SIGNAL REGION CHARACTERIZATION AND EXACT PHASE RECOVERY FOR CONSTANT ENVELOPE PRECODING IN SINGLE-USER LARGE-SCALE MISO CHANNELS

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ABSTRACT

This paper considers constant envelope (CE) precoding in single-user MISO downlink systems. CE precoding is a transmission scheme recently proposed for very large antenna arrays, in which the use of highly power-efficient RF amplifiers is a requirement. There are two important issues in CE precoding, namely the characterization of the region of all possible noise-free receive signals, and the recovery of the phases of the transmitting signal. An existing result by Mohammed and Larsson showed that the noise-free receive signal region can be geometrically interpreted as a region between two circles centered at the origin of the complex plane. However, this result did not prove the expression of the radius of the inner circle. We provide a new analysis approach to characterize the noise-free receive signal region. Our result shows that the radius of the inner circle has a simple closed-form expression, there by completing the result by Mohammed and Larsson. In addition, we propose an algorithm that can recover the phases of the transmitting signal exactly with a complexity linear in the number of antennas. Simulation results show that the proposed method can be significantly faster than an existing phase recovery algorithm.

Index Terms— Large antenna array, constant envelope, doughnut channel

1. INTRODUCTION

Recently, there has been an increasing interest in employing large antenna array at the base station in wireless communications [1, 2, 3, 4]. The large antenna array, which can have more than 100 antenna elements, enables higher spectral efficiency, better reliability, and simpler transmit/receive processing [5]. But the practical implementation of the large antenna array requires highly power-efficient RF amplifiers that can be cheaply implemented, as the large antenna array is equipped with a large number of RF amplifiers [6]. The power efficiency of an RF amplifier is largely limited by the linear range required by the transmitting signal. For the traditional precoding method, such as maximum ratio transmission (MRT), the instantaneous power can vary significantly depending on the channel conditions and the information symbols to be transmitted. Hence, the transmitting signal can have a very high peak-to-average power ratio (PAPR). The RF amplifier that is designed for such high PAPR transmitting signals must have a wide linear range to accommodate the large variations of the signal level, thus inevitably resulting in low power efficiency. In order to overcome this problem, it is proposed in [6, 7] to adopt the constant envelope (CE) precoding scheme. In CE precoding, the transmitting signal at each antenna is restricted to have a constant amplitude, and only the phases of the per-antenna transmitting signals are used to convey information to the receiver. Since the CE signal has a constant amplitude, the instantaneous power is fixed as a constant as well. Therefore, the RF amplifier for CE signals can be made highly power-efficient.

One important issue in CE precoding is the characterization of the region of all possible noise-free receive signals. This characterization is very important as the design of the input constellation generally requires knowledge of this region. The pioneering work [6] by Mohammed and Larsson shows that the noise-free receive signal region is a doughnut region, i.e. a region between two circles centered at the origin of the complex plane. However, the result in [6] only shows existence of the radius of the inner circle. In this paper, we use a different analysis approach to characterize the receive signal region, and prove the inner radius in *closed form*.

Another issue is how to design the phases of the CE signal corresponding to an information symbol. This amounts to solving a highly nonlinear equation. As an alternative way to directly handling this nonlinear equation, it is proposed in [6] to formulate an optimization problem whose optimal solution is a solution of the nonlinear equation. A combination of depth-first-search (DFS) and gradient descent is proposed in the same reference for tackling the optimization problem. Our second contribution is to derive an exact and efficient algorithm for the CE precoding design. The algorithm is a direct consequence of our first contribution, analysis of the CE receive signal region. Simulation results will show that the proposed algorithm is much faster than the gradient descent method. Moreover, CE precoding will be illustrated to provide about 2.5dB performance gain compared to MRT, when taking into account the different power efficiencies of RF amplifiers for CE and MRT.

