2D ORTHOGONAL SYMMETRIC WAVELET FILTERS USING ALLPASS FILTERS

Xi ZHANG

Dept. of Communication Engineering and Informatics
The University of Electro-Communications
Chofu-shi, Tokyo, 182-8585 Japan
E-mail: zhangxi@uec.ac.jp

ABSTRACT

This paper proposes a new class of 2D orthogonal symmetric wavelet filters using 2D nonseparable allpass filters. The proposed wavelet filters are based on the parallel structure of allpass filters with real-valued coefficients, which can be implemented with a low computational complexity and is robust to finite precision effects. The resulting wavelet bases are not only orthogonal, including perfect reconstruction (PR) condition, but also symmetric, whose analysis and synthesis filters have exactly linear phase response. It is also shown that the design problem of the proposed wavelet filters can be reduced to the phase approximation of the corresponding allpass filters. Therefore, it is easy to design this class of orthogonal symmetric wavelet filters by using the existing design methods of allpass filters. Finally, some examples are presented to demonstrate the effectiveness of the proposed orthogonal symmetric wavelet filters.

Index Terms— Allpass filter, orthogonality, symmetry, wavelets.

1. INTRODUCTION

Discrete wavelet transforms have been extensively used in digital signal processing applications. It is well-known [1]∼[3] that one desirable property for wavelets is orthogonality, and another is symmetry, which requires all analysis and synthesis filters to possess exactly linear phase response. It had been proven in [1] that except the Haar wavelet, there is not any orthogonal symmetric wavelets using one dimensional (1D) FIR filters, corresponding to compactly supported wavelets. To obtain orthogonal symmetric wavelets with more regularity than the Haar wavelet, a class of 1D IIR wavelet filters had been proposed by using 1D allpass filters in [5], [9] and [10], which are based on the parallel structure of allpass filters. Thus the allpass-based wavelet filters can be implemented with a low computational complexity and are robust to finite precision effects [2].

Two dimensional (2D) wavelet filters are often needed to decompose 2D signals into directional components in image and video processing applications, and have been discussed in [4], [6], [11], [12]. In [6], a class of two channel quincunx wavelet filters had been presented by mapping 1D biorthogonal wavelet filters to the 2D case, but the resulting 2D quincunx wavelet filters are biorthogonal, not orthogonal. In [11] and [12], a class of two channel quincunx and parallelogram QMF banks had been proposed by using 2D nonseparable allpass filters. However, the proposed QMF banks did not satisfy the perfect reconstruction (PR) condition, and an additional 2D allpass filter was needed to compensate the phase distortion as a phase equalizer. Moreover, the phase responses of the analysis and synthesis filters of QMF banks are not exactly linear.

In this paper, we propose a new class of 2D orthogonal symmetric wavelet filters using 2D nonseparable allpass filters. We will focus on two subsampling cases: quincunx and parallelogram. First of all, we generalize the 1D allpass-based orthogonal symmetric wavelet filters proposed in [9] to the 2D case by using 1D to 2D mapping on the polyphase components of wavelet filters. Thus the resulting 2D allpass-based wavelet filters are not only orthogonal, but also symmetric. Next, we show that the design problem of the proposed wavelet filters can be reduced to the phase approximation of the corresponding allpass filters. Therefore, it is easy to design this class of 2D allpass-based orthogonal symmetric wavelet filters by using the existing design methods of allpass filters in [7] and [8]. Finally, some examples are presented to demonstrate the effectiveness of the proposed orthogonal symmetric wavelet filters.

