KERNEL-INDUCED SAMPLING THEOREM FOR BANDPASS SIGNALS WITH UNIFORM SAMPLING

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ABSTRACT

In this paper, a sampling theorem for bandpass signals with uniformly spaced sampling points is discussed. We firstly show that a function space consisting of all functions with a specific bandpass property is a reproducing kernel Hilbert space and also give a closed-form of the corresponding reproducing kernel. Moreover, on the basis of the framework of the kernel-induced sampling theorem, we give a simple perfect reconstruction formula for the bandpass signals by uniformly spaced sampling points with the bandpass Nyquist rate, which is defined as twice the signal bandwidth, for the cases that the maximum frequency of the signals is identical to bandwidth multiplied by some positive integer.

***Index Terms*—** bandpass signals, reproducing kernel, kernel-induced sampling theorem, uniform sampling

1. INTRODUCTION

Bandpass sampling plays a crucial role in the field of various fields of signal processing such as in optics, radar, sonar, communications etc. as mentioned in [1, 2] for instance. Although nonuniform sampling scheme has been making great progress in the theory of bandpass sampling [3, 2], uniform sampling is still important from a practical point of view. Uniform sampling rate for a given band position achieving perfect reconstruction was thoroughly investigated in [2], and the same result was obtained by a more convincing way in [1]. According to their results, perfect reconstruction of bandpass signals by the bandpass Nyquist rate, defined as twice the signal bandwidth, is achieved only when the ratio $U/(U-L)$ is positive integer number, where U and L denote the maximum and the minimum frequencies of a given band; and we can never obtain perfect reconstruction by sampling rates lower than the bandpass Nyquist rate. Note that some literatures for this subject include errors as pointed out in [1]. Although the above limitation of passband setting exists, adopting the bandpass Nyquist rate is useful in practical problems in terms of efficiency of sampling since the bandpass Nyquist rate is the lowest sampling rate achieving perfect reconstruction. Therefore, we concentrate on bandpass sampling with the uniform bandpass Nyquist rate in this paper.

When only an insufficient set of sampling points, such as finite number of sampling points and lack of some sampling points, is available, we have to obtain the optimal approximation of a target function. However, we do not have efficient tools to deal with this problem. In [4], we constructed a unified framework for sampling problems including the optimal approximation called the kernel-induced sampling theorem. In this paper, we review the bandpass sampling on the basis of the framework of the kernel-induced sampling theorem. Specifically, we show that a function space consisting of all signals with bandpass property is a reproducing kernel Hilbert space and we also give a closed-form of the corresponding reproducing kernel; and give a formula for the optimal approximation with an insufficient set of sampling points. Moreover, we give a proof for bandpass sampling theorem with uniform bandpass Nyquist rate along with the framework of the kernel-induced sampling theorem, which yields a novel perfect reconstruction formula which is simpler than those obtained in conventional approach such as [2].

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2. MATHEMATICAL PRELIMINARIES FOR THE THEORY OF REPRODUCING KERNEL HILBERT SPACES

In this section, we prepare some mathematical tools concerned with the theory of reproducing kernel Hilbert spaces [5, 6].

Definition 1 *[5] Let $\mathbf{R}^n$ be an n-dimensional real vector space and let $\mathcal{H}$ be a class of functions defined on $\mathcal{D} \subset \mathbf{R}^n$, forming a Hilbert space of real-valued functions. The function $K(\boldsymbol{x}, \tilde{\boldsymbol{x}})$, $(\boldsymbol{x}, \tilde{\boldsymbol{x}} \in \mathcal{D})$ is called a reproducing kernel of $\mathcal{H}$, if*

1. *For every $\boldsymbol{x} \in \mathcal{D}$, $K(\cdot, \boldsymbol{x})$ is a function in $\mathcal{H}$.*
2. *For every $\boldsymbol{x} \in \mathcal{D}$ and every $f(\cdot) \in \mathcal{H}$,*

$$f(\boldsymbol{x}) = \langle f(\cdot), K(\cdot, \boldsymbol{x}) \rangle_{\mathcal{H}}, \tag{1}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ denotes the inner product of the Hilbert space $\mathcal{H}$.

