ROBUSTNESS OF ALIASING-TOLERANT SUB-NYQUIST SAMPLING WITH APPLICATION TO PARTICLE LOCALISATION

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ABSTRACT

This paper proposes a novel system design for signal acquisition in Particle Tracking Velocimetry (PTV) applications. For exploiting the Finite Rate of Innovation (FRI) of these signals an appropriate sampling kernel is designed and the reconstruction process is illustrated. The main contribution is to show that aliasing effects can be neglected under certain conditions and an exact reconstruction is possible in most cases. While the sampling rate is still at least twice the FRI, it can be a fraction of the sampling kernel bandwidth. Hence, the sampling rate can be adapted to the physical process under investigation without changing the kernel designed for the worst case scenario. Considering practical aspects of the reconstruction process it is shown that even in the presence of noise precise results can be obtained.

Index Terms— Particle Tracking Velocimetry (PTV), Finite Rate of Innovation (FRI), sampling, aliasing, ESPRIT

1. INTRODUCTION

The last decade raised up the novel theory of Compressed Sensing (CS) dealing with sparse signals. While CS handles signals which are sparse itself, a different theory dealing with signals which are sparse in their parametric definition has been created. Vetterli *et al.*defined signals having a Finite Rate of Innovation (FRI) per time interval whereby they can be completely described by a finite set of parameters, e.g. a stream of diracs is parametrised by the amplitudes and time delays for each impulse [1–5].

Such signals are not necessarily limited to a certain bandwidth but can be sampled at their rate of innovation without loss of information if an appropriate sampling kernel can be found. Early publications dealt with ideal or Gaussian lowpass kernels which suffered from numerical instabilities occurring during the reconstruction process or which could not deal with aperiodic or finite length signals. Recent results have shown that a filter kernel based on shifted sinc-functions in the spectral domain can handle both problems appropriately [6, 7]. Eldar *et al.*denoted this kernel as a Sum of Sinc (SoS) filter whereby the number of shifted and superposed sinc-functions has to be greater or at least equal to the number of signal parameters.

The reconstruction process can usually be solved by standard spectral estimation methods such as ESPRIT [8, 9] or the annihilating filter method [1–5]. In the absence of noise these techniques are exact up to numerical precision.

This paper considers the case of FRI signals and the reconstruction via standard spectral estimation methods. In detail, it is shown how an efficient 2-stage sampling scheme can be implemented which is motivated by the Particle Tracking Velocimetry (PTV) application [10, 11]. The main focus of this paper is to analyse and illustrate under which conditions aliasing effects have no negative influence. In contrast to [6, 7], this paper deals with sub-Nyquist sampling where the Nyquist criterion is violated not only with respect to the FRI signal's bandwidth but also with respect to the bandwidth of the SoS sampling kernel. By numerical simulations it is evaluated that the reconstruction process is successful with a probability close to one.

2. SYSTEM DESCRIPTION

2.1. Motivation

The goal of PTV is to estimate a motion vector field for tracer particles within flowing fluids such that turbulences can be identified. Therefore, a laser illuminates a plane within the volume and all particles located in that plane reflect light to a camera sensor. Considering consecutive images, the motion vector field can be estimated, e.g. by simple correlation. Fig. 1 illustrates the very basic principle of data acquisition for a PTV system. Since the flow velocity can be high and turbulences are of particular interest, there is a need to capture these images at high rates. Actual CMOS sensors acquire images with $1\mathrm{MPixel}$ at a rate of $7.5\mathrm{kHz}$ which leads to $7.5\mathrm{GSample/s}$. Thus, offline processing is required to handle that amount of data but the size of the memory where the data can be stored limits the observation time practically to only a few seconds.

