \mathbf{L}\mathbf{x}_2 + \mathbf{U}\mathbf{x}_l \quad (22)$$

$$\mathbf{y}_l \quad = \quad \mathbf{U}\mathbf{x}_{l-1} + \mathbf{E}\mathbf{x}_l + \mathbf{L}\mathbf{x}_1 \quad (23)$$

Therefore using the orthogonality conditions (9)-(11), it is straightforward to show that

$$\mathbf{x}_1 \quad = \quad \mathbf{L}'\mathbf{y}_l + \mathbf{E}'\mathbf{y}_1 + \mathbf{U}'\mathbf{y}_2 \quad (24)$$

$$\mathbf{x}_l \quad = \quad \mathbf{L}'\mathbf{y}_{l-1} + \mathbf{E}'\mathbf{y}_l + \mathbf{U}'\mathbf{y}_1 \quad (25)$$

i.e., the perfect reconstruction is achieved in all cases without any extra assumptions on the filter.

2. *Symmetric Extension*: Let $\mathbf{J}$ denote inverse diagonal matrix with ones only on the main anti-diagonal. Then,

$$\mathbf{y}_1 \quad = \quad (\mathbf{E} + \mathbf{U}\mathbf{J})\mathbf{x}_1 + \mathbf{L}\mathbf{x}_2 \quad (26)$$

$$\mathbf{y}_l \quad = \quad \mathbf{U}\mathbf{x}_{l-1} + (\mathbf{E} + \mathbf{L}\mathbf{J})\mathbf{x}_l \quad (27)$$

In general, this is not guaranteed to be reversible unless the filter is symmetric. However, if $(\mathbf{E} + \mathbf{U}\mathbf{J})$ and $(\mathbf{E} + \mathbf{L}\mathbf{J})$ are invertible, then $\mathbf{x}_1$ and $\mathbf{x}_l$ can be computed as

$$\mathbf{x}_1 \quad = \quad (\mathbf{E} + \mathbf{U}\mathbf{J})^{-1}(\mathbf{y}_1 - \mathbf{L}\mathbf{x}_2) \quad (28)$$

$$\mathbf{x}_l \quad = \quad (\mathbf{E} + \mathbf{L}\mathbf{J})^{-1}(\mathbf{y}_l - \mathbf{U}\mathbf{x}_{l-1}) \quad (29)$$

where $\mathbf{x}_2$ and $\mathbf{x}_{l-1}$ are among the middle components that are computed using (20).

3. *Zero Padding*: A similar analysis applies to the zero padding case. In this case the matrix model is $\mathbf{H}_t$ in (19). The perfect reconstruction is possible if $\mathbf{E}$ in (7) is full-rank. In this case,

$$\mathbf{x}_1 = \mathbf{E}^{-1}(\mathbf{y}_1 - \mathbf{L}\mathbf{x}_2) \quad (30)$$
$$\mathbf{x}_l = \mathbf{E}^{-1}(\mathbf{y}_l - \mathbf{U}\mathbf{x}_{l-1}) \quad (31)$$

4. *Generalized Extension*: In some applications, e.g., signal coding, we may extend the finite length signal by any segment of length $K$ from the middle $M - 2K$ samples to get better boundary behavior. Since these samples are recovered by direct filtering using (20), it does not affect the perfect reconstruction. In this case, the analysis/synthesis relations would be

$$\mathbf{y}_1 = \mathbf{U}\mathbf{x}_i + \mathbf{E}\mathbf{x}_1 + \mathbf{L}\mathbf{x}_2 \quad (32)$$
$$\mathbf{x}_1 = \mathbf{E}^{-1}(\mathbf{y}_1 - \mathbf{L}\mathbf{x}_2 - \mathbf{U}\mathbf{x}_i) \quad (33)$$

where $1 < i < l$.

In Table 1, we list the condition numbers of $\mathbf{E}$, $\mathbf{E}+\mathbf{UJ}$ and $\mathbf{E}+\mathbf{LJ}$ of some standard orthonormal wavelets: Daubechies wavelets of sizes 10, 14, and 18 (db10, db14, and db18); Coiflet wavelet of length 18 (cf18), Symmlet wavelet of size 14 (sm14), and Beylkin wavelet of size 18 (bl) [13, 14]. From the table we see that, some standard wavelets, e.g., sm14, could be used unmodified with zero padding while preserving perfect reconstruction.