The Hilbert space $\mathcal{H}$ that has a reproducing kernel is called a reproducing kernel Hilbert space (RKHS). The reproducing property Eq.(1) enables us to treat a value of a function at a point in $\mathcal{D}$. Note that reproducing kernels are positive definite [5]:

$$\sum_{i,j=1}^{N} c_i c_j K(\boldsymbol{x}_i, \boldsymbol{x}_j) \geq 0, \tag{2}$$

for any N, $c_1, \ldots, c_N \in \mathbf{R}$, and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N \in \mathcal{D}$. In addition, $K(\boldsymbol{x}, \tilde{\boldsymbol{x}}) = K(\tilde{\boldsymbol{x}}, \boldsymbol{x})$ for any $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in \mathcal{D}$ is followed [5]. If a reproducing kernel $K(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ exists, it is unique [5]. Conversely, every positive definite function $K(\boldsymbol{x}, \tilde{\boldsymbol{x}})$ has the unique corresponding RKHS [5].

Next, we introduce the Schatten product [7] that is a convenient tool to reveal the reproducing property of kernels.

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Definition 2 *[7] Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. The Schatten product of $g \in \mathcal{H}_2$ and $h \in \mathcal{H}_1$ is defined by*

$$(g \otimes h)f = \langle f, h\rangle_{\mathcal{H}_1} g, \quad f \in \mathcal{H}_1. \tag{3}$$

Note that $(g \otimes h)$ is a linear operator from $\mathcal{H}_1$ onto $\mathcal{H}_2$. It is easy to show that

$$(h \otimes g)^* = (g \otimes h), \tag{4}$$

holds, where the superscript * denotes the adjoint operator.

Finally, we introduce a kernel for a class of bandlimited (low-pass) signals along with the description in [8] as a preliminary for considering a class of bandpass signals. Let $\mathcal{B}_W$ be a class of functions whose Fourier transforms are supported by $[-W, W]$, with $W \geq 0$, forming a closed subspace in $L^2(-\infty, \infty)$. Note that $\mathcal{B}_W$ is trivially a Hilbert space. Let us consider the function

$$K_W(x, y) = 2W \operatorname{sinc}(2W(x-y)), \ \ x, y \in \mathbf{R}, \tag{5}$$

where

$$\operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x}. \tag{6}$$

Let $\chi_W(\xi)$ be the indicator function supported by $[-W, W]$,

$$(\mathcal{F}K_W(\cdot, y))(\xi) = \chi_W(\xi)\exp(-i2\pi\xi y) \tag{7}$$

holds for any $y \in \mathbf{R}$, where $\mathcal{F}$ denotes the Fourier transform operator, which implies $K_W(\cdot, y) \in \mathcal{B}_W$ for any $y \in \mathbf{R}$. Moreover,

$$\begin{aligned}
&\langle f(\cdot), K_W(\cdot, y)\rangle_{\mathcal{B}_W} \\
&= \int_{-\infty}^{\infty} f(x) K_W(x, y) dx \\
&= \int_{-\infty}^{\infty} (\mathcal{F}f(\cdot))(\xi)\overline{(\mathcal{F}K_W(\cdot, y))(\xi)} d\xi \\
&= \int_{-\infty}^{\infty} (\mathcal{F}f(\cdot))(\xi)\chi(\xi)\exp(i2\pi\xi y) d\xi \\
&= \int_{-\infty}^{\infty} (\mathcal{F}f(\cdot))(\xi)\exp(i2\pi\xi y) d\xi = f(y)
\end{aligned}$$

holds for any $y \in \mathbf{R}$ and any $f(\cdot) \in \mathcal{B}_W$. Therefore, $K_W(x, y)$ satisfies the two conditions in Definition 1 and it is concluded that $K_W(x, y)$ is the reproducing kernel of $\mathcal{B}_W$ [8].

3. REPRODUCING KERNEL FOR BANDPASS SIGNALS

In this section, we show that a class of functions with some specific bandpass property is an RKHS and also give a closed-form of the corresponding kernel.

Let $\mathcal{B}_{(L,U)}$ be a class of functions whose Fourier transforms are supported by $[-U, -L] \cup [L, U]$, with $0 \leq L \leq U$, forming a closed subspace in $L^2(-\infty, \infty)$. Note that $\mathcal{B}_{(L,U)}$ is also a Hilbert space. Let us consider the function

$$K_{(L,U)}(x, y) = K_U(x, y) - K_L(x, y), \ \ x, y \in \mathbf{R}, \tag{8}$$

then we have the following theorem.

