A GENERAL STRUCTURE FOR THE DESIGN OF ADJUSTABLE FIR FILTERS

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ABSTRACT

This paper proposes a general structure for FIR filters with adjustable magnitude and phase responses controlled by a few parameters. The Farrow structure which uses one parameter to control the fractional delay of an FIR filter can be viewed as special case. A filter bank structure consisting of different types of linear phase differentiators forms the basis of the structure. The filter bank outputs are combined with coefficients derived from a polynomial expansion of the desired frequency response. The magnitude and phase responses are controlled by synthesizing the polynomial coefficients from the small set of control parameters. A new optimal polynomial approximation strategy is also proposed to better approximate the family of target frequency responses.

Index Terms**—** Farrow structure, fractional delay filter, optimal polynomial approximation

1. INTRODUCTION

Digital FIR/IIR filter design is a well studied area. Various design specifications and the corresponding algorithms [1, 2] have been proposed or standardized. However, most of these design methods are optimized over a single fixed specification. To realize adjustable characteristics for those filters, we have to find an efficient way to update the coefficients of the digital filters rather than running a costly filter design routine in real time. Many interesting structures have been proposed to realize filters with adjustable cutoff frequencies [3] and sub band distributions [4]. Perhaps the most successful architecture is the so called Farrow structure [5] which controls the adjustable fractional delay (FD) of an FIR filter with a single parameter. FD filters are widely used in many applications such as timing adjustment in digital modems [6], speech coding and synthesis [7], digital wave guide modeling [8] and sampling rate converters [9]. A comprehensive study of the existing design methodologies is summarized in [10, 11].

Fig. 1 shows the diagram of the Farrow structure, where the filter bank $H_0(z)$ to $H_L(z)$ are FIR filters with the same order. The beauty of the Farrow structure lies in the fact that the adjustable characteristic, i.e., the fractional delay d of the filter structure, is controlled by the single parameter d. The

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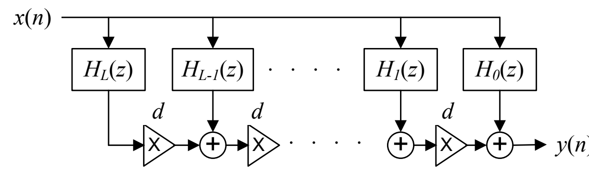

Fig. 1. The Farrow structure

Farrow structured FD filter represents a case where the slope of the linear phase response is adjustable. This paper proposes a general structure that is able to adjust both the magnitude and phase responses of the resulting filter simultaneously, with the Farrow structured FD filter as a special case. A new optimal polynomial approximation strategy is also proposed to better approximate the target frequency responses. The implementation of filters with adjustable cutoff frequencies and quadratic phases are also demonstrated.

2. SETUP FOR THE GENERALIZED ADJUSTABLE FILTER STRUCTURE

Suppose that the ideal frequency response of an FIR filter of order N can be written such that the magnitude and phase responses of the filter are expressed as polynomials of ω

$$H_{\text{id}}(e^{j\omega}) = \left(\sum_{p=0}^{P} a_p \omega^p\right) e^{-j\left[(N/2)\omega + \sum_{q=1}^{Q} b_q \omega^q\right]} \quad (1)$$

where P and Q are the order of amplitudes and phase responses, respectively. Then the frequency response of the filter can be controlled by directly adjusting the polynomial coefficients a_p and b_q. The ranges of adjustability for a_p and b_q are given by $a_p \in A_p$, $b_q \in B_q$. The bandwidth of interest is given by $\omega \in \Omega$.

In the first approximation stage, we can approximate each polynomial phase component $e^{-jb_q\omega^q}$ as a Taylor series of ω,

$$e^{-jb_q\omega^q} = \sum_{l=0}^{L_q} \frac{(-jb_q\omega^q)^l}{l!} + \varepsilon_q, \quad (2)$$

where L_q is the order of Taylor series for each polynomial phase component and the Taylor approximation error ε_q satisfies

$$|\varepsilon_q| \le \varepsilon_{q,\max} = \max_{\substack{b_q \in B_q \\ \omega \in \Omega}} \frac{(b_q \omega^q)^{L_q+1}}{(L_q+1)!} \tag{3}$$

