NEW MODEL FOR OPTIMIZED PERIODOGRAM-BASED SPECTRUM SENSING FOR COGNITIVE RADIOS

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ABSTRACT

This paper addresses the problem of spectrum sensing from a frequency domain approach using periodogram-based energy detection which naturally yields a lower probability of false alarm when compared with the time-domain detector. In this paper, the periodogram-based detector is optimized based on the likelihood ratio test and accurate expressions are derived for the probability of false alarm and the probability of detection. In addition, the case of sensing with multiple antennas is considered assuming that equal gain combining (EGC) is used, which yields a hypoexponential decision statistic. The high accuracy of the obtained models is verified by the obtained simulation results. The performance is enhanced by employing the optimized models. The receiver operator characteristics clearly show that better operation points are obtained as the SNR is increased.

***Index Terms*—** Periodogram, Spectrum Sensing, Hermitian quadratic forms, Equal gain combining, Fading.

1. INTRODUCTION

The cognitive radio (CR) technology offers more flexibility as it enables dynamic spectrum sharing (DSS) between license-exempt devices (secondary users) and license-holders (primary users) [1, 2]. A cognitive radio changes its communication parameters adaptively to locate and fit inside detected spectrum opportunities, and therefore spectrum sensing is a major function within a CR system. However, the problem of spectrum sensing is still an open issue for the application of cognitive radios.

In this paper we focus on energy detection from a frequency domain (FD) perspective. In previous work, the authors addressed the periodogram-based detector in [3], and the performance of Bartlett's Method in [4], and similarly Welch's method is considered in [5, 6] and the Multitaper method is analyzed in [7]. One major and attractive finding is that the FD detector naturally provides a lower probability of false alarm when compared wit the time domain (TD) conventional approach. Moreover, it is also shown that the periodogram detector is insensitive to the length of the observed signal, which is not the case for the TD detector. In contrast with what is previously provided, in this paper we extend the work in [3, 8] to obtain optimized models of the periodogram-based detector. Moreover, the case of sensing with multiple antennas is considered with equal gain combing (EGC).

The rest of this paper is organized as follows. In Section II, the system model is described including the mathematical notations employed throughout the paper and the investigated sensing scenario. The optimized detector is derived in Section III, and in Section IV, the case of sensing with multiple antenna is investigated. In Section V, numerical results are provided and finally conclusions are presented in Section VI.

2. SYSTEM MODEL

2.1. Mathematical Operators

Matrices will be denoted by boldfaced uppercase characters and vectors will be denoted by boldface lowercase characters throughout the paper. Other mathematical operators that will be used are listed as follows.

- $\oplus$ is the direct sum operator.
- $(.)^{\mathrm{H}}$ is the conjugate transpose.
- $|\,.\,|$ is the magnitude operator.
- $\mathbf{I}_n$ is the $n \times n$ identity matrix.
- $\mathrm{diag}(\mathbf{A})$ is a diagonal matrix consisting of the main diagonal elements of $\mathbf{A}$.
- $eig(\mathbf{A})$ is the set of eigenvalues of the matrix $\mathbf{A}$.
- $\widehat{(.)}$ is an estimated parameter.
- $\mathbb{E}[.]$ is the expectation operator.
- $p(x;\theta)$ the probability density function (PDF) of x with θ as a parameter.
- $L(.)$ is the likelihood ratio (LLR).
- $\mathcal{L}(.)$ is the log-likelihood ratio.
- $\mathcal{H}_0$ the null hypothesis (empty channel).
- $\mathcal{H}_1$ the alternate hypothesis (occupied channel).

The probability of false alarm and the probability of detection are denoted by

$$P_f \triangleq \mathrm{Prob}\left[\mathrm{choose}\,\mathcal{H}_1 \mid \mathcal{H}_0 = \mathrm{true}\right], \tag{1a}$$

$$P_d \triangleq \mathrm{Prob}\left[\mathrm{choose}\,\mathcal{H}_1 \mid \mathcal{H}_1 = \mathrm{true}\right], \tag{1b}$$

$\mathcal{CN}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is the multivariate complex normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. The notation $Gamma(a, b)$ is the gamma distribution with shape parameter a and scale parameter b. The notation $Hypo(s_1, \ldots, s_N)$ is the hypoexponential distribution with a number of N parameters [9].