Theorem 1 *$K_{(L,U)}(x, y)$ is the reproducing kernel of $\mathcal{B}_{(L,U)}$.*

Proof Since

$$\begin{aligned}
&(\mathcal{F}K_{(L,U)}(\cdot, y))(\xi) \\
&= (\mathcal{F}K_U(\cdot, y))(\xi) - (\mathcal{F}K_L(\cdot, y))(\xi) \\
&= \chi_U(\xi)\exp(-i2\pi\xi y) - \chi_L(\xi)\exp(-i2\pi\xi y) \\
&= (\chi_U(\xi) - \chi_L(\xi))\exp(-i2\pi\xi y)
\end{aligned}$$

holds and $(\chi_U(\xi) - \chi_L(\xi))$ is trivially the indicator function supported by $[-U, -L] \cup [L, U]$,

$$K_{(L,U)}(\cdot, y) \in \mathcal{B}_{(L,U)} \tag{9}$$

holds for any $y \in \mathbf{R}$.

Let $f(\cdot)$ be an arbitrary function in $\mathcal{B}_{(L,U)}$, then

$$\begin{aligned}
&\langle f(\cdot), K_{(L,U)}(\cdot, y)\rangle_{\mathcal{B}_{(L,U)}} \\
&= \int_{-\infty}^{\infty} f(x) K_{(L,U)}(x, y) dx \\
&= \int_{-\infty}^{\infty} (\mathcal{F}f(\cdot))(\xi)\overline{(\mathcal{F}K_{(L,U)}(\cdot, y))(\xi)} d\xi \\
&= \int_{-\infty}^{\infty} (\mathcal{F}f(\cdot))(\xi)(\chi_U(\xi) - \chi_L(\xi))\exp(i2\pi\xi y) d\xi \\
&= \int_{-\infty}^{\infty} (\mathcal{F}f(\cdot))(\xi)\exp(i2\pi\xi y) d\xi = f(y)
\end{aligned}$$

holds for any $y \in \mathbf{R}$. Thus, $K_{(L,U)}(x, y)$ satisfies the conditions in Definition 1, which concludes the proof. □

Note that L and U have no limitation except $0 \leq L \leq U$ in this theorem.

4. KERNEL-INDUCED SAMPLING THEOREM FOR BANDPASS SIGNALS

In this section, we discuss the optimal approximation of a function in $\mathcal{B}_{(L,U)}$ by a given set of sampling points; and also give an another proof for perfect reconstruction with bandpass Nyquist rate on the basis of the kernel-induced sampling theorem [4].

Firstly, we review some important results shown in [4]. Let $\mathcal{I} = \{1, \ldots, \ell\}$ be an index set, which may be an infinite set; and let $X = \{\boldsymbol{x}_i \mid \boldsymbol{x}_i \in \mathcal{D}, \ i \in \mathcal{I}\}$ be a given set of sampling points. Let $K(\boldsymbol{x}, \boldsymbol{y})$ be an adopted kernel. From the reproducing property Eq.(1),

$$f(\boldsymbol{x}_i) = \langle f(\cdot), K(\cdot, \boldsymbol{x}_i)\rangle_{\mathcal{H}_K} \tag{10}$$

holds for any $f(\cdot) \in \mathcal{H}_K$, where $\mathcal{H}_K$ denotes the corresponding RKHS of K. By vectorizing Eq.(10) for all $i \in \mathcal{I}$ and applying the Schatten product, we have

$$\boldsymbol{f} = \left(\sum_{i \in \mathcal{I}} [\boldsymbol{e}_i \otimes K(\cdot, \boldsymbol{x}_i)]\right) f(\cdot), \tag{11}$$

where $\boldsymbol{f}$ denotes the ℓ-dimensional vector whose i-th element is $f(\boldsymbol{x}_i)$ and $\boldsymbol{e}_i$ denotes the ℓ-dimensional unit vector whose i-th element is unity. For a convenience of description, we write

$$A_{K,X} = \sum_{i \in \mathcal{I}} [\boldsymbol{e}_i \otimes K(\cdot, \boldsymbol{x}_i)]. \tag{12}$$