Combining all the polynomial terms, we have the following approximated frequency response that is a polynomial of ω

$$\begin{aligned} H_{\text{poly}}(e^{j\omega}) =& e^{-j(N/2)\omega} \sum_{p=0}^{P} a_p \omega^p \prod_{q=1}^{Q} \sum_{l=0}^{L_q} \frac{(-jb_q\omega^q)^l}{l!} \\ =& e^{-j(N/2)\omega} \sum_{k=0}^{K} c_k \omega^k \end{aligned} \tag{4}$$

where the complex coefficients c_k of ω^k are given by the following convolutional relationship:

$$\begin{aligned} \mathbf{c} = & \begin{bmatrix} a_0 & a_1 & \cdots & a_P \end{bmatrix} * \\ & \begin{bmatrix} 1 & -jb_1 & \frac{(-jb_1)^2}{2!} & \cdots & \frac{(-jb_1)^{L_1}}{L_1!} \end{bmatrix} * \\ & \begin{bmatrix} 1 & 0 & -jb_2 & 0 & \frac{(-jb_2)^2}{2!} & 0 & \cdots & 0 & \frac{(-jb_2)^{L_2}}{L_2!} \end{bmatrix} * \cdots \\ & \begin{bmatrix} 1 & \underbrace{0 \cdots 0}_{Q-1} & -jb_Q & \underbrace{0 \cdots 0}_{Q-1} & \frac{(-jb_Q)^2}{2!} & \cdots & \frac{(-jb_Q)^{L_Q}}{L_Q!} \end{bmatrix} \end{aligned} \tag{5}$$

and

$$K = P + \sum_{q=1}^{Q} qL_q. \tag{6}$$

If we define the error

$$\varepsilon_A = \max_{\substack{a_p \in A_p \\ \omega \in \Omega}} a_p \omega^p \tag{7}$$

and ignore the high order error terms, we can bound the first stage approximation error by

$$|H_{\text{id}}(e^{j\omega}) - H_{\text{poly}}(e^{j\omega})| \lessapprox e_1 = \varepsilon_A \Big(\sum_{q=1}^{Q} \varepsilon_{q,\max} + \prod_{q=1}^{Q} \varepsilon_{q,\max}\Big) \tag{8}$$

If we separate the real and imaginary part of c_k as c_k^R and c_k^I and denote a pair of ideal k-th order linear phase FIR differentiators with filter order N as

$$G_k^R(e^{j\omega}) = \omega^k e^{-j(N/2)\omega}, \tag{9}$$

$$G_k^I(e^{j\omega}) = j\omega^k e^{-j(N/2)\omega}, \tag{10}$$

we can rewrite (4) as

$$H_{\text{poly}}(e^{j\omega}) = \sum_{k=0}^{K} c_k^R G_k^R(e^{j\omega}) + c_k^I G_k^I(e^{j\omega}) \tag{11}$$

In the second approximation stage, we can approximate the ideal differentiators with a pair of finite length differentiators $\hat{G}_k^R(e^{j\omega})$ and $\hat{G}_k^I(e^{j\omega})$ according to the minimax criterion. The approximation error is determined by the filter order N. The frequency response of the final realized filter is

$$\hat{H}(e^{j\omega}) = \sum_{k=0}^{K} c_k^R \hat{G}_k^R(e^{j\omega}) + c_k^I \hat{G}_k^I(e^{j\omega}) \tag{12}$$

If we optimize the differentiators $\hat{G}_k^R(e^{j\omega})$ and $\hat{G}_k^I(e^{j\omega})$ over $\omega \in \Omega$ according to the minimax criterion:

$$\left|\hat{G}_k^R(e^{j\omega}) - G_k^R(e^{j\omega})\right| \le \delta_k^R, \tag{13}$$

$$\left|\hat{G}_k^I(e^{j\omega}) - G_k^I(e^{j\omega})\right| \le \delta_k^I, \tag{14}$$

We can bound the second stage approximation error by

$$\left|\hat{H}(e^{j\omega}) - H_{\text{poly}}(e^{j\omega})\right| \le e_2 = \sum_{k=0}^{K} (|c_k^R|\delta_k^R + |c_k^I|\delta_k^I). \tag{15}$$

