CRLB FOR THE LOCALIZATION ERROR IN THE PRESENCE OF FADING

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ABSTRACT

Localization accuracy is crucial in sensor networks. A wireless sensor network (WSN) with M anchors and one node is considered in this paper. The estimation is based on time of arrival (TOA) in the presence of fading channels. The Cramer-Rao lower bound (CRLB) for localization error in the presence of fading is derived under different scenarios. Firstly, fading coefficients are considered as unknown random parameters with a prior distribution. The ML estimator for this case is also derived. If the distribution of fading is unknown to the estimator then the modified CRLB (MCRLB) is applied and shown to be equal to the CRLB in the absence of fading. This is used to conclude that fading always deteriorates the CRLB in localization. It is shown that there is a loss of about 5dB in CRLB due to Rayleigh fading.

Index Terms— time of arrival, Cramer-Rao lower bound, fading, ML estimator, localization, wireless sensor networks

1. INTRODUCTION

In many applications, measured sensor data is meaningful only when the location of sensors is accurately known. In localization problems, sensor nodes at known locations, anchors, transmit signals to sensor nodes at unknown locations, nodes, and use these transmissions to estimate the TOA, which leads to location estimation [1–3]. In [4] the CRLB on the variance of localization error is derived by assuming Gaussian TOA measurements. Also, the CRLB in a multipath environment is derived with TOA measurements in [5]. Reference [6] considers a cooperative sensor network in the absence of fading, and derives the CRLB. In [7], the variance of TOA measurements is assumed to be a function of the distance between the node and the anchor, and the CRLB is derived, whereas [8] considers biased measurements. Despite the prevalence of fading in practice, none of these work have studied the CRLB for localization under fading environments. Although [9] has studied the CRLB in the presence of Rayleigh fading under received signal strength (RSS) estimation method, and some existing work has considered fading environments for TOA measurements [10] [11], CRLB for localization error under fading environments for the TOA method has not been studied.

The authors are funded in part by the SenSIP Center. The authors are also supported in part by the grant NSF FRP 1231034.

In this paper, we consider localization in the presence of fading. The fading coefficients are considered as either unknown random parameters with a prior distribution or without any prior distribution known at the estimator. CRLBs are derived for both one dimensional (1-D) and two dimensional (2-D) localization problems with TOA measurements. Our results are compared with the CRLBs in the absence of fading that were derived in [4]. The ML estimator in the presence of fading is derived. Also, the MCRLB in fading is shown to be equal to the CRLB in the AWGN case.

The rest of this paper is organized as follows. In Section 2, the system model is presented, and the CRLB in the presence of fading for localization error is derived under different scenarios. The ML estimator in fading environments is also derived. Following this, in Section 3, simulations are used to compare the CRLBs that are derived in Section 2, and the fading ML estimator is compared with the ML estimator that was derived in [4]. In Section 4, concluding remarks are presented.

2. LOCALIZATION IN THE PRESENCE OF FADING

2.1. System Model

Assume that a non-cooperative wireless sensor network, in which nodes do not communicate with each other, contains M anchors and N nodes in $\mathbb{R}^n$, where $n = 1, 2$. The vector $\mathbf{p} = [\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_M]$ contains the cartesian coordinates of all anchors, and $\mathbf{z} = [\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_N]$ is the vector of node locations. In 1-D, the location of the i^{th} anchor, $\mathbf{p}_i = x_i$, and the j^{th} node, $\mathbf{z}_j = x_j$ are scalars. In 2-D, $\mathbf{p}_i = [x_i, y_i]^T$ and $\mathbf{z}_j = [x_j, y_j]^T$ are vectors. As we assume the network is non-cooperative, the CRLB on the variance of the location error for each node is independent of the other nodes. Therefore, to simplify the problem statement, we only assume one node exists in the network at location $\mathbf{z}$.

We assume the node communicates with all anchors. Each anchor transmits a signal to a node, and the node sends back the signal immediately after receiving it. Each anchor measures the round-trip time and halves it to obtain the TOA estimates $\hat{\tau}_i$ between the i^{th} anchor and the node. Using these

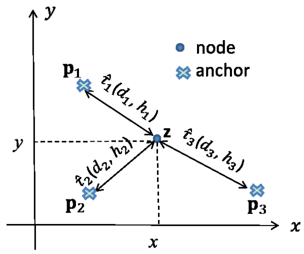

Fig. 1. Three anchors are present at positions $\mathbf{p} = [\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3]$, and 1 node at the position $\mathbf{z} = [x, y]^T$. Variables $\hat{\tau}_i$, d_i and h_i are the TOA measurement, distance and the channel fading coefficient between the node and the i^{th} anchor respectively, where $\hat{\tau}_i$ is the function of d_i and h_i in the presence of fading.