Note that $A_{K,X}$ is a linear operator from $\mathcal{H}_K$ onto $\mathbf{R}^\ell$. Thus, sampling process of $f(\cdot)$ by X is written as

$$\boldsymbol{f} = A_{K,X} f(\cdot) \tag{13}$$

and reconstruction of $f(\cdot)$ from $\boldsymbol{f}$ and X can be regarded as a linear inversion problem of Eq.(13). The solution subspace of the linear inversion problem Eq.(13) is $\mathcal{R}(A^*_{K,X})$, that is, the range space of $A^*_{K,X}$. Note that any function $f(\cdot) \in \mathcal{R}(A^*_{K,X})$ can be represented by

$$f(\cdot) = \sum_{i \in \mathcal{I}} \alpha_i K(\cdot, \boldsymbol{x}_i) \tag{14}$$

with some coefficient vector $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_\ell]' \in \mathbf{R}^\ell$, where the superscript $'$ denotes the transposition operator, since

$$A^*_{K,X}\boldsymbol{\alpha} = \sum_{i\in\mathcal{I}}[K(\cdot,\boldsymbol{x}_i)\otimes \boldsymbol{e}_i]\boldsymbol{\alpha} = \sum_{i\in\mathcal{I}}\alpha_i K(\cdot,\boldsymbol{x}_i)$$

holds. Accordingly, the optimal solution of the linear inversion problem is given by the orthogonal projection of $f(\cdot)$ onto $\mathcal{R}(A^*_{K,X})$ and its closed-form is given as

$$\hat{f}(\cdot) = \sum_{i,j\in\mathcal{I}} f(\boldsymbol{x}_i)(G^+_{K,X})_{ij}K(\cdot,\boldsymbol{x}_j) \tag{15}$$

as shown in [4], where $G_{K,X}$ denotes the Gramian matrix of K with X defined as $G_{K,X} = (g_{ij}),\ g_{ij} = K(\boldsymbol{x}_i,\boldsymbol{x}_j)$; and the superscript $^+$ denotes the Moore-Penrose generalized inverse [9].

When $\mathcal{R}(A^*_{K,X})$ is identical to $\mathcal{H}_K$, it means that any function in $\mathcal{H}_K$ can be perfectly reconstructed by the set of sampling points X, that is, the sampling theorem for $\mathcal{H}_K$ with X holds. The next theorem clarifies a necessary and sufficient condition for $\mathcal{R}(A^*_{K,X}) = \mathcal{H}_K$.

Theorem 2 *[4] $\mathcal{H}_K = \mathcal{R}(A^*_{K,X})$ if and only if*

$$K(\boldsymbol{y},\boldsymbol{y}) = \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{I}} K(\boldsymbol{y},\boldsymbol{x}_j)(G^+_{K,X})_{i,j}K(\boldsymbol{y},\boldsymbol{x}_i) \tag{16}$$

holds for any $\boldsymbol{y}\in\mathcal{D}$.

On the basis of the above preliminaries, we discuss sampling theorem and the optimal approximation for bandpass signals. As mentioned in Section 1, in order to achieve perfect reconstruction by the bandpass Nyquist rate, $m = U/(U-L)$ must be a positive integer number, which is identical to

$$L = \frac{m-1}{m}U. \tag{17}$$

So we assume Eq.(17) with some positive integer number m. Note that the bandwidth of functions in $\mathcal{B}_{(L,U)}$ is U/m and the bandpass Nyquist interval for this case is $m/(2U)$.

Firstly, we give the following theorem.

Theorem 3 *Let $X = \{mk/(2U) \mid k\in\mathbf{Z}\}$, where $\mathbf{Z}$ denotes the set of integer numbers. For any $f(\cdot)\in\mathcal{B}_{(L,U)}$,*

$$f(x) = \sum_{k\in\mathbf{Z}} f\left(\frac{m}{2U}k\right)K_{(L,U)}\left(x,\frac{m}{2U}k\right) \tag{18}$$

holds.