2. 1D ORTHOGONAL SYMMETRIC WAVELET FILTERS USING ALLPASS FILTERS

It is well-known [1]∼[3] that 1D wavelet basis can be generated by 1D two channel perfect reconstruction (PR) filter bank $\{H(z), G(z)\}$, where $H(z)$ is a lowpass filter and $G(z)$ is highpass. The orthogonality condition of $\{H(z), G(z)\}$ is given by

$$\begin{cases} H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = 1 \\ G(z)G(z^{-1}) + G(-z)G(-z^{-1}) = 1 \\ H(z)G(z^{-1}) + H(-z)G(-z^{-1}) = 0 \end{cases}, \tag{1}$$

in which the PR condition has been included. If a wavelet basis is required to be symmetric, $H(z)$ and $G(z)$ must have exactly linear phase responses.

In [9], a class of 1D orthogonal symmetric wavelet filters has proposed by using allpass filters as follows;

$$\begin{cases} H(z) = \frac{1}{2}[A(z^2) + z^{-2K-1}A(z^{-2})] \\ G(z) = \frac{1}{2}[A(z^2) - z^{-2K-1}A(z^{-2})] \end{cases}, \tag{2}$$

where K is integer, and $A(z)$ is 1D allpass filter of order N defined by

$$A(z) = z^{-N} \frac{\sum_{n=0}^{N} a_n z^n}{\sum_{n=0}^{N} a_n z^{-n}}, \tag{3}$$

where a_n are a set of real-valued coefficients and $a_0 = 1$.

Assume that the phase response of $A(z)$ is $\theta(\omega)$, then we have

$$\theta(\omega) = -N\omega + 2\tan^{-1}\frac{\sum_{n=0}^{N} a_n \sin n\omega}{\sum_{n=0}^{N} a_n \cos n\omega}. \tag{4}$$

Thus the frequency responses of $H(z)$ and $G(z)$ can be given from Eq.(2) by

$$\begin{cases} H(e^{j\omega}) = e^{-j(K+\frac{1}{2})\omega}\cos\{\theta(2\omega) + (K+\frac{1}{2})\omega\} \\ G(e^{j\omega}) = je^{-j(K+\frac{1}{2})\omega}\sin\{\theta(2\omega) + (K+\frac{1}{2})\omega\} \end{cases}. \tag{5}$$

It is clear in Eq.(5) that $H(z)$ and $G(z)$ have exactly linear phase responses and satisfy the following power-complementary relation;

$$|H(e^{j\omega})|^2 + |G(e^{j\omega})|^2 = 1. \tag{6}$$

As shown in Fig.1, $H(z)$ and $G(z)$ are a pair of lowpass and highpass filters, and then the desired magnitude response of $H(z)$ is given by

$$|H_d(e^{j\omega})| = \begin{cases} 1 & (0 \le \omega \le \omega_p) \\ 0 & (\omega_s \le \omega \le \pi) \end{cases}, \tag{7}$$

where ω_p and ω_s are the cutoff frequencies in passband and stopband of $H(z)$, respectively, and $\omega_p + \omega_s = \pi$.

Therefore, it is clear from Eq.(5) that to get a pair of lowpass and highpass filters, the phase response of $A(z)$ must satisfy

$$\theta(2\omega) + (K+\frac{1}{2})\omega = \begin{cases} 0 & (0 \le \omega \le \omega_p) \\ \pm\frac{\pi}{2} & (\omega_s \le \omega \le \pi) \end{cases}. \tag{8}$$

Due to the antisymmetry property of the phase response, the desired phase response of $A(z)$ can be reduced to

$$\theta_d(\omega) = -(\frac{K}{2} + \frac{1}{4})\omega \qquad (0 \le \omega \le 2\omega_p). \tag{9}$$

Therefore, the design problem of the allpass-based orthogonal symmetric wavelet filters shown in Fig.1 becomes the phase approximation of $A(z)$ to the desired phase response in Eq.(9).

There are many design methods for 1D allpass filters to approximate the desired phase response, for example, the maximally flat, equiripple approximations, and so on [7]∼[9]. Therefore, these existing design methods can be used to design the allpass-based orthogonal symmetric wavelet filters.