The total approximation error is bounded by the sum of the first and second stage approximation errors

$$e_{\max} = \left|\hat{H}(e^{j\omega}) - H_{\text{id}}(e^{j\omega})\right| \le e_1 + e_2 \tag{16}$$

Fig. 2 shows the corresponding structure for the generalized adjustable filter. The magnitude and phase responses of the filter are directly controlled by the input vectors $\mathbf{a} = \begin{bmatrix} a_0 & a_1 & \cdots & a_P \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 & b_2 & \cdots & b_Q \end{bmatrix}$. The synthesizing coefficient vectors $\mathbf{c}^R$ and $\mathbf{c}^I$ are determined according to (5).

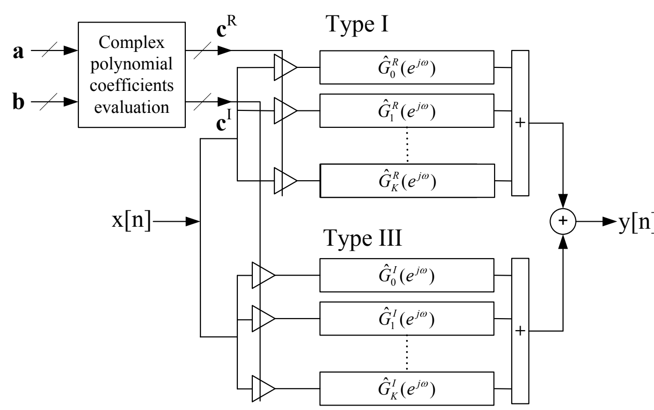

Fig. 2. A general filter structure with adjustable magnitude and phase responses.

We can increase the sub-filter order N to decrease δ_k^R and δ_k^I in order to reduce the second stage approximation error. Or, we can increase the order of the polynomials L_q to decrease the first stage approximation error.

The Farrow structured FD filter in Fig. 1 can be viewed as a special case of the proposed general structure. An ideal fractional delay filter of order N has the following frequency response

$$H_{\text{id}}(e^{j\omega}) = e^{-j(N/2)\omega} e^{-j\omega d}, \tag{17}$$

where $d \in [-0.5, 0.5)$ is the fractional delay. Comparing with the general expression in (1), we have $a_0 = 1$, $b_1 = d$. The synthesizing coefficients are

$$c_k^R = \begin{cases} \frac{1}{k!}d^k \\ 0 \\ \frac{-1}{k!}d^k \\ 0 \end{cases} \qquad c_k^I = \begin{cases} 0 & k \bmod 4 = 0 \\ \frac{-1}{k!}d^k & k \bmod 4 = 1 \\ 0 & k \bmod 4 = 2 \\ \frac{1}{k!}d^k & k \bmod 4 = 3 \end{cases} \tag{18}$$

Therefore, only one filter out of the differentiator pair $\hat{G}_k^I(e^{j\omega})$ and $\hat{G}_k^R(e^{j\omega})$ is active for each k. Each subfilter $H_k(e^{j\omega})$ in the Farrow structure in Fig. 1 approximates

$$H_k(e^{j\omega}) \to \frac{j^k}{k!} e^{j(N/2)\omega}(\omega^k). \tag{19}$$

3. OPTIMAL POLYNOMIAL APPROXIMATION

For the Farrow structured FD filter, the Taylor series approximation is accurate at the point of expansion, when $\omega d = 0$. The approximation degrades significantly as the product ωd moves away from 0, which suggests poor performance for a larger value of d, or at a higher frequency ω. Alternatively, an optimal polynomial approximation that minimizes the Chebyshev norm of the approximation error can be applied to evenly distribute the errors over the entire range of ωd

$$e^{-j\omega d} = \sum_{k=0}^{L} c_k (\omega d)^k + \varepsilon, \quad |\varepsilon| \le \varepsilon_{\max} \tag{20}$$

so that the first stage approximation error is bounded by $\varepsilon_{\max}$.