values, the location of the node is estimated. Figure 1 shows an example of three anchors and one node. The channels between anchors and the node are subject to additive noise and fading. In the absence of fading, the fading coefficients h_i in Figure 1 are set to 1. The distribution of $\hat{\tau}_i$ in this case is known to be [4]

$$\hat{\tau}_i \sim \mathcal{N}\left(\frac{d_i}{c}, \sigma_T^2\right), \tag{1}$$

where d_i is the true distance between the i^{th} anchor and the node, which is denoted as $||\mathbf{p}_i - \mathbf{z}||_2$. Here, $||\cdot||_2$ is the Euclidean norm, c is the speed of propagation, and σ_T^2 is the variance of the TOA estimates on the channel.

In the presence of fading, the measurement $\hat{\tau}_i$ is a function of d_i and h_i (as shown in Figure 1), and the following scenarios will be considered. Firstly, fading coefficients are assumed to be random with a prior distribution known at the estimator. In this case, the unconditional distribution of the TOA measurements can be calculated by averaging across the fading to derive the CRLB [12]. In the second scenario, fading coefficients as still considered as random parameters; however, the prior distribution is unknown to the estimator so that the fading effect in the previous scenario cannot be averaged out. In this case, the modified CRLB (MCRLB), which does not need the full knowledge of the fading distribution, is applied [13].

2.2. Fading coefficients as random parameters

Assume fading is not changing during all TOA measurements. Also, fading coefficients are random parameters with a prior Nakagami distribution. The TOA measurements $\hat{\tau}_i$ are assumed to be i.i.d. and conditioned on the fading coefficients satisfy

$$\hat{\tau}_i \Big| |h_i|^2 \sim \mathcal{N}\left(\frac{d_i}{c}, \frac{\sigma_T^2}{|h_i|^2}\right), \tag{2}$$

where the fading power is Gamma distributed and given by [14]

$$f_{|h_i|^2}(x) = m^m x^{m-1}\Gamma(m)^{-1}\exp(-mx), \tag{3}$$

where m is the fading parameter, and $\mathrm{E}_{h_i}\left(|h_i|^2\right)$ is the average received power at the node and is fixed to 1. When $m = 1$, the fading $|h_i|$ is Rayleigh distributed, and as $m \to \infty$, the channel exhibits no fading so that the AWGN dominates.

The unconditioned distribution of $\hat{\tau}_i$ can be calculated by using the formula

$$f_{\hat{\tau}_i}(\hat{\tau}_i|\mathbf{z}) = \int_0^\infty f\left(\hat{\tau}_i \Big| |h_i|^2, \mathbf{z}\right) f_{|h_i|^2}(x) d|h_i|^2. \tag{4}$$

By substituting (2) and (3) into (4), using the formula in [15] and carrying out the integration we obtain

$$f_{\hat{\tau}_i}(\hat{\tau}_i|\mathbf{z}) = \frac{m^m(m-\frac{1}{2})!}{2\sqrt{2\pi\sigma_T^2}\Gamma(m)\left[\frac{1}{2\sigma_T^2}(\hat{\tau}_i - \frac{d_i}{c})^2 + m\right]^{(m+\frac{1}{2})}}. \tag{5}$$

Define the vector $\mathbf{h} = [h_1, h_2, \ldots, h_M]$ containing all fading coefficients between the node and M anchors, and $\mathbf{T} = [\hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_M]$ is the vector of TOA measurements. Also define $l(\mathbf{T}|\mathbf{z}) = \sum_{i=1}^M \ln f_{\hat{\tau}_i}(\hat{\tau}_i|\mathbf{z})$ as the log-likelihood function of $f(\mathbf{T}|\mathbf{z})$, where $f(\mathbf{T}|\mathbf{z}) = \prod_{i=1}^M f_{\hat{\tau}_i}(\hat{\tau}_i|\mathbf{z})$. Based on the log-likelihood, the CRLB can be expressed as [12]

$$\text{CRLB} = \left[\mathrm{E}_{\mathbf{T}}\left\{\left[\left(\frac{\partial l(\mathbf{T}|\mathbf{z})}{\partial \mathbf{z}}\right)^2\right]\right\}\right]^{-1}. \tag{6}$$