2. SYSTEM MODEL AND PROBLEM FORMULATION

2.1. System Model and Problem Statement

A standard single-user MISO downlink model is considered:

$$y = d + \nu, \tag{1}$$

$$d = \boldsymbol{h}^T \boldsymbol{x}, \tag{2}$$

where $y \in \mathbb{C}$ is the receive signal, d the noise-free receive signal, $\boldsymbol{h} = [h_1, \ldots, h_N]^T \in \mathbb{C}^N$ the channel vector, $\boldsymbol{x} = [x_1, \ldots x_N]^T \in \mathbb{C}^N$ the transmitting signal, and $\nu \in \mathbb{C}$ AWGN with zero mean and variance σ_ν^2. Here, N is the number of transmit antennas.

From (1) we can see that from the perspective of the receiver, the model (1) is essentially an SISO channel with d being the channel input. Suppose that the equivalent SISO channel input constellation has been chosen as $\mathcal{Q} = \{d^{(m)}\}_{m=1}^M$ (e.g., QAM). For convenience, we also write $\mathcal{Q} = \alpha\mathcal{S}$, where $\alpha > 0$ is a coefficient and $\mathcal{S}$ is a normalized constellation with unit power, i.e. $\mathbb{E}_{s\in\mathcal{S}}[|s|^2] = 1$. The goal

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of a precoder is to design a transmitting signal $\boldsymbol{x}^{(m)}$ corresponding to each $d^{(m)}$ such that $d^{(m)} = \boldsymbol{h}^T\boldsymbol{x}^{(m)}$ is satisfied.

A simple, convenient way to carry out the precoding task mentioned above is maximum ratio transmission (MRT). MRT is a linear scheme where the transmitting signal $\boldsymbol{x}_{\text{MRT}}$ and the coefficient α_{MRT} are given by

$$\boldsymbol{x}_{\text{MRT}} = \sqrt{P_T}\frac{\boldsymbol{h}^*}{\|\boldsymbol{h}\|_2}s, \ \ \alpha_{\text{MRT}} = \sqrt{P_T}\|\boldsymbol{h}\|_2, \tag{3}$$

where $s \in \mathcal{S}$ is an information symbol, and P_T is the average total transmission power. We can see that the average per-antenna power is equal to a constant $\mathbb{E}[|x_{\text{MRT},i}|^2] = \frac{P_T}{N}$ for an i.i.d fading channel. However, the *instantaneous per-antenna power*, depending on the realization of $\boldsymbol{h}$ and s, may vary dramatically from zero to $\max_{s\in\mathcal{S}} P_T|s|^2$. In order to accommodate the large variations of the instantaneous per-antenna power, the RF amplifier built for MRT signals must have a very wide linear region, inevitably leading to a low power efficiency. The power efficiency for such highly linear RF amplifier is typically about $0.15 - 0.25$ [6, 8].

The difficulty in using highly power-efficient RF amplifiers for large antenna array systems has recently motivated the use of constant envelope (CE) signals for transmission [6]. CE precoding is a nonlinear scheme (with respect to the information symbol). In essence, we constrain the transmitting signal x_i of each antenna to take the form of

$$x_i = \sqrt{\frac{P_T}{N}}e^{\boldsymbol{j}\theta_i}, \text{ for } i = 1,\dots,N, \tag{4}$$

where $\boldsymbol{j} = \sqrt{-1}$, and $\theta_i \in [0, 2\pi)$ is the phase of x_i. In contrast to MRT, the *instantaneous power* of CE signal x_i is fixed at $|x_i|^2 = \frac{P_T}{N}$, which is independent of the channel realization and information message. Hence, the RF amplifiers for CE signals can have a high power efficiency ranging from 0.75 to 0.85 [6, 8].

While the CE signal (4) enables the use of highly power-efficient RF amplifiers, it also presents new challenges. The first challenge is the characterization of the set of all possible noise-free receive signal, which is defined as

$$\mathcal{D} \triangleq \left\{ \sqrt{\frac{P_T}{N}}\sum_{i=1}^{N} h_i e^{\boldsymbol{j}\theta_i} \;\middle|\; \theta_i \in [0, 2\pi), i = 1,\dots,N \right\}. \tag{5}$$

The motivation for characterizing $\mathcal{D}$ is that the design of the input constellation $\mathcal{Q}$ depends on $\mathcal{D}$, since $\mathcal{Q}$ must belong to $\mathcal{D}$. If $\mathcal{D}$ is not known, then it would become unclear how to choose $\mathcal{Q}$.