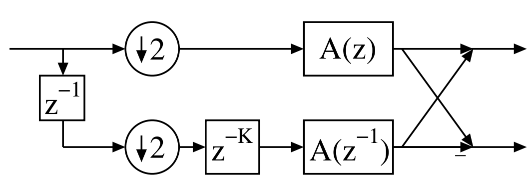

Fig. 1. 1D allpass-based orthogonal symmetric wavelet filters.

3. 2D ORTHOGONAL SYMMETRIC WAVELET FILTERS USING ALLPASS FILTERS

In this section, we will generalize the 1D allpass-based orthogonal symmetric wavelet filters discussed in the preceding section to the 2D case.

It is seen in Eq.(9) that the phase response of $A(z)$ is required to be a fractional delay $K/2 + 1/4$. Thus, the phase response of $A(z)$ is the same as that of $z^{-K-\frac{1}{2}}A(z^{-1})$. We first replace the 1D allpass filter $A(z)$ with the 2D allpass filter $A(z_1, z_2)$ that is defined by

$$A(z_1, z_2) = z_1^{-N} z_2^{-M} \frac{\sum_{n=0}^{N}\sum_{m=0}^{M} a_{nm} z_1^{n} z_2^{m}}{\sum_{n=0}^{N}\sum_{m=0}^{M} a_{nm} z_1^{-n} z_2^{-m}}, \tag{10}$$

where a_{nm} are a set of real-valued coefficients and $a_{00} = 1$. Then we replace $z^{-K-\frac{1}{2}}A(z^{-1})$ with $(z_1z_2)^{-K-\frac{1}{2}}A(z_1^{-1}, z_2^{-1})$.

Similarly to the 1D case, the phase response of $A(z_1, z_2)$ is also required to be the same as that of $(z_1z_2)^{-K-\frac{1}{2}}A(z_1^{-1}, z_2^{-1})$. Thus, the desired phase response of $A(z_1, z_2)$ is given by

$$\theta_d(\omega_1, \omega_2) = -(\frac{K}{2} + \frac{1}{4})(\omega_1 + \omega_2) \quad \begin{pmatrix} -\omega_{1p} \le \omega_1 \le \omega_{1p} \\ -\omega_{2p} \le \omega_2 \le \omega_{2p} \end{pmatrix}. \tag{11}$$

where ω_{1p} and ω_{2p} are the cutoff frequencies for ω_1 and ω_2, respectively.

In the simplest case, we can choose a 2D separable allpass filter as $A(z_1, z_2) = A(z_1)A(z_2)$. Therefore, the desired phase response of the 1D allpass filter $A(z)$ is the same as in the 1D case (see Eq.(9)), and then the existing design methods of 1D allpass filters in [7] and [8] can be used to design this class of wavelet filters. In the following, we will discuss two subsampling cases: quincunx and parallelogram.

3.1. Quincunx Case

In the quincunx subsampling case, the decimation/interpolation matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \tag{12}$$

then $A(z_1, z_2)|_{\uparrow\mathbf{M}} = A(z_1z_2, z_1^{-1}z_2)$ and $(z_1z_2)^{-K-\frac{1}{2}}|_{\uparrow\mathbf{M}} = z_2^{-2K-1}$. Therefore, we have a pair of lowpass and highpass filters $H(z_1, z_2)$ and $G(z_1, z_2)$ as follows;

$$\begin{cases} H(z_1, z_2) = \frac{1}{2}[A(z_1z_2, z_1^{-1}z_2) + z_2^{-2K-1}A(z_1^{-1}z_2^{-1}, z_1z_2^{-1})] \\ G(z_1, z_2) = \frac{1}{2}[A(z_1z_2, z_1^{-1}z_2) - z_2^{-2K-1}A(z_1^{-1}z_2^{-1}, z_1z_2^{-1})] \end{cases}, \tag{13}$$