For convenience, Let $l(\hat{\tau}_i|\mathbf{z}) = \ln f_{\hat{\tau}_i}(\hat{\tau}_i|\mathbf{z})$ be the log likelihood function of each TOA measurement. Then the Fisher information matrix (FIM), $\mathbf{F}_\mathbf{z}$, of the node location in $\mathbb{R}^n$ is denoted as $[\mathbf{F}_\mathbf{z}]_{jk} = f_{jk}$, and

$$f_{jk} = \begin{cases} \sum_{i=1}^M \mathrm{E}_{\hat{\tau}_i}\left[\left(\frac{\partial l(\hat{\tau}_i|\mathbf{z})}{\partial \mathbf{z}_j}\right)^2\right] & j = k \\ -\sum_{i=1}^M \mathrm{E}_{\hat{\tau}_i}\left[\frac{\partial^2 l(\hat{\tau}_i|\mathbf{z})}{\partial \mathbf{z}_j \partial \mathbf{z}_k}\right] & j \neq k \end{cases}. \tag{7}$$

In 1-D, the location of the node is $\mathbf{z} = x$ and the distance between the node and the i^{th} anchor is $d_i = ||x_i - x||_2 = |x_i - x|$. Using $\mathbf{F}_\mathbf{z} = \sum_{i=1}^M \mathrm{E}_{\hat{\tau}_i}\left[\left(\frac{\partial l(\hat{\tau}_i|\mathbf{z})}{\partial \mathbf{z}}\right)^2\right]$, $l(\hat{\tau}_i|\mathbf{z})$ can be calculated using (5), as

$$\mathrm{E}_{\hat{\tau}_i}\left[\left(\frac{\partial l(\hat{\tau}_i|\mathbf{z})}{\partial \mathbf{z}}\right)^2\right] = \frac{m^m(m-\frac{1}{2})!(m+\frac{1}{2})^2}{\Gamma(m)\sqrt{2\pi}c^2\sigma_T^5} X(d_i), \tag{8}$$

where

$$X(d_i) = \int_0^\infty \frac{(\hat{\tau}_i - \frac{d_i}{c})^2}{\left[\frac{1}{2\sigma_T^2}(\hat{\tau}_i - \frac{d_i}{c})^2 + m\right]^{\frac{5}{2}+m}} d\hat{\tau}_i. \tag{9}$$

Unlike the AWGN case, the Fisher Information (FI) depends on d_i through $X(d_i)$ in (9). However, using [15] it is possible to express it as

$$X(d_i) \leq \frac{\sqrt{2}\sigma_T^3\Gamma(\frac{3}{2})\Gamma(m+1)}{m^{1+m}\Gamma(m+\frac{5}{2})} + \frac{\left(\frac{d_i}{c}\right)^2}{\left[\frac{1}{2\sigma_T^2}(\frac{d_i}{c})^2+m\right]^{\frac{5}{2}+m}}. \tag{10}$$

Since the second term in (10) is small ($d_i/c \approx 0$), it is clear that $X(d_i)$ can be approximated by the first term, and therefore approximately independent of d_i. Whether we use the term $X(d_i)$, or its approximation in (10), the CRLB in the presence of Nakagami fading is

$$\text{CRLB}_{\text{1-D}}(\mathbf{z}) = \frac{\Gamma(m)\sqrt{2\pi}c^2\sigma_T^5}{m^m(m-\frac{1}{2})!(m+\frac{1}{2})^2\sum_{i=1}^M X(d_i)}. \tag{11}$$

Using only the first term in (10), the loss due to fading can be expressed as

$$\frac{\text{CRLB}_{\text{1-D}}(\mathbf{z})}{\text{CRLB}_{\text{1-D}}^{\text{AWGN}}} \approx k = \frac{\sqrt{\pi}\Gamma(m+\frac{5}{2})}{\Gamma(\frac{3}{2})(m+\frac{1}{2})^2\left(m-\frac{1}{2}\right)!}, \tag{12}$$

where we recall that $\text{CRLB}_{\text{1-D}}^{\text{AWGN}} = c^2\sigma_T^2/M$ [4]. When $m \to \infty$, the second term in (10) goes to 0 and k in (12) goes to 1 so that the CRLB in the presence of fading converges to the AWGN case.

When $m = 1$, the fading is Rayleigh distributed, (11) is simplified as

$$\text{CRLB}_{\text{1-D}}(\mathbf{z}) = 9\left[8\sqrt{2}c^2\sigma_T^5\right]^{-1}\left(\sum_{i=1}^M X(d_i)\right), \tag{13}$$

and $X(d_i)$ is

$$X(d_i) = \left(\frac{d_i}{c}\right)^3\left[\frac{\left(\frac{d_i}{c}\right)^2}{\sigma_T^2}+5\right]\left(15\left[\frac{\left(\frac{d_i}{c}\right)^2}{2\sigma_T^2}+1\right]^{5/2}\right)^{-1}. \tag{14}$$

Simplifying (13) for small σ_T^2 (high SNR), we obtain

$$\text{CRLB}_{\text{1-D}}(\mathbf{z}) = \frac{\sigma_T^2c^2}{M}\frac{10}{3} + o\left(\sigma_T^2\right). \tag{15}$$

This shows that the loss in SNR due to Rayleigh fading is a factor of $k \triangleq \frac{10}{3}$, which is about 5dB.