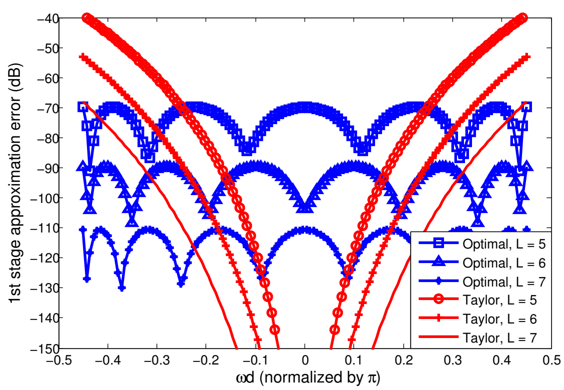

Fig. 3. Approximation error comparison between the optimal polynomial approximation and the Taylor series approximation strategies for $d \in [-0.5, 0.5]$, $\omega_c = 0.9\pi$.

Unlike the Taylor series case, there is no closed form formula for the coefficients c_k as they depend on the range of ωd. Given the range of $[(\omega d)_{\min}, (\omega d)_{\max}]$, the optimal coefficients that minimize the Chebyshev norm of the approximation error can be solved using the Remez exchange algorithm or convex optimization.

A comparison between the approximation errors with the optimal polynomial and the Taylor series approximation for various polynomial orders L is shown in Fig. 3, where the fractional delay $d \in [-0.5, 0.5]$ and cutoff frequency $\omega_c = 0.9\pi$. For the optimal polynomial approximation, the errors exhibit equal ripples across the range of ωd while the error associated with the Taylor series approximation is zero at $\omega d = 0$ and increases monotonically as $|\omega d|$ increases. In terms of the maximum approximation error, the optimal polynomial approximation is always superior to the Taylor series approximation with the same polynomial order L.

The adjustable fractional delay filter structure with the optimal polynomial approximation strategy is shown in Fig. 4. The overall approximation error over the frequency band $[0, \omega_c]$ for fixed values of d is shown in Figs. 5 and 6. Using the optimal polynomial approximation strategy, the approximation errors exhibit an equal ripple behavior in $[0, \omega_c]$ when d is fixed. However, when using Taylor series approximation strategy, the approximation error increases significantly in the high frequency band. This is caused by the fact that a large ωd makes the first stage approximation error dominate the overall approximation error.

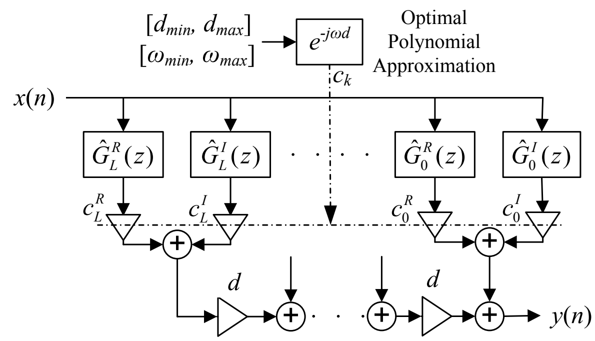

Fig. 4. An adjustable fractional delay filter

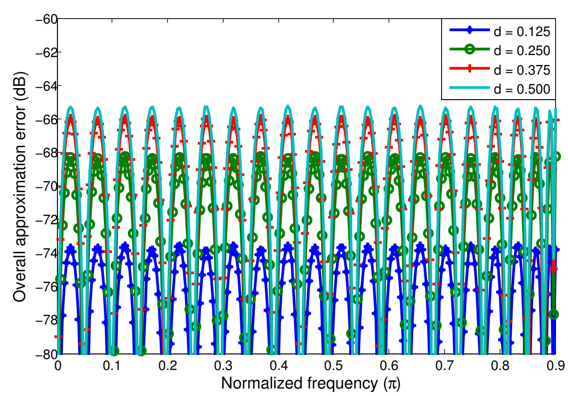

Fig. 5. Approximation error over $[0, \omega_c]$ with the optimal polynomial approximation strategy where $d \in [0, 0.5]$, $\omega_c = 0.9\pi$, $N = 40$, $L = 5$.