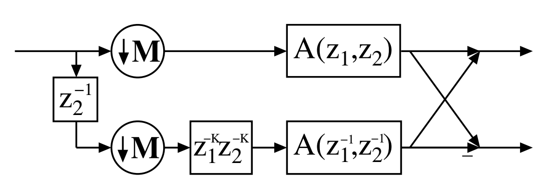

Fig. 2. 2D allpass-based orthogonal symmetric wavelet filters.

whose structure is shown in Fig.2 and whose frequency responses are given by

$$\begin{cases} H(e^{j\omega_1}, e^{j\omega_2}) = e^{-j(K+\frac{1}{2})\omega_2} \times \\ \qquad \cos\{\theta(\omega_1+\omega_2, \omega_2-\omega_1) + (K+\frac{1}{2})\omega_2\} \\ G(e^{j\omega_1}, e^{j\omega_2}) = je^{-j(K+\frac{1}{2})\omega_2} \times \\ \qquad \sin\{\theta(\omega_1+\omega_2, \omega_2-\omega_1) + (K+\frac{1}{2})\omega_2\} \end{cases}. \quad (14)$$

It is clear in Eq.(14) that $H(z_1, z_2)$ and $G(z_1, z_2)$ have exactly linear phase responses and satisfy the following power-complementary relation;

$$|H(e^{j\omega_1}, e^{j\omega_2})|^2 + |G(e^{j\omega_1}, e^{j\omega_2})|^2 = 1. \quad (15)$$

From the desired phase response of $A(z_1, z_2)$ in Eq.(11) and the antisymmetry property of the phase response, we can obtain a lowpass filter $H(z_1, z_2)$ with the passband $(|\omega_1+\omega_2| \leq \omega_{1p}, |\omega_1 - \omega_2| \leq \omega_{2p})$ and stopband $(\omega_1+\omega_2 \geq 2\pi - \omega_{1p}, \omega_1 \leq \pi, \omega_2 \leq \pi)$, $(\omega_1+\omega_2 \leq -2\pi + \omega_{1p}), \omega_1 \geq -\pi, \omega_2 \geq -\pi)$, $(\omega_1 - \omega_2 \geq 2\pi - \omega_{2p}, \omega_1 \leq \pi, \omega_2 \geq -\pi)$, and $(\omega_1 - \omega_2 \leq -2\pi + \omega_{2p}, \omega_1 \geq -\pi, \omega_2 \leq \pi)$, as shown in Fig.3(a).

3.2. Parallelogram Case

In the parallelogram subsampling case, the decimation/interpolation matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \quad (16)$$

then $A(z_1, z_2)|_{\uparrow\mathbf{M}} = A(z_1, z_1^{-1}z_2^2)$ and $(z_1z_2)^{-K-\frac{1}{2}}|_{\uparrow\mathbf{M}} = z_2^{-2K-1}$. Therefore, we have a pair of lowpass and highpass filters $H(z_1, z_2)$ and $G(z_1, z_2)$ as follows;

$$\begin{cases} H(z_1, z_2) = \frac{1}{2}[A(z_1, z_1^{-1}z_2^2) + z_2^{-2K-1}A(z_1^{-1}, z_1z_2^{-2})] \\ G(z_1, z_2) = \frac{1}{2}[A(z_1, z_1^{-1}z_2^2) - z_2^{-2K-1}A(z_1^{-1}, z_1z_2^{-2})] \end{cases}, \quad (17)$$

whose structure is the same as shown in Fig.2. Then we have the frequency responses of $H(z_1, z_2)$ and $G(z_1, z_2)$ as

$$\begin{cases} H(e^{j\omega_1}, e^{j\omega_2}) = e^{-j(K+\frac{1}{2})\omega_2} \times \\ \qquad \cos\{\theta(\omega_1, 2\omega_2-\omega_1) + (K+\frac{1}{2})\omega_2\} \\ G(e^{j\omega_1}, e^{j\omega_2}) = je^{-j(K+\frac{1}{2})\omega_2} \times \\ \qquad \sin\{\theta(\omega_1, 2\omega_2-\omega_1) + (K+\frac{1}{2})\omega_2\} \end{cases}. \quad (18)$$