The second challenge is the phase recovery problem. Once $\mathcal{Q}$ has been designed, for each d in $\mathcal{Q}$, we need to recover a corresponding phase vector $\boldsymbol{\theta} = [\theta_1,\dots,\theta_N]^T$, i.e. a phase vector $\boldsymbol{\theta}$ satisfying the following equation

$$d = \sqrt{\frac{P_T}{N}}\sum_{i=1}^{N} h_i e^{\boldsymbol{j}\theta_i}. \tag{6}$$

Unlike the MRT which is a linear precoding scheme, CE precoding has a highly nonlinear relationship between the noise-free receive signal d and the phase vector $\boldsymbol{\theta}$. This nonlinear phase recovery problem introduces a challenge in efficient CE precoding in practice.

2.2. Prior Works and Contributions

The pioneering work [6] by Mohammed and Larsson shows that $\mathcal{D}$ is a doughnut region given by

$$\mathcal{D} = \{d \in \mathbb{C} \ | \ r \le |d| \le R\}, \tag{7}$$

where r and R are scalars depending on $\boldsymbol{h}$. Moreover, r and R are shown in the same reference to satisfy

$$r \le \sqrt{\frac{P_T}{N}}\|\boldsymbol{h}\|_\infty, \quad R = \sqrt{\frac{P_T}{N}}\|\boldsymbol{h}\|_1. \tag{8}$$

However, the exact value of r is not known. In this paper, by resorting to an induction argument, we show that r can be computed in closed form, thereby completing the result of [6]. Moreover, we show that with very high probability, r is zero in large antenna array systems with i.i.d circular complex Gaussian channel.

The authors in [6] also considered the phase recovery problem (6). Instead of handling (6) directly, the phase recovery problem is formulated as an optimization problem

$$\min_{\boldsymbol{\theta}} \left| d - \sqrt{\frac{P_T}{N}}\sum_{i=1}^{N} h_i e^{\boldsymbol{j}\theta_i} \right|. \tag{9}$$

To solve problem (9), it is proposed in [6] to use the gradient descent method for large N, and for small N ($N \le 10$), to use a two-step algorithm which includes a discrete depth-first-search (DFS) and a gradient descent method. In this paper, our proof of the characterization of $\mathcal{D}$ reveals a much more straightforward way for CE precoding — an exact closed-form solution exists for the phase recovery problem (6). In particular, the complexity of this new closed-form solution is $\mathcal{O}(N)$, which can be very efficiently implemented in practice.

3. MAIN RESULTS

In this section we provide the characterization of the noise-free receive signal region $\mathcal{D}$ and propose an exact phase recovery algorithm. For notational convenience in the subsequent development, denote for $i = 1,\dots,N$,

$$g_i = \sqrt{\tfrac{P_T}{N}}|h_i|, \quad \phi_i = \theta_i + \alpha_i,$$

where α_i is the argument of h_i. Then, (6) and (5) can be equivalently expressed as

$$d = \sum_{i=1}^{N} g_i e^{\boldsymbol{j}\phi_i} \tag{10}$$

$$\mathcal{D} = \left\{ \sum_{i=1}^{N} g_i e^{\boldsymbol{j}\phi_i} \;\middle|\; \phi_i \in [0, 2\pi), i = 1,\dots,N \right\}. \tag{11}$$

Without loss of generality, we assume that g_1 and g_2 are respectively the first and the second largest elements in $\{g_i\}_{i=1}^N$, i.e. $g_1 \ge g_2 \ge g_i \ge 0$ for $i = 3,\dots,N$. We also define, for $i = 1,\dots,N$,

$$\mathcal{D}_i \triangleq \left\{ d_i = \sum_{j=1}^{i} g_j e^{\boldsymbol{j}\phi_j} \;\middle|\; \phi_j \in [0, 2\pi), j = 1,\dots,i \right\}. \tag{12}$$

Physically, $\mathcal{D}_i$ can be interpreted as the noise-free receive signal region when only the first i antennas are used. Note that $\mathcal{D}_N = \mathcal{D}$.