In 2-D, the distance between the node and the i^{th} anchor is $d_i = \sqrt{(x_i-x)^2+(y_i-y)^2}$, and let $Y(m) = m^m(m-\frac{1}{2})!(m+\frac{1}{2})^2\left[\Gamma(m)\sqrt{2\pi}c^2\sigma_T^5\right]^{-1}$, and the FIM is

$$\mathbf{F}_{\mathbf{z}} = Y(m)\sum_{i=1}^M\begin{bmatrix}\frac{(x_i-x)^2X(d_i)}{d_i^2} & \frac{(y_i-y)(x_i-x)X(d_i)}{d_i^2}\\ \frac{(y_i-y)(x_i-x)X(d_i)}{d_i^2} & \frac{(y_i-y)^2X(d_i)}{d_i^2}\end{bmatrix}. \tag{16}$$

Therefore, the CRLB on the variance of the localization error in 2-D is shown as [4]

$$\text{CRLB}_{\text{2-D}}(\mathbf{z}) = \text{tr}\left(\mathbf{F}_{\mathbf{z}}^{-1}\right). \tag{17}$$

By comparing (17) with the CRLB in the absence of fading in [4], both CRLBs in 2-D depend on the true location of the node. However, it is still possible to see that $\text{CRLB}_{\text{2-D}}(\mathbf{z})$ is also a factor of k higher than the AWGN counterpart. Meanwhile, as $m \to \infty$, the CRLB in 2-D converges to the AWGN case as well.

The ML estimator for location estimation in the presence of fading is denoted as

$$\hat{\mathbf{z}} = \underset{\mathbf{z}}{\text{argmax}}\prod_{i=1}^M f_{\hat{\tau}_i}(\hat{\tau}_i|\mathbf{z}). \tag{18}$$

Substituting (5) into (18), we have

$$\hat{\mathbf{z}} = \underset{\mathbf{z}}{\text{argmin}}\sum_{i=1}^M \log\left[\frac{1}{2\sigma_T^2}\left(\hat{\tau}_i-\frac{d_i}{c}\right)^2+m\right], \tag{19}$$

where $d_i = ||\mathbf{p}_i-\mathbf{z}||_2$.

In the absence of fading, the ML estimator which is derived in [4] is

$$\hat{\mathbf{z}} = \underset{\mathbf{z}}{\text{argmin}}\sum_{i=1}^M\left(\hat{\tau}_i-\frac{d_i}{c}\right)^2. \tag{20}$$

By comparing (19) and (20) one can see that the ML estimator in the presence of fading does require the knowledge of m, and σ_T, whereas the ML estimator in the absence of fading does not require the knowledge of σ_T. In Section 3 we illustrate that it is more appropriate to use (19) compared to (20) in the presence of fading.

2.3. Unknown fading distribution

In the previous section we assumed the fading is Nakagami distributed and the effect of fading is averaged out. However, in some cases the prior distribution of the fading is unknown to the estimator. In such cases, the modified CRLB (MCRLB), which is

$$\text{MCRLB}(\mathbf{z}) = \left[\text{E}_{\mathbf{T},\mathbf{h}}\left\{\left[\left(\frac{\partial\ln f(\mathbf{T}|\mathbf{h},\mathbf{z})}{\partial\mathbf{z}}\right)^2\right]\right\}\right]^{-1} \tag{21}$$

can be applied [13].

When computing the CRLB in Section 2.2, the FIM uses the distribution in (5). For the MCRLB, from the PDF in (2), the FI can be calculated for the 1-D case as

$$\mathbf{F}_{\mathbf{z}} = \left(c^2\sigma_T^2\right)^{-1}\sum_{i=1}^M \text{E}_{h_i}\left(|h_i|^2\right). \tag{22}$$

Therefore, (21) can be calculated as

$$\text{MCRLB}_{\text{1-D}}(\mathbf{z}) = c^2\sigma_T^2\left[\sum_{i=1}^M \text{E}_{h_i}\left(|h_i|^2\right)\right]^{-1}. \tag{23}$$

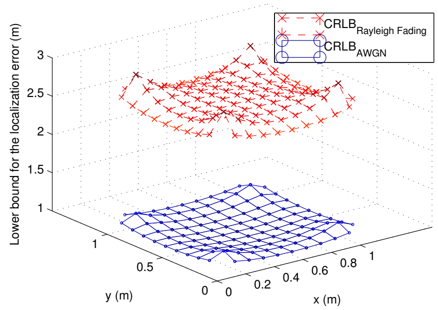

Fig. 2. CRLB comparison in a 1m $\times$ 1m square with $\sigma_T = \frac{1}{c}$.