**Table 1**. Condition numbers of standard wavelets

| Wavelet | $K$ | $\kappa(\mathbf{E})$ | $\kappa(\mathbf{E}+\mathbf{UJ})$ | $\kappa(\mathbf{E}+\mathbf{LJ})$ |
|---|---|---|---|---|
| db10 | 4 | 88 | 88 | 88 |
| db14 | 6 | 56 | 554 | 8.02e+3 |
| db18 | 8 | 370 | 6.07e+5 | 8.7e+6 |
| cf18 | 8 | 21 | 31 | 64 |
| sm14 | 6 | 1.9 | 3.6 | 2.4 |
| bl | 8 | 478 | 8.6e+5 | 7.8e+5 |

## 4. WAVELET DESIGN PROCEDURE

Assume we have a given orthogonal wavelet filter $\{h_0(n)\}$ with an ill-conditioned $\mathbf{E}$ matrix. The objective of wavelet filter design is to find another orthogonal wavelet filter $\{h(n)\}$ so that the corresponding $\mathbf{E}$ matrix is well-conditioned (i.e., has a relatively small condition number) while minimizing the distance between $\{h(n)\}$ and $\{h_0(n)\}$ in some meaningful sense. The condition number of $\mathbf{E}$ is defined as [8]

$$\kappa(\mathbf{E}) \triangleq \frac{\sigma_1(\mathbf{E})}{\sigma_K(\mathbf{E})} \quad (34)$$

where $\sigma_1(\mathbf{E})$ and $\sigma_K(\mathbf{E})$ are respectively the maximum and minimum singular values of $\mathbf{E}$. From [10], we have

$$\sigma_K(\mathbf{E}) = \sqrt{1 - \sigma_1^2(\mathbf{L})} \quad (35)$$
$$\sigma_1(\mathbf{E}) \approx 1 \quad (36)$$

where $\sigma_1(\mathbf{L})$ is the largest singular value of $\mathbf{L}$ in (8). Hence if $\kappa(\mathbf{E})$ is required (for numerical stability) to be less than a certain value $\varepsilon$, it is equivalent to

$$\sigma_1(\mathbf{L}) \leq \sqrt{1 - \varepsilon^{-2}} \quad (37)$$

Note that, if a matrix has distinct singular values, then the mapping of the singular values and the entries of the singular vectors to the matrix entries is unique [8]. Furthermore, from the rank-one representation of the singular value decomposition (e.g., in (12)) this mapping is continuous and differentiable. Therefore, by the inverse function theorem [15], the inverse function, i.e., the mapping from the matrix entries to the singular values and the entries of the singular vectors, is also continuous and differentiable. Therefore, the mapping from the matrix entries to the singular values is differentiable. Furthermore, by applying the chain-rule [15] on the maximum of the singular values, we conclude that, the largest singular value is a differentiable function of the matrix entries. Hence, $\sigma_1(\mathbf{L})$ in (37) is a differentiable function of $\{h(n)\}$. From (15), $\{h(n)\}$ is a differentiable function of the vector $[\mathbf{v}' \ \sigma]'$ (of length $N/2+1$) that parameterizes $\mathbf{\Gamma}$ in (15). Hence, by the chain rule, $\sigma_1(\mathbf{L})$ is also a differentiable function of $[\mathbf{v}' \ \sigma]'$. Similarly, $\kappa(\mathbf{E})$ is also a differentiable function of the decision variables.

The filter perturbation problem can now be put in a standard form. Starting from a template wavelet filter $\{h_0(n)\}$, find a wavelet filter $\{h(n)\}$ to minimize $\|\mathbf{h}_0 - \mathbf{h}\|^2$ under the constraint in (37), where $\mathbf{h}$ satisfies (15) and the decision variables are $[\mathbf{v}' \ \sigma]'$ that parameterize $\mathbf{\Gamma}$. This is a constrained nonlinear optimization problem with differentiable objective functions and constraints of the decision variables. Standard nonlinear programming techniques [11] could be used to solve it. In particular, the log-barrier algorithm could be used to convert the problem to an unconstrained nonlinear optimization problem. In this case, the objective is to minimize the modified objective function

$$\mathbf{J} = \|\mathbf{h} - \mathbf{h}_0\|^2 - \log[\sqrt{1 - \varepsilon^{-2}} - \sigma_1(\mathbf{L})] \quad (38)$$

or equivalently

$$\mathbf{J} = \|\mathbf{h} - \mathbf{h}_0\|^2 - \log[\xi - \kappa(\mathbf{E})] \quad (39)$$

and the solution is done using standard gradient search or the Newton algorithm [11]. The initial interior point of the algorithm could be obtained by using random initial values of the decision vector $[\mathbf{v}' \ \sigma]'$ in (15), and computing the corresponding $\sigma_1(\mathbf{L})$ until condition (37) is fulfilled.