3.1. Characterization of $\mathcal{D}$

Theorem 1. *For $i = 1,\dots,N$,*

$$\mathcal{D}_i = \{d_i \in \mathbb{C} \ | \ r_i \le |d_i| \le R_i\}, \tag{13}$$

where $R_i = \sum_{j=1}^{i} g_j$ and $r_i = \max\{g_1 - \sum_{j=2}^{i} g_j,\ 0\}$. In particular,

$$\mathcal{D} = \{d \in \mathbb{C} \ | \ r \le |d| \le R\}$$

where $R = \sum_{j=1}^{N} g_j$ and $r = \max\{g_1 - \sum_{j=2}^{N} g_j,\ 0\}$.

Theorem 1 states that the radius r_i is actually equal to the difference between the largest channel coefficient g_1 and the sum of the remaining channel coefficients $\sum_{j=2}^{i} g_j$. With the complete knowledge of $\mathcal{D}$, one may design the constellation $\mathcal{Q}$, which lies in $\mathcal{D}$, for optimizing the system utility. For example, one may optimize $\mathcal{Q}$ for higher achievable data rate or smaller error rate performance. As a side benefit of the characterization of $\mathcal{D}$, one can easily check whether or not (10) has a solution by checking the condition $r \le |d| \le R$.

The proof of Theorem 1 is as follows. The main idea is to show by induction from $i = 1$ to $i = N$ that $\mathcal{D}_i = \{d_i \in \mathbb{C} \mid r_i \le |d_i| \le R_i\}$. For $i = 1$, this holds true obviously. For $i = 2$, from the definition of $\mathcal{D}_2$ in (12), we can see that if $d_2 \in \mathcal{D}_2$, then $r_2 = g_1 - g_2 \le |d_2| \le g_1 + g_2 = R_2$ by triangular inequality. Conversely, if d_2 satisfies $g_1 - g_2 \le |d_2| \le g_1 + g_2$, then we need to find (ϕ_1, ϕ_2) satisfying

$$d_2 = g_1 e^{\boldsymbol{j}\phi_1} + g_2 e^{\boldsymbol{j}\phi_2}. \tag{14}$$

It can be easily verified that the (ϕ_1, ϕ_2) given below is a solution,

$$\begin{aligned} \phi_1 &= \arccos\left(\frac{g_1^2 + |d_2|^2 - g_2^2}{2g_1|d_2|}\right) + \omega_2 \\ \phi_2 &= \arccos\left(\frac{g_1^2 + g_2^2 - |d_2|^2}{2g_1 g_2}\right) + \phi_1 + \pi \end{aligned} \tag{15}$$

where ω_2 is the argument of d_2. Hence, (13) is true for $i = 2$.

For $i \ge 3$, we need to invoke the following lemma which reveals the relationship between $\mathcal{D}_i$ and $\mathcal{D}_{i-1}$.

Lemma 1. *Let*

$$\begin{aligned} \mathcal{A} &= \{\, x \in \mathbb{C} \mid r_a \le |x| \le R_a \,\}, \\ \mathcal{B} &= \{\, y \in \mathbb{C} \mid |y| = r_b \,\}, \\ \mathcal{C} &= \{\, z \in \mathbb{C} \mid z = x + y, x \in \mathcal{A}, y \in \mathcal{B} \,\}, \end{aligned}$$

and suppose that

$$R_a - r_a \ge 2r_b. \tag{16}$$

Then, $\mathcal{C}$ is a doughnut region

$$\mathcal{C} = \{\, z \in \mathbb{C} \mid r_c \le |z| \le R_c \,\},$$

with

$$r_c = \max\{r_a - r_b, 0\}, \quad R_c = R_a + r_b.$$

Moreover, for any $z \in \mathcal{C}$, we can construct $x \in \mathcal{A}, y \in \mathcal{B}$ such that $z = x + y$ holds. Specifically, such (x, y) is obtained by setting

$$y = \begin{cases} r_b e^{\boldsymbol{j}\phi_z}, & |z| \ge R_a - r_b \\ r_b e^{\boldsymbol{j}(\phi_z + \pi)}, & |z| < R_a - r_b \end{cases} \tag{17}$$

and $x = z - y$, where ϕ_z denotes the argument of z.