Proof Firstly, we calculate the Gramian matrix $G_{K_{(L,U)},X}$. For any $k\in\mathbf{Z}$,

$$\begin{aligned}
&K_{(L,U)}\left(\frac{m}{2U}k,\frac{m}{2U}k\right)\\
&= K_U\left(\frac{m}{2U}k,\frac{m}{2U}k\right) - K_L\left(\frac{m}{2U}k,\frac{m}{2U}k\right)\\
&= 2U\mathrm{sinc}(0) - 2L\mathrm{sinc}(0) = 2(U-L) = \frac{2U}{m}
\end{aligned}$$

holds and for any $k,j\in\mathbf{Z}$ with $k\neq j$ and $d = k-j$,

$$\begin{aligned}
&K_{(L,U)}\left(\frac{m}{2U}k,\frac{m}{2U}j\right)\\
&= K_U\left(\frac{m}{2U}k,\frac{m}{2U}j\right) - K_L\left(\frac{m}{2U}k,\frac{m}{2U}j\right)\\
&= 2U\frac{\sin(\pi md)}{\pi md} - 2L\frac{\sin(\pi(m-1)d)}{\pi(m-1)d}\\
&= \frac{2U}{\pi md}(\sin(\pi md) - \sin(\pi(m-1)d))\\
&= \frac{2U}{\pi md}(\sin(\pi md) - \sin(\pi md)\cos(\pi d)) = 0
\end{aligned}$$

holds, which implies that $G_{K_{(L,U)},X} = (2U/m)I$, where I denotes the infinite dimensional identity matrix.

Next, we confirm that Eq.(16) holds for $K_{(L,U)}$ and X. The left-hand side of Eq.(16) is reduced to

$$K_{(L,U)}(y,y) = \frac{2U}{m} \tag{19}$$

for any $y\in\mathbf{R}$ as seen in the calculation of the diagonal elements of the Gramian matrix. The right-hand side of Eq.(16) is reduced to

$$\frac{m}{2U}\sum_{k\in\mathbf{Z}}\left(K_{(L,U)}\left(y,\frac{m}{2U}k\right)\right)^2 \tag{20}$$

since $G^{-1}_{K_{(L,U)},X} = (m/(2U))I$. Therefore, we have to show that

$$\sum_{k\in\mathbf{Z}}\left(K_{(L,U)}\left(y,\frac{m}{2U}k\right)\right)^2 = \frac{4U^2}{m^2} \tag{21}$$

holds. When $y\in X$, Eq.(21) trivially holds. When $y\notin X$, we have

$$\begin{aligned}
&\sum_{k\in\mathbf{Z}}\left(K_{(L,U)}\left(y,\frac{mk}{2U}\right)\right)^2\\
&= \sum_{k\in\mathbf{Z}}\left(2U\mathrm{sinc}\left(2U\left(y-\frac{mk}{2U}\right)\right)\right.\\
&\qquad \left.-2L\mathrm{sinc}\left(2L\left(y-\frac{mk}{2U}\right)\right)\right)^2\\
&= \sum_{k\in\mathbf{Z}}\left(\frac{\sin(2\pi Uy-\pi mk)}{\pi\left(y-\frac{mk}{2U}\right)}\right.\\
&\qquad \left.-\frac{\sin\left(2\pi Uy\frac{m-1}{m}-\pi(m-1)k\right)}{\pi\left(y-\frac{mk}{2U}\right)}\right)^2\\
&= \sum_{k}\frac{1}{\pi^2\left(y-\frac{mk}{2U}\right)^2}\left(\sin^2(2\pi Uy)\right.\\
&\qquad +\sin^2\left(2\pi Uy\frac{m-1}{m}\right)\\
&\qquad \left.-2(-1)^k\sin(2\pi Uy)\sin\left(2\pi Uy\frac{m-1}{m}\right)\right).
\end{aligned}$$

Gathering the terms for $k = 2j$ yields

$$\begin{aligned} T_e &= \sum_{j\in\mathbf{Z}} \frac{\left(\sin(2\pi U y) - \sin\left(2\pi U y \frac{m-1}{m}\right)\right)^2}{\pi^2\left(y - \frac{2mj}{2U}\right)^2} \\ &= \frac{U^2\left(2\cos\left(\pi U y \frac{2m-1}{m}\right)\sin\left(\pi U y \frac{1}{m}\right)\right)^2}{m^2 \sin^2\left(\pi U y \frac{1}{m}\right)} \\ &= \frac{4U^2}{m^2}\cos^2\left(\pi U y \frac{2m-1}{m}\right) \end{aligned}$$

and gathering the terms for $k = 2j + 1$ yields

$$\begin{aligned} T_o &= \sum_{j\in\mathbf{Z}} \frac{\left(\sin(2\pi U y) + \sin\left(2\pi U y \frac{m-1}{m}\right)\right)^2}{\pi^2\left(y - \frac{2mj+m}{2U}\right)^2} \\ &= \frac{U^2\left(2\sin\left(\pi U y \frac{2m-1}{m}\right)\cos\left(\pi U y \frac{1}{m}\right)\right)^2}{m^2 \cos^2\left(\pi U y \frac{1}{m}\right)} \\ &= \frac{4U^2}{m^2}\sin^2\left(\pi U y \frac{2m-1}{m}\right). \end{aligned}$$