However, the images obtained for PTV are sparse in that there are only few particles within the illuminated plane. Furthermore, these particles can be parametrised by a few coefficients, i.e. their position and their size which influences the reflected amount of light. The main idea of this paper is to

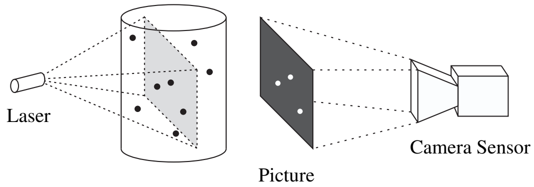

Fig. 1. Model for PTV system

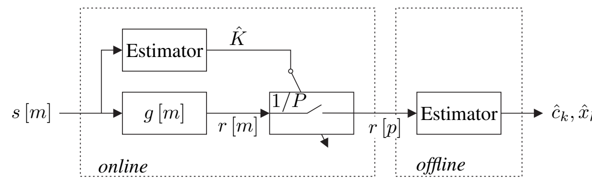

Fig. 2. Model for sampling FRI signals with adaptive rate

compress the acquired image by exploiting the FRI nature of these signals and, therefore, reduce the amount of data to be stored onto memory which increases the possible observation time. The reconstruction can be done with spectral estimation methods within an offline processing. For simplicity, the rest of this paper is restricted to the one-dimensional case which can be easily extended to the two-dimensional case for the PTV application.

2.2. FRI Signals & Sampling Scheme

A signal with a finite rate of innovation, i.e. the set of spatial weights and positions $\{c_k, x_k\}_{k=0}^{K-1}$ of K particles with impulse form $h(x)$

$$s(x) = \sum_{k=0}^{K-1} c_k \cdot h(x - x_k) \tag{1}$$

is restricted to a finite space τ_{x} and acquired by the camera sensor. Usually, a pixel of a camera sensor accumulates the light impinging onto its surface. Assuming the pixel size is small, the FRI signal after spatial sampling with rate M can be approximated as

$$s[m] \approx s\left(x = m\frac{\tau_{\mathrm{x}}}{M}\right) \tag{2}$$

which is the light impinging on a certain point with $m \in \mathbb{M} = \{0, ..., M-1\}$ and $M = |\mathbb{M}|$ as the number of pixels of the sensor in the considered dimension. According to (1) the FRI signal has a bandwidth depending on impulse shape $h(x)$ and requires, regarding to Nyquist, a sampling rate twice as large as that bandwidth. Practically, the considered FRI signal (in PTV applications) is observed with a rate given by the resolution of usual CCD or CMOS sensors.
However, with respect to the rate of innovation the sampling rate can be further decreased. Therefore, the authors use a second sampling stage to compress the signal. In the discrete spatial domain an appropriate sampling kernel $g[m]$ is applied leading via cyclic convolution to the lowpass signal $r[m]$.

$$r[m] = s[m] \circledast g[m] \tag{3}$$

This paper uses the Sum of Sinc (SoS) kernel [6, 7]. Conventionally, the kernel's design depends on the rate of innovation. However, the number of particles inside the plane are not known a priori. Moreover, it is assumed that the filter kernel can hardly be adapted in real time. Hence, a worst case implementation of the sampling kernel has to be chosen. Adapting now the sampling rate to the kernel's bandwidth [6, 7] would require a sampling rate much higher than the rate of innovation in many pictures. This paper proposes a sampling scheme with a fixed SoS sampling kernel while using a sampling rate P below the corresponding Nyquist rate. If the number of impulses (or particles) can be estimated appropriately, the sampling rate can be adapted to the rate of innovation without changing the kernel. Fig. 2 illustrates the author's proposal for an adaptive sampling scheme.