2.2. System Setup

Let us consider the hypothesis test for the spectrum sensing problem which is given by

$$\mathcal{H}_0: \quad x(t) = n(t), \qquad t = 0, \ldots, N-1, \quad \text{(2a)}$$
$$\mathcal{H}_1: \quad x(t) = hs(t) + n(t), \quad t = 0, \ldots, N-1, \quad \text{(2b)}$$

where $x(t)$ denotes the received signal at the secondary user, $s(t)$ is the transmitted primary user signal which is assumed zero mean and decomposable into symmetric inphase and quadphase components each with variance $\sigma_s^2/2$. h denotes a slow fading channel. Finally, $n(t)$ is a complex (AWGN) noise process with variance σ_n^2. Let $\mathbf{x} \in \mathbb{C}^{N\times 1}$ denote the column vector of observed samples, i.e., $\mathbf{x} = \{x(t)\}_{t=0}^{N-1}$, and let $\mathbf{h}$ denote the corresponding channel vector. Similarly, let $\widehat{S}_{\mathbf{x}}(f)$ denote the periodogram estimated from the vector $\mathbf{x}$ for the fth frequency index, where $f = 0, 1, \ldots, N-1$. For both cases of the hypothesis test we have

$$\mathbf{x} \sim \begin{cases} \mathcal{CN}\left(0, \sigma_n^2 \mathbf{I}_N\right), & \mathcal{H}_0, \\ \mathcal{CN}\left(0, \sigma_s^2 \mathrm{diag}\left(\mathbf{h}\mathbf{h}^{\mathrm{H}}\right) + \sigma_n^2 \mathbf{I}_N\right), & H_1. \end{cases} \quad (3)$$

The decision statistic for the hypothesis test can be written into as a positive semi-definite hermitian quadratic form representation. We have [3]

$$\mathbf{x}^{\mathrm{H}} \mathbf{Q}_f \mathbf{x} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma, \quad f = 0, 1, \ldots, N-1, \quad (4)$$

where γ is the sensing threshold, and $\mathbf{Q}_f$ is a singular $N \times N$ matrix defined by [3]

$$\mathbf{Q}_f = \frac{1}{N} \begin{pmatrix} 1 & \xi_f^{-1} & \xi_f^{-2} & \cdots & \xi_f^{N-1} \\ \xi_f & 1 & \xi_f^{-1} & \cdots & \xi_f^{N-2} \\ \xi_f^2 & \xi_f & 1 & \cdots & \xi_f^{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_f^{N-1} & \xi_f^{N-2} & \xi_f^{N-3} & \cdots & 1 \end{pmatrix}, \quad (5)$$

where ξ_f is an Nth primitive root of unity for the employed f-th frequency index. The statistics of the variable given by (4) as a central positive-semidefinite hermitian quadratic form depends on the eigenvalues of the product of the covariance matrix of the vector $\mathbf{x}$ and the matrix of the quadratic form $\mathbf{Q}_f$ given in (5). The characteristics of the product $\mathbf{Q}_f \boldsymbol{\Sigma}_x$ are studied in [3, 8] and the product is found to be rank-1 with a single nonzero eigenvalue given by

$$eig\left(\mathbf{Q}_f \boldsymbol{\Sigma}_x\right) = \begin{cases} \sigma_n^2, & \mathcal{H}_0, \\ \sigma_n^2 + \frac{\sigma_s^2}{N} \mathbf{h}^{\mathrm{H}} \mathbf{h}, & \mathcal{H}_1. \end{cases} \quad (6)$$

The probabilities of false alarm and detection are given by [3, 8]

$$P_f(\gamma) = \exp\left(\frac{-\gamma}{\sigma_n^2}\right), \quad (7)$$

$$P_d(\gamma) = \exp\left(\frac{-\gamma}{\sigma_n^2 + \frac{1}{N}\sigma_s^2 \mathbf{h}^{\mathrm{H}} \mathbf{h}}\right). \quad (8)$$