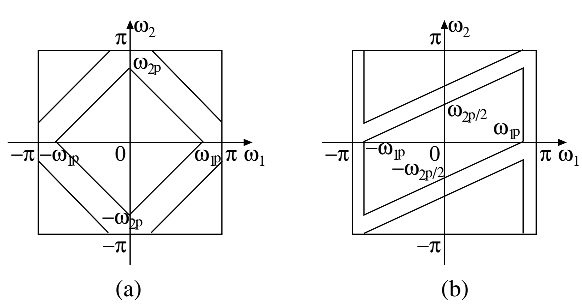

Fig. 3. Frequency supports for 2D wavelet filters: (a) quincunx case, (b) parallelogram case, where $\omega_{1p} = \omega_{2p}$.

It is clear in Eq.(18) that $H(z_1, z_2)$ and $G(z_1, z_2)$ have exactly linear phase responses and satisfy the following power-complementary relation in Eq.(15). Similarly, we can obtain a lowpass filter $H(z_1, z_2)$ with the passband $(|\omega_1| \leq \omega_{1p}, |\omega_1 - 2\omega_2| \leq \omega_{2p})$ and stopband $(|\omega_1| \leq \omega_{1p}, 2\pi - \omega_{2p} \leq \omega_1 - 2\omega_2 \leq 2\pi + \omega_{2p}, \omega_2 \geq -\pi)$ and $(|\omega_1| \leq \omega_{1p}, -2\pi - \omega_{2p} \leq \omega_1 - 2\omega_2 \leq -2\pi + \omega_{2p}, \omega_2 \leq \pi)$, as shown in Fig.3(b). Therefore, a class of 2D parallelogram orthogonal symmetric wavelet filters has been obtained.

4. DESIGN EXAMPLES

In this section, we present two design examples to demonstrate the effectiveness of the proposed 2D allpass-based orthogonal symmetric wavelet filters.

Example 1 Firstly, we have designed an 1D allpass filter $A(z)$ of $N = 2$ and $K = 0$ with the maximally flat phase response by using the existing design method of allpass filters in [7]. We then considered the simplest case where 2D separable allpass filter is chosen as $A(z_1, z_2) = A(z_1)A(z_2)$. It should be noted that 2D allpass filter is separable in the decimated domain, but the corresponding analysis and synthesis filters are nonseparable. In the quincunx subsampling case, the magnitude responses of the resulting quincunx lowpass and highpass filters are shown in Fig.4 and Fig.5, respectively. It is clear in Fig.4 and Fig.5 that a pair of quincunx lowpass and highpass filters with the maximally flat magnitude responses has been obtained.

Example 2 We have used the same allpass filter $A(z)$ as in Example 1, and considered the parallelogram subsampling case. Then the resulting magnitude responses of the parallelogram lowpass and highpass filters are shown in Fig.6 and Fig.7, respectively. It is seen in Fig.6 and Fig.7 that the parallelogram lowpass and highpass filters have the maximally flat magnitude responses and satisfy the power-complementary relation in Eq.(15). Note that the phase responses of all lowpass and highpass filters are exactly linear, as shown in Eq.(18).

5. CONCLUSION

In this paper, we have discussed two subsampling cases: quincunx and parallelogram, and proposed a new class of 2D orthogonal symmetric wavelet filters using 2D nonseparable allpass filters, which can be extended to other subsampling cases also. We have used 1D to 2D mapping on the polyphase components of wavelet filters to generalize the 1D allpass-based orthogonal symmetric wavelet filters in [9] to the 2D case. Thus the resulting 2D allpass-based wavelet filters are not only orthogonal, including PR condition, but also symmetric, whose analysis and synthesis filters have exactly linear phase response. Since the proposed wavelet filter is based on the parallel structure of allpass filters, it can be implemented with a low computational complexity and is robust to finite precision effects. It is also shown that the design of the proposed wavelet filters can be reduced to the phase approximation of the corresponding allpass filters. Therefore, it is easy to design this class of 2D allpass-based orthogonal symmetric wavelet filters by using the existing design methods of allpass filters. Finally, some examples are presented to demonstrate the effectiveness of the proposed 2D allpass-based orthogonal symmetric wavelet filters.