2.3. Sampling Kernel & Spectral Description

Consider the discrete signal in (2), its spectral description is given by

$$S[m] = H[m] \cdot \frac{1}{\tau_{\mathrm{x}}} \sum_{k=0}^{K-1} c_k \cdot e^{-j2\pi m x_k/\tau_{\mathrm{x}}} \tag{4}$$

where $m \in \mathbb{M}$ defines the spectral position and $H[m]$ are the spectral coefficients for the impulse shape $h(t)$. At the spectral estimator at least $2K$ coefficients $S[m]/H[m]$ are considered to retrieve the signal parameters. Therefore, a sampling kernel is applied retaining these coefficients. In the continuous spectral domain this is the SoS kernel

$$G(f) = \frac{\tau_{\mathrm{x}}}{\sqrt{2\pi}} \sum_{l \in \mathbb{L}} \operatorname{sinc}(f \cdot \tau_{\mathrm{x}} - l) \tag{5}$$

with sinc-functions centred at multiples of $2\pi/\tau_{\mathrm{x}}$. Hence, after filtering different spectral coefficients will not interfere with each other. To ensure a real valued filter kernel the spectrum has to be symmetric and therefore, the coefficients $l \in \mathbb{L}$ have to be chosen from a set $\mathbb{L} = \left\{-\frac{L-1}{2}, ..., \frac{L-1}{2}\right\}$ where $L = |\mathbb{L}|$ has to be an odd integer.
Performing a cyclic convolution in spatial domain (3) leads to a discrete spectrum for the kernel

$$G[m] = \begin{cases} \text{nonzero} & m \in \mathbb{L} \\ 0 & m \in \mathbb{Z} \backslash \mathbb{L} \end{cases} \tag{6}$$

and a multiplication between signal and filter coefficients

$$R[m] = G[m] \cdot S[m] \quad . \tag{7}$$

2.4. Conventional Subsampling & Reconstruction

Recall the fact that the amount of data has to be decreased, the lowpass signal $r\left[m\right]$ has to be re-sampled in the spatial domain to obtain $r\left[p\right]$ where $p \in \mathbb{P} \subseteq \mathbb{M}$. Only if $P = |\mathbb{P}| \ll M$ holds, the amount of data to be stored for the offline processing can be decreased significantly.

Now, two different cases are distinguished. Critical sampling at the kernel's Nyquist rate is achieved with $P = L$. In this case the spectral coefficients $R\left[l\right]$ can be obtained from the spatial samples $r\left[p\right]$ by Discrete Fourier Transform (DFT).

If $r\left[m\right]$ is re-sampled at a rate above the Nyquist rate, $P > L$, an over-determined equation system has to be solved to obtain the spectral coefficients $R\left[l\right]$. This requires a larger complexity than a DFT but can increase the performance in the presence of noise. Details on the equation system can be found in [6, 7, 11].

Once the coefficients $R\left[l\right]$ are obtained, a linear equation system can be solved to obtain the coefficients $S\left[l\right]/H\left[l\right]$ which in turn are considered by a spectral estimator, e.g. ESPRIT [8, 9], to estimate the signal parameters $\{\hat{c}_k, \hat{x}_k\}_{k=0}^{K-1}$.

Due to this reconstruction procedure two different bounds have been defined, recently.

$$P \geq L \geq 2K \tag{8}$$

First, $L \geq 2K$ ensures that the number of spectral coefficients is sufficient to estimate $2K$ parameters. Second, $P \geq L$ has been defined in order to avoid aliasing. The case $P < L$ is investigated in Sec. 3.

3. SUB-NYQUIST SAMPLING

With respect to (8), the highest compression for the FRI signal is only achieved with sampling at the critical rate $P = 2K$. Conventionally, the sampling kernel is adapted to that rate of innovation as well ($P = L$). Instead, this paper proposes sub-Nyquist sampling with $L \geq P \geq 2K$. Thus, the number of samples P can be adapted to the rate of innovation $2K$ while the kernel size L is fixed.