3. THE NEYMAN-PEARSON DETECTOR

3.1. The Likelihood Ratio Test

The probability of detection can be maximized for a given probability of false alarm using the Neyman-Pearson criteria [10]. By making use of (7) and (8) the test is rewritten in terms of the likelihood ratio such that the alternate hypothesis $\mathcal{H}_1$ is accepted if the condition

$$L(\widehat{S}(f)) = \frac{p(\widehat{S}(f); \mathcal{H}_1)}{p(\widehat{S}(f); \mathcal{H}_0)} > \gamma_L \quad (9)$$

is satisfied, where γ_L is the threshold chosen to satisfy a predefined probability of false alarm of α, and $\rho = \frac{\sigma_s^2}{\sigma_n^2}$. We have

$$\frac{\sigma_n^2}{\sigma_n^2 + \frac{\sigma_s^2}{N} \mathbf{h}^{\mathrm{H}} \mathbf{h}} \exp\left(\frac{\rho \widehat{S}(f) \mathbf{h}^{\mathrm{H}} \mathbf{h}}{N\sigma_n^2 + \sigma_s^2 \mathbf{h}^{\mathrm{H}} \mathbf{h}}\right) > \gamma_L. \quad (10)$$

Hence, based on the LLR the test is rewritten in the form $\mathcal{L}(\widehat{S}(f)) > \log(\gamma_L)$, and after further simplification we get the condition $\widehat{S}(f) > \gamma_L^*$ where γ_L^* is the modified threshold given by

$$\gamma_L^* = \frac{\sigma_n^2 N + \sigma_s^2 \mathbf{h}^{\mathrm{H}} \mathbf{h}}{\rho \mathbf{h}^{\mathrm{H}} \mathbf{h}} \log\left(\gamma_L + \frac{\rho}{N} \gamma_L \mathbf{h}^{\mathrm{H}} \mathbf{h}\right). \quad (11)$$

Based on the modified threshold, the probability of false alarm is given by

$$P_f(\gamma_L; N, \rho, \mathbf{h}) = \exp\Bigg(- \left(1 + \frac{N}{\rho \mathbf{h}^{\mathrm{H}} \mathbf{h}}\right) \times \log\left[\gamma_L \left(1 + \frac{\rho}{N} \mathbf{h}^{\mathrm{H}} \mathbf{h}\right)\right] \Bigg), \quad (12)$$

and the probability of detection is obtained by

$$P_d(\gamma_L; N, \rho, \mathbf{h}) = \exp\left(- \frac{N}{\rho \mathbf{h}^{\mathrm{H}} \mathbf{h}} \log\left[\gamma_L \left(1 + \frac{\rho}{N} \mathbf{h}^{\mathrm{H}} \mathbf{h}\right)\right] \right). \quad (13)$$

3.2. Average Probabilities Over Rayleigh fading

Let us consider a Rayleigh fading channel, such that the channel instants are independent and identically distributed, with $\Omega = \mathbb{E}\left[\mid h \mid^2 \right]$. In this case, we have $\mathbf{h}^{\mathrm{H}}\mathbf{h} \sim \mathrm{Gamma}\left(N, \Omega\right)$. The average probabilities of false alarm and detection are obtained by averaging (12) and (13) over the probability distribution function of $\mathbf{h}^{\mathrm{H}}\mathbf{h}$. The probability of false alarm is given by

$$P_{f,Ray}(\gamma_L) = \frac{1}{\Gamma(N)\Omega^N} \times \int_0^\infty \left(\gamma_L + \frac{\rho}{N}\gamma_L z\right)^{-1-\frac{N}{\rho z}} z^{N-1} \exp\left(-\frac{z}{\Omega}\right) \mathrm{d}z, \quad (14)$$

where $\Gamma(N)$ is the gamma function [11]. By applying Jensen's inequality [12], $P_{f,Ray}(\gamma_L)$ will have a minimum bound given by

$$P_{f,Ray}(\gamma_L) \geq \left(\frac{1}{\gamma_L(1+\rho\Omega)}\right)^{1+\frac{1}{\rho\Omega}}. \quad (15)$$

It can be seen that the minimum probability over Rayleigh fading is not affected by the vector length N. This is similar to what is found in [3], which shows that the performance of the periodogram-based detector (the non optimized model) is not affected by the vector length used to compute the estimate. Similarly, the average probability of detection is obtained by

$$P_{d,Ray}(\gamma_L) = \frac{1}{\Gamma(N)\Omega^N} \times \int_0^\infty \left(\gamma_L + \frac{\rho}{N}\gamma_L z\right)^{-\frac{N}{\rho z}} z^{N-1} \exp\left(-\frac{z}{\Omega}\right) \mathrm{d}z, \quad (16)$$

and the minimum bound for the probability of detection is

$$P_{d,Ray}(\gamma_L) \geq \left(\frac{1}{\gamma_L(1+\rho\Omega)}\right)^{\frac{1}{\rho\Omega}}. \quad (17)$$