By the definitions of $\mathcal{D}_i$ and $\mathcal{D}_{i-1}$, $\mathcal{D}_i$ can be written as

$$\mathcal{D}_i = \{d_i \in \mathbb{C} \mid d_i = d_{i-1} + \tilde{d}_i, d_{i-1} \in \mathcal{D}_{i-1}, |\tilde{d}_i| = g_i\}. \tag{18}$$

Suppose that $\mathcal{D}_{i-1}$ is a doughnut region with radii $r_{i-1} = \max\{g_1 - \sum_{j=2}^{i-1} g_j, 0\}$ and $R_{i-1} = \sum_{j=1}^{i-1} g_j$. Then,

$$R_{i-1} - r_{i-1} \ge R_2 - r_2 = 2g_2 \ge 2g_i,$$

which satisfies the premise (16) of Lemma 1. Applying Lemma 1 to (18), we have that $\mathcal{D}_i$ is a doughnut region with radii $r_i = \max\{r_{i-1} - g_i, 0\} = \max\{g_1 - \sum_{j=2}^{i} g_j, 0\}$ and $R_i = R_{i-1} + g_i = \sum_{j=1}^{i} g_j$.

3.2. A probability bound on $\Pr\{r > 0\}$

From the expression of r in Theorem 1, one may expect that for an i.i.d. fading channel, r is actually zero with high probability when N is large. Indeed, this is true for the i.i.d circular complex Gaussian channel.

Proposition 1. *Suppose that each element of $\boldsymbol{h}$ follows an i.i.d circular complex Gaussian distribution with zero mean and unit variance. Then,*

$$\frac{1}{N^{N-2}} \le \Pr\{r > 0\} \le \frac{1}{(N-1)!}.$$

Proof. The proof is based on the direct integration of the distribution of the ordered statistics of $\{|h_i|\}_{i=1}^N$ [9]. Details are omitted due to space limit. □

The pioneering work [6] has provided a similar result that $\Pr\{r \ge c(\log N)/\sqrt{N}\}$ converges to zero as N goes to infinity for all $c > 0$. We can see that the result in Proposition 1 provides a better guarantee of r being zero. Proposition 1 states that $\Pr\{r > 0\}$ decays factorially fast in N. For example, for $N = 10$, we have $\Pr\{r > 0\} \le 3 \times 10^{-7}$. For very large array systems, where N could be more than 100, it is expected that $\Pr\{r > 0\}$ is virtually zero. This indicates that with high probability, the doughnut region is essentially a disk region. Therefore, the constellation $\mathcal{Q} = \alpha\mathcal{S}$ can be simply designed by choosing $\mathcal{S}$ as a commonly used constellation such as QAM, and choosing α as the largest positive number such that $\alpha\mathcal{S}$ belongs to $\mathcal{D}$. More sophisticated methods of designing $\mathcal{Q}$ can also be found in [10, 6, 11, 12].

3.3. Exact Phase Recovery

In this subsection, we propose an exact phase recovery algorithm for (10) which has a linear complexity in the problem size N.

The main idea of the proposed algorithm can be derived from our proof of Theorem 1. Assume that $r \le |d| \le R$, for otherwise (10) has no solution by Theorem 1. Let $d_N \triangleq d$. Observe from (18) that if d_N belongs to $\mathcal{D}_N$, then a ϕ_N exists such that $d_N - g_N e^{\boldsymbol{j}\phi_N}$ belongs to $\mathcal{D}_{N-1}$; again, a ϕ_{N-1} exists such that $(d_N - g_N e^{\boldsymbol{j}\phi_N}) - g_{N-1} e^{\boldsymbol{j}\phi_{N-1}}$ belongs to $\mathcal{D}_{N-2}$. Repeating this argument, it can be seen that d_N can be decomposed in the form of $d_N = \sum_{i=1}^N g_i e^{\boldsymbol{j}\phi_i}$. Hence, it suffices to choose ϕ_i such that

$$d_{i-1} \triangleq d_i - g_i e^{\boldsymbol{j}\phi_i} \in \mathcal{D}_{i-1}. \tag{19}$$

from $i = N$ down to $i = 2$. At the end of the process, the resultant $\boldsymbol{\phi}$ is a solution[1] of (10).