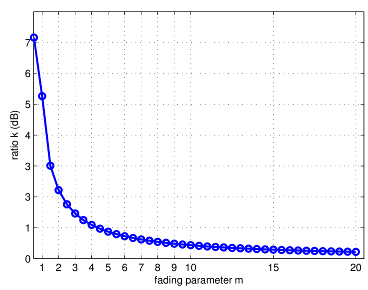

Fig. 3. The ratio k in (12) versus the Nakagami m parameter.

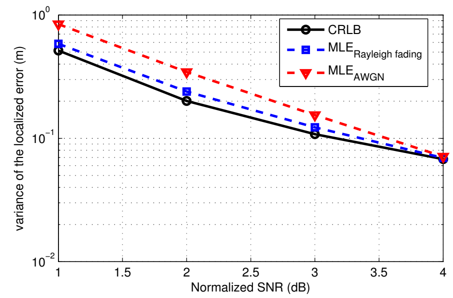

Fig. 4. ML estimators comparison.

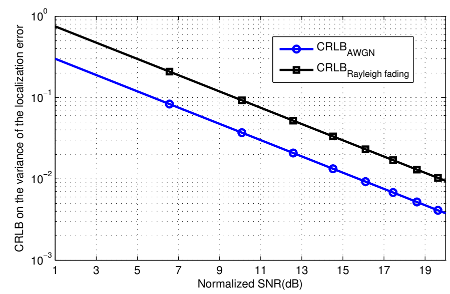

Fig. 5. CRLB comparison when SNR is large.

From (23) one can see that although the fading distribution is unknown at the estimator, the MCRLB of the localization error can be calculated if the second moment of fading is known. since $\mathrm{E}_{h_i}\left(|h_i|^2\right) = 1$, then (23) can be simplified as $\mathrm{MCRLB}_{\text{1-D}}(\mathbf{z}) = c^2\sigma_T^2/M$, and the MCRLB for the localization error equals the AWGN case in [4]. Since the MCRLB is known to be a lower bound on the CRLB in (11), we can conclude that the presence of fading will always degrade performance of *any* fading distribution. For the MCRLB in 2-D, the derivation is very similar as 1-D, and it turns out the MCRLB in 2-D is the same as the 2-D AWGN case as well. The details are omitted for brevity.

3. NUMERICAL RESULTS

Consider a sensor network with four anchors in the corner of a square, and one node within the square. The fading is Rayleigh distributed. In Figure 2 the area of the square is 1m $\times$ 1m. We observe that the loss due to fading is about 2.5 everywhere within the square.

Figure 3 shows the loss due to fading as a function of the Nakagami m parameter. As expected, the loss decreases with increasing m and converges to 1.

In Figure 4 we consider a 1m $\times$ 1m square, 4 anchors are in the corners, and 1 node is in the middle of the square. We compare estimators (19) and (20) both in the presence of fading by plotting the normalized SNR (with respect to c^2) vs. the variance of localization error in Figure 4. We observe that the fading ML estimator (19) performs better than the AWGN ML estimator (20) in the presence of fading.

Figure 5 shows the CRLB comparison in 1-D between the AWGN case and the presence of Rayleigh fading. From the fading one can see that in the high SNR regime, to maintain the same variance of localization error, CRLB in the presence of Rayleigh fading needs about 5dB more power than the AWGN case.

4. CONCLUSIONS

In this paper, we derived CRLBs in the presence of fading under TOA measurements. Fading coefficients are first considered as random parameters with a prior Nakagami distribution, the CRLB is derived by averaging out the effect of fading. Also, by comparing the CRLB in the presence of Rayleigh fading and the AWGN case, it is shown in both 1-D and 2-D that an SNR of about 5dB is present. when the variance of noise is close to 0 the SNR loses around 5dB due to fading. Also, the CRLB in the presence of fading converges to the AWGN case as the fading parameter increases. Meanwhile, the ML estimator in the presence of fading is derived, and is different than the ML estimator for the AWGN case. Secondly, the MCRLB is derived when the prior fading distribution is not known at the estimator. In this case, the MCRLB is the same as the CRLB in the absence of fading, proving that fading always leads to loss in performance.

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