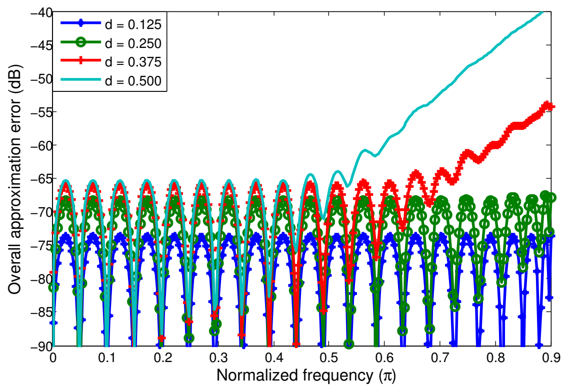

Fig. 6. Approximation error over $[0, \omega_c]$ with the Taylor series approximation strategy where $d \in [0, 0.5]$, $\omega_c = 0.9\pi$, $N = 40$, $L = 5$.

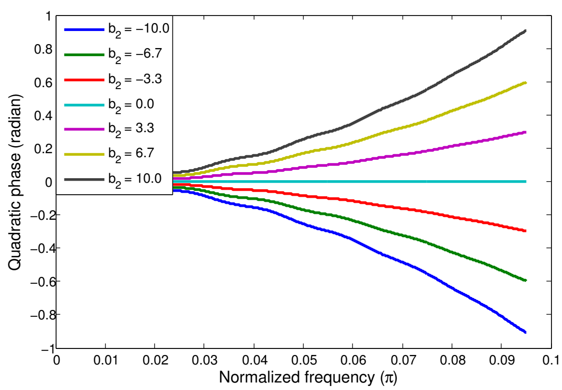

Fig. 7. Family of quadratic phase responses of the adjustable quadratic phase filter, controlled by the parameter b_2.

4. FILTERS WITH ADJUSTABLE QUADRATIC PHASE RESPONSES

The Farrow structured fractional delay filter represents a case where the single parameter d controls the linear phase response of the filter. The generalized adjustable filter structure can also be used to realize filters with an adjustable quadratic phase response, which is applicable in the RF pulse design problem [12, 13] in the MRI systems. For example, consider a lowpass filter of order 256 with an adjustable quadratic phase response,

$$H_{\text{id}}(e^{j\omega}) = \begin{cases} e^{-j\left[(N/2)\omega + b_2\omega^2\right]} & \omega \in [0, \omega_p] \\ 0 & \omega \in [\omega_s, \pi] \end{cases} \tag{21}$$

where the passband and stopband cutoff frequencies are given by $\omega_p = 0.095\pi$ and $\omega_s = 0.11\pi$. The range of the quadratic phase coefficient $b_2 \in [-10, 10]$. The polynomial order is given by $L = 3$. The corresponding phase responses are shown in Fig. 7, where the linear phase component $e^{-j(N/2)\omega}$ has already been removed so that only the quadratic phase component is present.

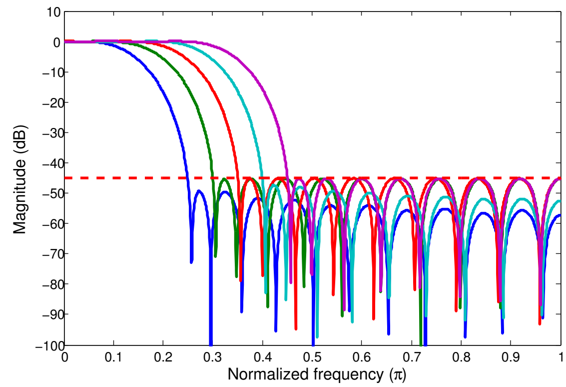

Fig. 8. Magnitude responses of LPF with adjustable cutoff frequencies. Filter order $N = 24$, polynomial order $L = 4$ and cutoff frequency (control) parameter $b \in [0.15\pi, 0.35\pi]$.