Re-sampling of $r\left[p\right]$ equidistantly with P samples at positions $x_p = p\frac{\tau_x}{P}$ generates periodic repetitions at multiples of the sampling frequency. Due to $P < L$, aliasing occurs and these repetitions interfere with the original spectrum.

$$R\left[l\right] = \sum_{\alpha \in \mathbb{Z}} G\left[l - \alpha P\right] \cdot S\left[l - \alpha P\right] \tag{9}$$

Since the SoS kernel is zero at discrete frequencies beyond its bandwidth (6), only a finite set of repetitions has to be considered for the aliasing effect while all others can be neglected. Considering the case where L is an integer multiple of P, $L = \nu \cdot P$, $\nu \in \mathbb{N}$, exactly $\nu - 1$ shifted copies at negative and positive multiples of the sampling frequency as well as

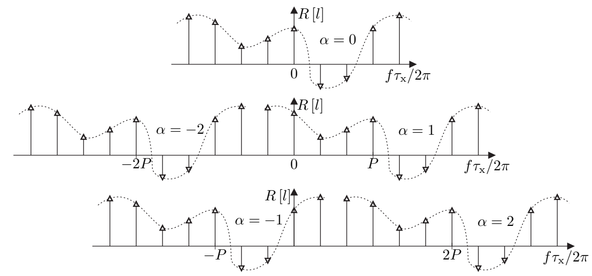

Fig. 3. Aliasing of the original spectrum with $L = 9$ nonzero coefficients due to subsampling with $P = 3$ ($\nu = 3$). Periodic repetitions ($\alpha \neq 0$) appear at multiples of the normalised sampling frequency P

the original spectrum (without spectral shift) remain.

$$R\left[l\right] = \sum_{\alpha=-\nu+1}^{\nu-1} G\left[l - \alpha P\right] \cdot S\left[l - \alpha P\right] \tag{10}$$

Fig. 3 illustrates such a case with $L = 9$, $P = 3$ and $\nu = 3$. As can be seen, the left and the right hand side of two linearly shifted versions touch each other ($\alpha = \{+1, -2\}$ and $\alpha = \{-1, +2\}$) and each pair forms one cyclically shifted spectrum within the range of $l \in \mathbb{L}$. Since a circular shift by αP in the spectral domain leads to a factor $\exp\left(-j2\pi\alpha Pp/P\right) = \exp\left(-j2\pi\alpha p\right)$ in the original domain, the spatially sampled signal can be reconstructed via Inverse DFT from its aliased spectrum and has the form

$$\hat{r}\left[p\right] = \left(r\left[p\right] \circledast g\left[p\right]\right) \cdot \sum_{\alpha=0}^{\nu-1} e^{-j2\pi\cdot\alpha p} \quad . \tag{11}$$

With α and p being integers, (11) becomes

$$\hat{r}\left[p\right] = \nu \cdot r\left[p\right] \quad . \tag{12}$$

Except a constant scaling factor, the reconstructed spatial signal is represented by the same samples as in the absence of any aliasing effects. If this is the case, the aliased spectrum can also be considered for spectral estimation techniques to reconstruct the signal parameters. Hence,

$$L = \nu P \geq P \geq 2K \tag{13}$$

can be defined for sub-Nyquist sampling. On the one hand, (13) ensures that there are a sufficient number of spectral coefficients for the reconstruction of $2K$ parameters. On the other hand, the relation $L = \nu P$ ensures that each two linearly shifted spectra complement each other to one circular shift and, thus, aliasing effects can be neglected.

4. SIMULATIONS

Since (12) validates that a reconstruction is possible, it does not ensure that there is a unique solution. However, numerical simulations showed that with a probability close to one the correct solution can be found. Beside that, the reconstruction fails if there is a signal with different parameters requiring less energy. This is caused by the least squares criterion exploited by the reconstruction algorithm, e.g. ESPRIT.