The above time-domain optimization results in general in poor behavior in the frequency domain. Nevertheless, it is useful when the kernel wavelet is designed to match a template function in the time domain, e.g., [16]. In the general case however, we are more interested in matching the kernel behavior in the frequency domain. If $\hat{\mathbf{h}}$ and $\hat{\mathbf{h}}_0$ denote the magnitude of the discrete Fourier transform (DFT) of $\mathbf{h}$ and $\mathbf{h}_0$ respectively, then the objective function for the frequency-domain optimization could be expressed as

$$\mathbf{J} = (\hat{\mathbf{h}} - \hat{\mathbf{h}}_0)'\mathbf{W}(\hat{\mathbf{h}} - \hat{\mathbf{h}}_0) \quad (40)$$

where $\mathbf{W}$ is a diagonal weighting matrix, whose entries are inversely proportional to the corresponding entries in $\hat{\mathbf{h}}_0$. Further, to allow for both symmetric extension and zero extension at the boundaries, two extra constraints should be included:

$$\kappa(\mathbf{E}+\mathbf{UJ}) \leq \xi \quad (41)$$
$$\kappa(\mathbf{E}+\mathbf{LJ}) \leq \xi \quad (42)$$

to allow for numerically stable reconstruction in (28) and (29). In Fig. 1, we show few examples of applying the above constrained optimization problem to three standard orthonormal wavelets (whose condition numbers are listed in Table 1). The resulting filters are illustrated in Fig. 1 and the condition numbers of the resulting optimized wavelet kernels are listed in Table 2.

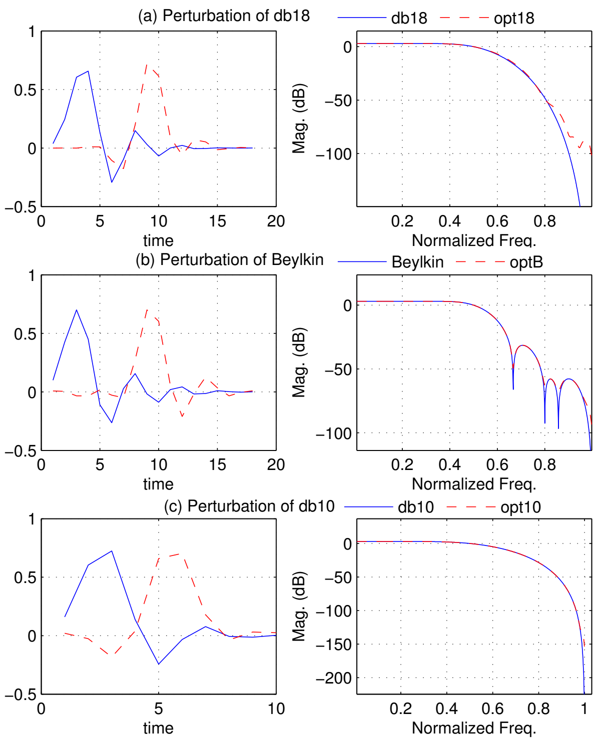

**Fig. 1**. Perturbation examples of some standard orthonormal wavelets: Daubechies-18 (db18), Beylkin, and Daubechies 10 (db10)

**Table 2**. Condition numbers of perturbed Wavelets

| Wavelet | $\kappa(\mathbf{E})$ | $\kappa(\mathbf{E}+\mathbf{UJ})$ | $\kappa(\mathbf{E}+\mathbf{LJ})$ |
|---|---|---|---|
| opt18 | 1.05 | 1.38 | 1.30 |
| optB | 1.04 | 1.27 | 1.21 |
| opt10 | 1.001 | 1.001 | 1.001 |