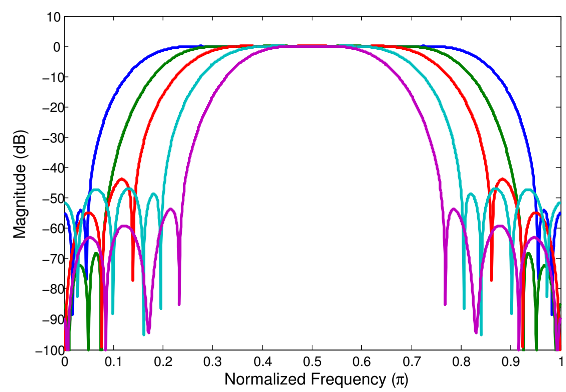

Fig. 9. Magnitude responses of a bandpass filter with two adjustable cutoff frequencies

5. FILTERS WITH ADJUSTABLE MAGNITUDE RESPONSES

The generalized adjustable filter structure can also be used to realize filters with adjustable magnitude responses in a similar way as the Farrow structure. The feasibility of such a structure was initially studied in [3]. For example, we can use a single parameter b to control the cutoff frequency of an adjustable lowpass filter(LPF). We can write the magnitude response of the variable LPF as $A(\omega) = \sum_{p=0}^{P} f_p(b)\omega^p$, where polynomial function $f_p(b)$ are produced in a design process. Fig. 8 shows a design example with the range of cutoff frequencies $b \in [0.15\pi, 0.35\pi]$. It is easy to introduce another control branch to formulate a bandpass filter with adjustable cutoff frequencies as shown in Fig. 9.

6. REFERENCES

[1] C.S. Burrus, J.A. Barreto, and I.W. Selesnick, "Iterative reweighted least-squares design of FIR filters," *Signal Processing, IEEE Transactions on*, vol. 42, no. 11, pp. 2926 –2936, Nov. 1994.

[2] L.J. Karam and J.H. McClellan, "Complex Chebyshev approximation for FIR filter design," *Circuits and Systems II: Analog and Digital Signal Processing, IEEE Transactions on*, vol. 42, no. 3, pp. 207 –216, Mar. 1995.

[3] P. Löwenborg and H. Johansson, "Minimax design of adjustable-bandwidth linear-phase FIR filters," *Circuits And Systems Part I, IEEE Transactions on*, vol. 53, no. 2, pp. 431–439, 2006.

[4] Y. Wei and D. Liu, "A design of digital FIR filter banks with adjustable subband distribution for hearing aids," in *Information, Communications and Signal Processing (ICICS) 2011 8th International Conference on*, Dec. 2011, pp. 1 –5.

[5] C.W. Farrow, "A continuously variable digital delay element," in *IEEE International Symposium on Circuits and Systems*, 1988.

[6] F.M. Gardner, "Interpolation in digital modems. I. Fundamentals," *Communications, IEEE Transactions on*, vol. 41, no. 3, pp. 501 –507, Mar. 1993.

[7] P. Kroon and B.S. Atal, "On the use of pitch predictors with high temporal resolution," *Signal Processing, IEEE Transactions on*, vol. 39, no. 3, pp. 733 –735, Mar. 1991.

[8] V. Valimaki and M. Karjalainen, "Implementation of fractional delay waveguide models using allpass filters," in *Acoustics, Speech, and Signal Processing, 1995. ICASSP-95., 1995 International Conference on*, May 1995, vol. 2, pp. 1524 –1527.

[9] H. Johansson and O. Gustafsson, "Linear-phase FIR interpolation, decimation, and mth-band filters utilizing the farrow structure," *Circuits and Systems I: Regular Papers, IEEE Transactions on*, vol. 52, no. 10, pp. 2197 – 2207, Oct. 2005.

[10] T.I. Laakso, V. Valimaki, M. Karjalainen, and U.K. Laine, "Splitting the unit delay [FIR/all pass filters design]," *Signal Processing Magazine, IEEE*, vol. 13, no. 1, pp. 30 –60, Jan. 1996.

[11] S. Samadi, M.O. Ahmad, and M.N.S. Swamy, "Results on maximally flat fractional-delay systems," *Circuits and Systems I: Regular Papers, IEEE Transactions on*, vol. 51, no. 11, pp. 2271 – 2286, Nov. 2004.

[12] R.F. Schulte, J. Tsao, P. Boesiger, and K.P. Pruessmann, "Equi-ripple design of quadratic-phase RF pulses.," *Journal of Magnetic Resonance*, vol. 166, no. 1, pp. 111–122, 2004.

[13] R.F. Schulte, A. Henning, J. Tsao, P. Boesiger, and K.P. Pruessmann, "Design of broadband RF pulses with polynomial-phase response.," *Journal of Magnetic Resonance*, vol. 186, no. 2, pp. 167–175, 2007.