3.3. Average Probabilities Over Rician fading

Let us consider a Rician fading process with a K-factor of $\kappa = \frac{\nu^2}{a}$. In this case the sum of the instantaneous squared channel magnitudes is chi-square distributed with $2N$ degrees of freedom and noncentrality parameter of $\frac{2N\nu^2}{a}$. The average probability of false alarm is obtained by

$$P_{f,Rice}(\gamma_L) = \frac{1}{a}\int_0^\infty \left(\gamma_L + \frac{\rho}{N}\gamma_L z\right)^{-1-\frac{N}{\rho z}} \times \exp\left(-\frac{z+N\nu^2}{a}\right)\left(\frac{z}{N\nu^2}\right)^{\frac{N-1}{2}} I_{N-1}\left(\frac{2\nu}{a}\sqrt{Nz}\right) \mathrm{d}z, \quad (18)$$

and the minimum bound for the probability of false alarm is given by

$$P_{f,Rice}(\gamma_L) \geq \left(\frac{1}{\gamma_L\left(1+\rho(2+2\nu^2\right)}\right)^{1+\dfrac{1}{\rho(2+2\nu^2)}} \quad (19)$$

and the average probability of detection is given by

$$P_{d,Rice}(\gamma_L) = \frac{1}{a}\int_0^\infty \left(\gamma_L + \frac{\rho}{N}\gamma_L z\right)^{-\frac{N}{\rho z}} \times \exp\left(-\frac{z+N\nu^2}{a}\right)\left(\frac{z}{N\nu^2}\right)^{\frac{N-1}{2}} I_{N-1}\left(\frac{2\nu}{a}\sqrt{Nz}\right) \mathrm{d}z, \quad (20)$$

where $I_n(.)$ is the nth order modified Bessel function of the first type [11]. The minimum bound for the probability of detection is given by

$$P_{d,Rice}(\gamma_L) \geq \left(\frac{1}{\gamma_L\left(1+\rho(2+2\nu^2\right)}\right)^{\dfrac{1}{\rho(2+2\nu^2)}}. \quad (21)$$

4. MULTIPLE ANTENNA SENSING WITH EGC

In this part, it is assumed that the secondary user employs multiple antennas to perform spectrum sensing. Let K denote the number of antennas, and let $\widehat{S}^{(i)}(f)$ denote the decision variable computed at the ith branch, where $i = 1, \ldots, K$ and let $\mathbf{x}_i$ and $\mathbf{h}_i$ denote the corresponding observations and channel vectors respectively. For simplicity, but without loss of generality, all branches are assumed independent and identically distributed. Assuming that EGC is employed to make the final test variable, which is in this case can be written as

$$\widehat{S}_K(f) = \bigoplus_{i=1}^{K} \mathbf{x}_i^{\mathrm{H}}\mathbf{Q}_f\mathbf{x}_i. \quad (22)$$

It can be seen that $\widehat{S}_K(f)$ is another hermitian quadratic form, but in this case the rank of the product of the covariance matrix and the matrix of the quadratic form is K. This can be verified by treating $\widehat{S}_K(f)$ as a block-diagonal matrix and applying the theorems within [13, Sec. 0.9.2] Therefore, in this case there is a number of K nonzero eigenvalues. For the case of $\mathcal{H}_0$ all eigenvalues are identical and equal to the noise variance, but for the case of the hypothesis $\mathcal{H}_1$, the nonzero eigenvalues are different and equal to $\sigma_n^2 + \frac{\sigma_s^2}{N}\mathbf{h}_i^{\mathrm{H}}\mathbf{h}_i$ where $i = 1, \ldots, K$. Henceforth, for each case of the hypothesis test we have the following statistical properties,

$$\widehat{S}_K(f) \sim \begin{cases} Gamma\left(K, \sigma_n^2\right), & \mathcal{H}_0, \\ Hypo\left(\sigma_n^2 + \frac{\sigma_s^2}{N}\mathbf{h}_1^{\mathrm{H}}\mathbf{h}_1, \ldots, \sigma_n^2 + \frac{\sigma_s^2}{N}\mathbf{h}_K^{\mathrm{H}}\mathbf{h}_K\right), & \mathcal{H}_1. \end{cases}, \quad (23)$$