6. ACKNOWLEDGEMENTS

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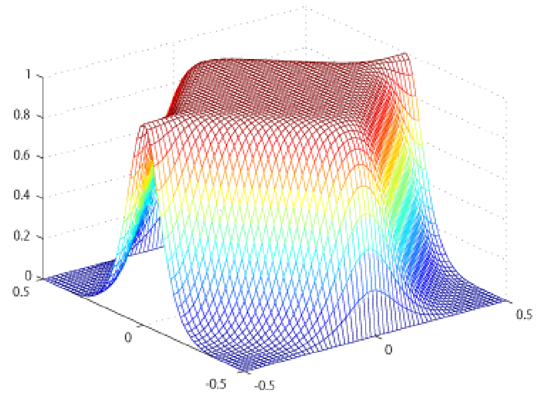

Fig. 4. Magnitude response of quincunx lowpass filter.

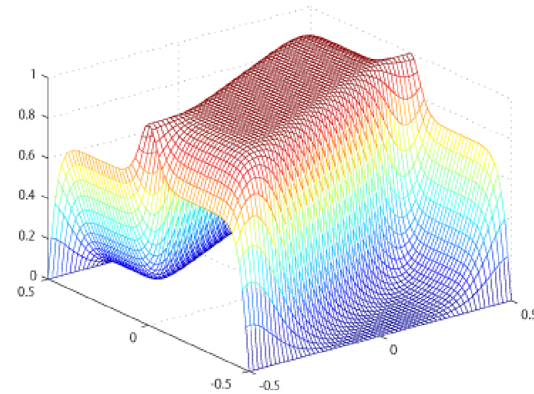

Fig. 6. Magnitude response of parallelogram lowpass filter.

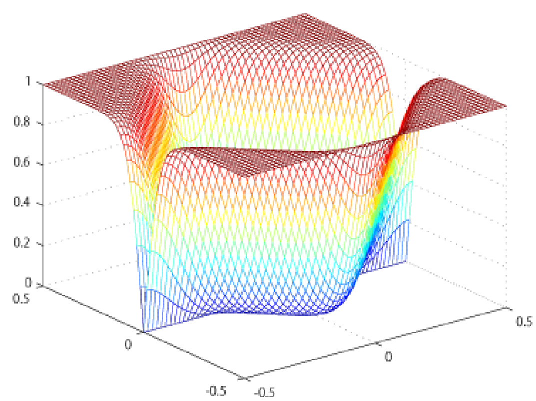

Fig. 5. Magnitude response of quincunx highpass filter.

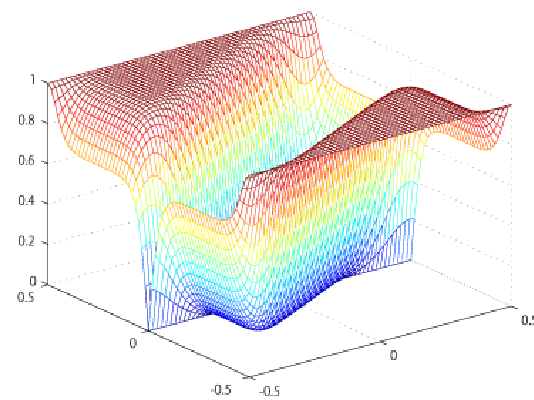

Fig. 7. Magnitude response of parallelogram highpass filter.

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