Note that we used the identical equation

$$\sum_{k\in\mathbf{Z}} \frac{1}{\pi^2(k+a)^2} = \frac{1}{\sin^2 \pi a} \tag{22}$$

shown in [10] for obtaining T_e and T_o. Thus, we have

$$\sum_{k\in\mathbf{Z}} \left(K_{(L,U)}\left(y, \frac{mk}{2U}\right)\right)^2 = T_e + T_o = \frac{4U^2}{m^2},$$

and it is concluded that Eq.(21) holds for any $y \in \mathbf{R}$. Accordingly, on the basis of Theorem 2, $\mathcal{B}_{(L,U)} = \mathcal{R}(A^*_{K_{(L,U)},X})$ holds, which implies that the orthogonal projection of any $f(\cdot) \in \mathcal{B}_{(L,U)}$ onto $\mathcal{R}(A^*_{K_{(L,U)},X})$, written as

$$\hat{f}(x) = \sum_{k\in\mathbf{Z}} f\left(\frac{m}{2U}k\right) K_{(L,U)}\left(x, \frac{m}{2U}k\right), \tag{23}$$

is identical to $f(x)$ itself, which concludes the proof. □

The newly obtained perfect reconstruction formula Eq.(18) for $\mathcal{B}_{(L,U)}$ is much simpler than those obtained in existing literatures such as [2].

When some of sampling points are missing in X or only finite subset of X are available, perfect reconstruction can not be achieved. However, omitting the terms corresponding to these missing sampling points from Eq.(18) still yields the orthogonal projection of a target function onto the linear subspace $\mathcal{R}(A^*_{K_{(L,U)},\tilde{X}})$, where $\tilde{X} \subset X$, which implies that Eq.(18) with $\tilde{X}$ is the optimal approximation of the target function.

Since $G_{K_U,X} = 2UI$ and $G_{K_L,X} = 2LI$,

$$f_U(x) = \sum_{k\in\mathbf{Z}} f\left(\frac{m}{2U}k\right) K_U\left(x, \frac{m}{2U}k\right) \tag{24}$$

and

$$f_L(x) = \sum_{k\in\mathbf{Z}} f\left(\frac{m}{2U}k\right) K_L\left(x, \frac{m}{2U}k\right) \tag{25}$$

are also the orthogonal projections of $f(\cdot)$ onto $\mathcal{R}(A^*_{K_U,X})$ and $\mathcal{R}(A^*_{K_L,X})$, respectively; and it is trivial that $K_L(\cdot, y)$ is orthogonal to $K_{(L,U)}(\cdot, z)$ for any $y, z \in \mathbf{R}$. Therefore,

$$f_U(x) = f_L(x) + f(x) \tag{26}$$

gives an orthogonal decomposition for $f_U(x)$. Note that $f_U(x)$ can be regarded as a reconstruction formula for $[-U, U]$-bandlimited (lowpass) signal with aliasing effect caused by an incomplete set of sampling points. Accordingly, it is concluded that $f_L(x)$ is the aliasing term in $f_U(x)$ that is orthogonal to the true function $f(\cdot) \in \mathcal{B}_{(L,U)}$, which is one of newly obtained aspects by our analyses.

5. CONCLUSION

In this paper, we discussed sampling theories for bandpass signals. We reformulated a class of bandpass signals as a reproducing kernel Hilbert space and gave a closed-form of the corresponding kernel functions, that is, the difference of ordinary sinc kernels. Moreover, on the basis of the theory of kernel-induced sampling theorem, we gave an another proof for sampling theorem of bandpass signals with bandpass Nyquist rate under a specific band structure; and also gave the optimal approximation formula with insufficient set of sampling points.

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