The proof of Theorem 1 already offers a way to determine a ϕ_i for (19). By (17) in Lemma 1, we can see that for $i \ge 3$, ϕ_i can be chosen as

$$\phi_i = \begin{cases} \omega_i, & \text{if } |d_i| \ge R_{i-1} - g_i, \\ \omega_i + \pi, & \text{if } |d_i| < R_{i-1} - g_i, \end{cases} \tag{20}$$

where ω_i is the argument of d_i. For $i = 2$, by noting $r_1 = R_1 = g_1$, it can be seen that (19) is equivalent to the equation in (14). Then ϕ_2 and ϕ_1 can be chosen as (15).

We can see that the proposed algorithm only involves N steps of operations. Hence, the complexity of the proposed algorithm is $\mathcal{O}(N)$. The description of the proposed algorithm is complete, and we provide the pseudo code in Algorithm 1.

[1] Note that ϕ_1 is automatically obtained when choosing ϕ_2 such that $d_2 - g_2 e^{\boldsymbol{j}\phi_2} = d_1$, since d_1 is of the form of $d_1 = g_1 e^{\boldsymbol{j}\phi_1}$.

Algorithm 1: Exact phase recovery

input : $d = |d|e^{\boldsymbol{j}\omega}$, $\{g_i\}_{i=1}^N$ with $g_1 \geq g_2 \geq g_i \geq 0$, $\forall\, i \geq 3$.

$R = \sum_{j=1}^N g_j$;
$r = \max\{g_1 - \sum_{j=2}^N g_j, 0\}$;
if $|d| > R$ **or** $|d| < r$ **then**
    return. (There is no solution);
end
$d_N = |d|$;
for $i \leftarrow N$ **to** 3 **do**
    $R_{i-1} = \sum_{j=1}^{i-1} g_j$;
    if $d_i \geq R_{i-1} - g_i$ **then**
        $\phi_i = \omega$;
        $d_{i-1} = d_i - g_i$;
    else
        $\phi_i = \omega + \pi$;
        $d_{i-1} = d_i + g_i$;
    end
end
$\phi_1 = \omega + \arccos \frac{g_1^2 + d_2^2 - g_2^2}{2g_1 d_2}$;
$\phi_2 = \phi_1 + \pi + \arccos \frac{g_1^2 + g_2^2 - d_2^2}{2g_1 g_2}$;
output: $\{\phi_i\}_{i=1}^N$

	Problem size N				
	10	50	100	150	200
Gradient descent	3.1e-3	1.1e-2	2.2e-2	3.3e-2	3.7e-2
Proposed	7.8e-5	1.3e-4	1.9e-4	2.5e-4	3.1e-4

Table 1. Average running time (in seconds) of the gradient descent and the proposed methods.

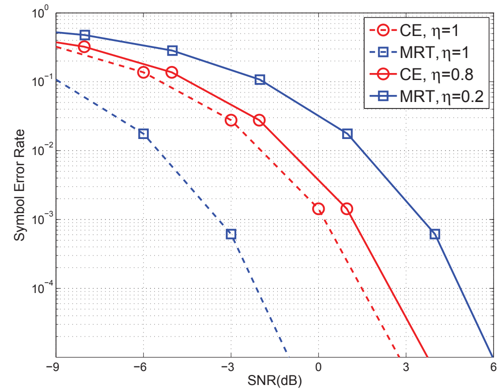

Fig. 1. Symbol error rate comparison.