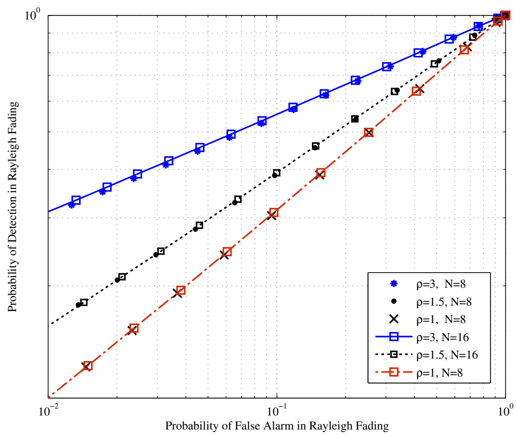

Fig. 1. Receiver operator characteristics for detection over Rayleigh fading

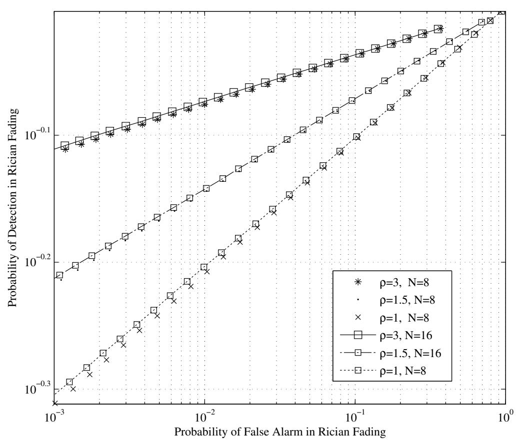

Fig. 2. Receiver operator characteristics for detection over Rician fading

i.e., we have

$$p(x;\mathcal{H}_0) = \frac{1}{\Gamma(N)\sigma_n^{2N}} x^{N-1} \exp\left(-\frac{x}{\sigma_n^2}\right), \tag{24}$$

$$p(x;\mathcal{H}_1) = \sum_{i=1}^{K} \frac{\mathfrak{L}_i(0)}{\sigma_n^2 + \frac{\sigma_s^2}{N}\mathbf{h}_i^{\mathrm{H}}\mathbf{h}_i} \exp\left(\frac{-x}{\sigma_n^2 + \frac{\sigma_s^2}{N}\mathbf{h}_i^{\mathrm{H}}\mathbf{h}_i}\right), \tag{25}$$

where $\mathfrak{L}_i(0)$ can be seen as the Lagrange basis polynomial associated with the ith eigenvalue of the quadratic form, and $\mathfrak{L}_i(0)$is given by

$$\mathfrak{L}_i(0) = \prod_{\substack{j=1\\ j\neq i}}^{K} \frac{\frac{N}{\rho} + \mathbf{h}_i^{\mathrm{H}}\mathbf{h}_i}{\mathbf{h}_i^{\mathrm{H}}\mathbf{h}_i - \mathbf{h}_j^{\mathrm{H}}\mathbf{h}_j} \tag{26}$$

In this case, the likelihood ratio test is obtained such that the hypothesis H_1 is accepted when the condition given by

$$L(\widehat{S}_K(f)) = \frac{p(\widehat{S}_K(f);\mathcal{H}_1)}{p(\widehat{S}_K(f);\mathcal{H}_0)} > \gamma_L \tag{27}$$

is satisfied. The test for the case of EGC is given by

$$\Gamma(N)\sigma_n^{2N}\left(\frac{N}{\sigma_s^2}\right)^K \sum_{i=1}^{K}\prod_{\substack{j=1\\ j\neq i}}^{K} \frac{1}{\mathbf{h}_i^{\mathrm{H}}\mathbf{h}_i - \mathbf{h}_j^{\mathrm{H}}\mathbf{h}_j} \times \exp\left(-\frac{\widehat{S}_K(f)\rho\mathbf{h}_i^{\mathrm{H}}\mathbf{h}_i}{N\sigma_n^2 + \sigma_s^2\mathbf{h}_i^{\mathrm{H}}\mathbf{h}_i}\right) \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma. \tag{28}$$

In this case, obtaining the expressions for the probabilities of false alarm and detection is complex, which is beyond the scope of this paper.