4.1. Estimation Reliability

To avoid reconstruction failures the authors propose a 3-stage improvement process.
First, ESPRIT considers a Hankel matrix containing the coefficients $R[p]$. If the order of K is not known at the spectral estimator, the rank of that matrix can be considered to estimate K. The Hankel matrix can be modified in order to decrease the degrees of freedom for this algorithm. Therefore, the probability for reconstruction failures is decreased as well. However, the estimated rank must not be smaller than K, otherwise the reconstruction process certainly fails.
Second, the roots of the eigenvalues of the estimated subspace can be considered. Since the weights c_k are real-valued, all roots have to lie on the unit circle. Otherwise, it is a priori known that the obtained estimated value leads to a failure.
Finally, the reconstructed weights can be considered. The authors observed that for a false reconstruction the weights will be very small, usually close to numerical precision of the considered machine. This is caused by the least squares criterion which minimises the weights c_k with respect to the observed samples $r[p]$. If very small weights are not of interest, as for PTV, or if the occurrence probability for small weights is negligible, these reconstructed pulses can be rejected while others remain correct.

4.2. Results

Simulations have been run with 10^4 iterations, equally distributed positions $x_k \in (0, \tau_\mathrm{x}]$ and Rayleigh[1] distributed weights with variance one. Furthermore, the samples $\tilde{s}[m] = s[m] + n[m]$ have been disturbed by additive white Gaussian noise n with zero mean and variance σ_N^2. Fig. 4 illustrates the MSE, which is the normalised distance between the original and the reconstructed impulses for $K = \{5, 10\}$. Furthermore, different sampling schemes without a) and with b), c) aliasing have been applied.
The figure illustrates that there is only a very slight performance degradation between those sampling schemes with and without aliasing. This means, that an increasing kernel bandwidth does not lead to a noise amplification and, therefore, no worth mentioning performance degradation occurs. In practice, this slight difference might be negligible. Furthermore, it is shown that noise can be combated by oversampling. Again, the analysed aliasing effects lead only to a very slight loss in terms of the MSE (curves without aliasing are not shown for oversampling).

[1]It is assumed that $c_k \geq 0$ holds for the PTV problem, since there is no negative light intensity. Furthermore, only significant particles are of interest and, therefore, Rayleigh distribution has been chosen.

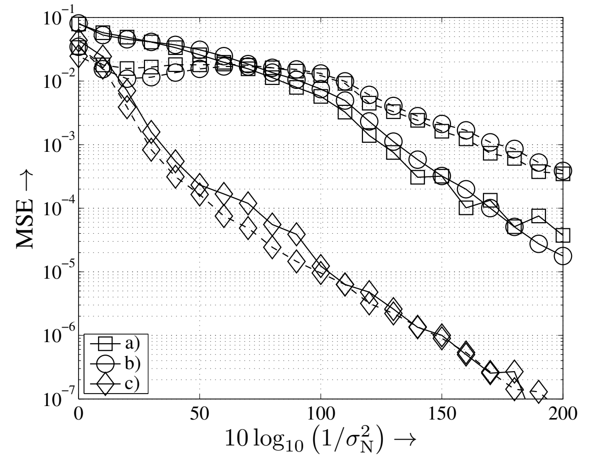

Fig. 4. MSE for the reconstructed particle position for $K = 5$ (solid lines) and $K = 10$ (dashed lines) with a) minimum sampling rate $P = 2K + 1$ without aliasing, b) sampling at the minimum rate $P = 2K + 1$ with aliasing $\nu = 3$ and c) oversampling by factor three ($P = 6K + 1$) with aliasing effects $\nu = 3$

Further analysis considering the error between the real and the estimated number of particle, K and $\hat{K}$ respectively, show that the error $\left|K - \hat{K}\right|$ for the model order decreases significantly with decreasing noise variance and is even negligible for very small noise values.
Moreover, it has been observed, that an appropriate rank estimation can improve the performance significantly. The MSE as well as the error for the model order decreases significantly compared to simulation without rank estimation.

5. CONCLUSION

This paper proposed a new design for sampling FRI signals with the SoS kernel. Furthermore, it was shown that aliasing effects can be neglected under certain conditions and exact reconstruction results can be obtained. Simulations showed that noise has no influence on the analysed aliasing effects and can be treated by oversampling.

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