4. SIMULATIONS

In this section, we use simulations to demonstrate the performance advantage of CE precoding over MRT, as well as the computational efficiency of the proposed method over the gradient descent method. In the simulations, the channel $\boldsymbol{h}$ is generated following an i.i.d circular complex Gaussian distribution with zero mean and unit variance. The constellation $\mathcal{Q} \subset \mathcal{D}$ is chosen as a scaled version of the 16-QAM constellation. The SNR is defined as $\mathrm{SNR} = \frac{P_T}{\eta \sigma_\nu^2}$, where η is the power efficiency of the RF amplifiers. We use the Armijo rule [13] for gradient descent, and we stop the algorithm when the objective value of (9) is smaller than $\epsilon = 0.01$.

In Table. 1, we compare the average running times of the proposed method and the gradient descent method for solving (9). We can see that the proposed algorithm is at least 50 times faster than that of the gradient descent for all problem sizes tested.

Fig. 1 shows the symbol error rate for problem size $N = 128$. The result is obtained by averaging over 10^6 channel realizations. It can be seen that if the power efficiencies of MRT and CE are both equal to one, MRT is better than CE precoding. This is expected since CE precoding is a more restrictive way of transmission. However, we must take into account the different power efficiencies for fair comparison. As suggested by [6, 8], $\eta = 0.2$ is chosen for MRT and $\eta = 0.8$ is chosen for CE precoding, which amounts to about 7dB and 1dB SNR penalty for MRT and CE precoding, respectively. We can see that under this setting, CE precoding outperforms MRT by about 2.5dB. This demonstrates that CE precoding is a very promising transmission technique in very large antenna array communication.

5. CONCLUSION

In this paper, we provided a complete characterization of the doughnut region of CE precoding for single-user MISO channels, and showed that the inner radius of the doughnut region is zero with very high probability. We further proposed a very simple method to exactly recover the phases of the CE signals. An interesting extension is to apply CE precoding to multi-user MISO downlink channels [7]. In this multi-user scenario, the characterization of receive signals of all users and the phase recovery of the CE signal remain open problems.

6. APPENDIX: PROOF OF LEMMA 1

First, we show that any $z \in \mathcal{C}$ must satisfy $r_c \leq |z| \leq R_c$. For any $x \in \mathcal{A}$, $y \in \mathcal{B}$, we have that

$$|x + y| \leq |x| + |y| \leq R_a + r_b = R_c$$

and that

$$|x + y| \geq \max\{0, |x| - |y|\} \geq \max\{0, r_a - r_b\}.$$

This means that $r_c \leq |z| \leq R_c$ must hold.

Next, we show that any $z \in \mathbb{C}$, $r_c \leq |z| \leq R_c$ must lie in $\mathcal{C}$. The proof is by construction. We consider two cases, namely $|z| \geq R_a - r_b$, and $|z| < R_a - r_b$. For the case of $|z| \geq R_a - r_b$, set

$$y = r_b e^{\boldsymbol{j}\phi_z}, \quad x = (|z| - r_b)e^{\boldsymbol{j}\phi_z}.$$

It holds true that $z = x + y$, and that $y \in \mathcal{B}$. The question left is whether $x \in \mathcal{A}$. We first observe that $|x| \geq |z| - r_b \geq R_a - 2r_b \geq r_a$, where the last inequality is due to (16). Moreover, we have $|x| = |z| - r_b \leq R_c - r_b = R_a$. Hence, x lies in $\mathcal{A}$. For the case of $|z| < R_a - r_b$, set

$$y = r_b e^{\boldsymbol{j}(\phi_z + \pi)}, \quad x = (|z| + r_b)e^{\boldsymbol{j}\phi_z}.$$

Again, since $z = x + y$ and $y \in \mathcal{B}$, we seek to show $x \in \mathcal{A}$. One can easily verify that $|x| = |z| + r_b \geq r_c + r_b \geq r_a$ and $|x| = |z| + r_b < R_a - r_b + r_b = R_a$. Hence, $x \in \mathcal{A}$ is true. We therefore conclude that any $z \in \mathbb{C}$, $r_c \leq |z| \leq R_c$, satisfies $z = x + y$ for some $x \in \mathcal{A}, y \in \mathcal{B}$, or equivalently $z \in \mathcal{C}$. It is also clear from the above proof that such (x, y) can be constructed via (17) and $x = y - z$.

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