## 5. EXAMPLES

Zero-padding of the input signal is of particular importance in real-time implementation of the wavelet transform because it is a *causal* transform [17]. The proposed design procedure provides an explicit procedure for designing wavelts that support zero-padding among other extensions with a single wavelet kernel. In Fig. 2, we give an illustrative example of a single-stage wavelet decomposition using different signal extensions. We compare the standard Daubechies wavelet of order 18 (with periodic extension of the signal), to the new designed filters: filter perturbation opt18 (with $\kappa(\mathbf{E}) \approx 1.05$) with zero-padding. The proposed wavelet filter behaves almost the same as the original wavelet but with a much better condition number that allows a more numerically stable synthesis. Also, note that the periodic extension introduces high frequency components that is emphasized at the border of the wavelet coefficients.

The performance of the proposed algorithms in image coding was evaluated using many standard test images with the Set Partitioning In Hierarchical Trees (SPIHT) coding algorithm [18]. The PSNRs of the tests are listed in Table 3 for Daubechies wavelets of size 10 and 18, and the corresponding perturbed wavelets opt10, and opt18 as in Fig. 1. Note that, both the zero-padding and the symmetric extension (using the perturbed wavelet) provide consistent improvement over periodic extension with the original orthonormal wavelet even though the original wavelets have slightly better frequency behavior.

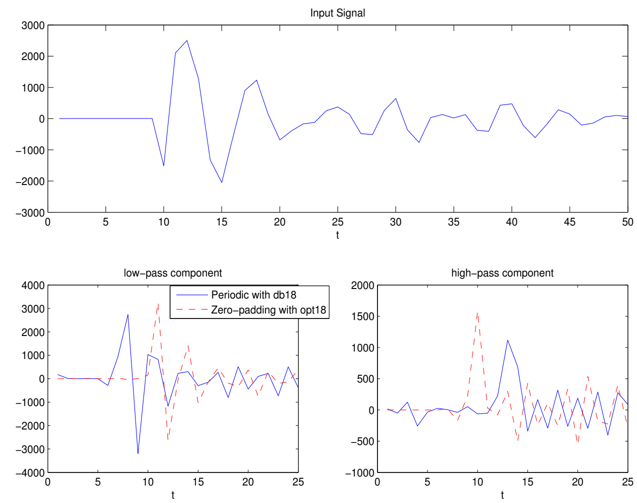

**Fig. 2**. Example of a single-stage wavelet decomposition of a segment of an ultrasound scan line using db18 (with periodic extension) and opt18 (with zero padding)

**Table 3**. PSNR of proposed algorithm with SPIHT coding of standard images at 1 bit/pixel

| Image | Periodic | | Symmetric | | Zero Padding | |
|---|---|---|---|---|---|---|
| | db10 | db18 | opt10 | opt18 | opt10 | opt18 |
| Bridge | 29.56 | 29.50 | 29.75 | 29.82 | 29.68 | 29.72 |
| Lena | 39.59 | 39.50 | 39.97 | 40.03 | 39.66 | 39.72 |
| Cameraman | 44.49 | 44.82 | 44.66 | 45.10 | 44.11 | 44.55 |
| Pirate | 35.80 | 35.63 | 36.25 | 36.31 | 36.00 | 36.04 |
| Barbara | 35.47 | 35.77 | 35.93 | 36.29 | 35.66 | 36.03 |
| Boat | 35.59 | 35.57 | 35.90 | 36.00 | 35.72 | 35.81 |
| Peppers | 36.09 | 35.40 | 36.64 | 36.51 | 36.47 | 36.35 |

## 6. CONCLUSION

We presented a novel procedure for designing orthonormal wavelets of compact support that is suited for finite-length wavelet transform. The perfect reconstruction conditions for the finite-length transform (with the required signal extensions) are set as constraints in the wavelet design problem. We demonstrated the design procedure by examples to match the frequency or time response of standard wavelets while satisfying the perfect reconstruction conditions of the finite-length transform. The procedure can be generalized straightforwardly to handle other desired wavelet features that could be added as additional constraints to the wavelet design problem.

The proposed algorithm provides a viable solution to support perfect reconstruction orthogonal wavelet transform in real-time systems that operate on a sample-by-sample basis without compromising the coding performance. The effectiveness of the proposed algorithm is established using practical signal processing examples.

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