5. SIMULATION RESULTS

The performance is inspected in terms of the receiver operator characteristics (P_d vs P_f). Figure1 depicts the receiver operator characteristics for transmission over fading channels, and Figure2 depicts the performance for the case of Rican fading. Both figures are simulated for vector lengths of $N = \{8, 16\}$ and various cases for the variable ρ. For Rayleigh fading $\Omega = 1$ and for Rician fading $a = 3$ and $\nu = 2$. Monte Carlo simulations are repeated for a number of 10^5 trials, and an adaptive Gaussian quadrature is employed for numerical integration. It can be seen from the results in both figures that the performance is enhanced when ρ is increased. In addition, it can be seen that for the same set of parameters, changing the length of the vector used to obtain the periodogram does not affect the performance, i.e, changing the value of N will always yield the same receiver operator characteristics, which is similar to the trend found in [3] for the case of raw periodograms.

6. CONCLUSION

In this paper, optimization of the periodogram-based energy detector is investigated. An Optimized model is derived based on the LRT for both cases of Rayleigh and Rican fading, and the impact of employing multiple antenna for spectrum sensing with EGC is also addressed. The results reveal the accuracy of the derived mathematical models and simulations show that the performance is enhanced when the SNR is increased. For the case of EGC, the PDF of the test statistic is hypoexponetial with a number of phases equal to the employed number of branches. Therefore, in this case it is very difficult to derive an expression for the average probabilities of false alarm and detection because of the presence of the Lagrange basis polynomials. For Future work, this issue will be considered in addition to other methods such as selection combining and switch-and-stay combining.

7. REFERENCES

[1] J. Mitola, "Cognitive Radio architecture evolution," *Proc. IEEE*, vol. 97, no. 4, pp. 626–641, Apr 2009.

[2] I. F. Akyildiz, Won-Yeol Lee, M. C. Vuran, and S. Mohanty, "A survey on spectrum management in cognitive radio networks," *IEEE Commun. Mag.*, vol. 46, no. 4, pp. 40–48, 2008.

[3] E. H. Gismalla and E. Alsusa, "Performance analysis of the periodogram-based energy detector in fading channels," *IEEE Trans. Signal Process.*, vol. 59, no. 8, pp. 3712–3721, 2011.

[4] Ebtihal Haider Gismalla and Emad Alsusa, "Modeling and optimization of spectrum sensing for unknown signals based on bartlett's power spectrum estimator," in *IEEE WCNC 2012*, Paris, France, Apr. 2012, pp. 797–801.

[5] M. Matinmikko, H. Sarvanko, M. Mustonen, and A. Mammela, "Performance of spectrum sensing using welch's periodogram in rayleigh fading channel," in *Proc. 4th International Conference on Cognitive Radio Oriented Wireless Networks and Communications (CROWNCOM'09)*, Jun 2009, pp. 1–5.

[6] E. H. Gismalla and E. Alsusa, "On the detection of unknown signals using welch overlapped segmented averaging method," in *Proc. IEEE Vehicular Technology Conf. (VTC Fall)*, 2011, pp. 1–5.

[7] Ebtihal Haider Gismalla and Emad Alsusa, "New and accurate results on the performance of the Multitaper-Based detector," in *IEEE ICC 2012 (ICC'12 CRN)*, Ottawa, Ontario, Canada, June 2012.

[8] E. H. Gismalla and E. Alsusa, "A generalized system model and performance analysis for the periodogram-based energy detector," in *Proc. IEEE Global Telecommunications Conf. (GLOBECOM 2011)*, 2011, pp. 1–5.

[9] W.J. Stewart, *Probability, Markov chains, queues, and simulation: the mathematical basis of performance modeling*, Princeton Univ Press, 2009.

[10] Steven M. Kay, *Fundamentals of Statistical Signal Processing, Volume 2: Detection Theory*, Prentice-Hall, Englewood Cliffs, NJ, 1998.

[11] I.S. Gradshteyn and I.M. Ryzhik, *Tables of integrals, sums, series and products*, Elsevier, seventh edition, 2007.

[12] J.E. Pečarić, F. Proschan, and Y.L. Tong, *Convex functions, partial orderings, and statistical applications*, Academic Press, 1992.

[13] R.A. Horn and C.R. Johnson, *Matrix analysis*, Cambridge Univ Press, 1990.