ZIV-ZAKAI LOWER BOUND FOR UWB BASED TOA ESTIMATION WITH MULTIUSER INTERFERENCE

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ABSTRACT

The aim of this paper is to derive a Ziv-Zakai lower bound (ZZLB) for the time of arrival (TOA) estimation in single-path (SP) additive white Gaussian noise (AWGN) channels and in the presence of multiuser interference (MUI). Time-hopping pulse position modulated (TH-PPM) ultra-wideband (UWB) signals are considered. Following a classical approach and extending it for the scenario under consideration, we compute the ZZLB by converting the related estimation problem into a binary detection one. To obtain the bit error rate (BER) of the corresponding detection problem, we both consider the exact handling of the MUI and the modeling of the MUI by means of a Gaussian approximation (GA). We compare the performance of the single-user maximum likelihood (ML) TOA estimator with the bounds.

***Index Terms*—** Time of arrival estimation, Ultra wideband communication, Multiple access interference, Ziv-Zakai lower bound

1. INTRODUCTION

Ultra-wideband (UWB) signals provide very accurate positioning given their short duration pulses. The interest in the use of UWB for localization has grown since the Federal Communications Commission (FCC) allowed unlicensed operations between 3.1 and 10.6 GHz in the United States, in 2002.

Different authors investigated lower bounds for UWB signals. For instance, in [1] the baseline expressions for the Cramèr-Rao lower bound (CRLB) and the Ziv-Zakai Lower Bound (ZZLB) are given for both single-path (SP) and multipath (MP) channels. In [2] the authors propose lower bounds for realistic UWB channels. Specifically, they report ZZLB averaged over several channel impulse response (CIR) realizations and they present an approximation of the error probability. Also for realistic channels, in [3], the CRLBs are given when up to three overlapping MP components (MPCs) are considered. In [4] the ZZLB is derived for convolutive random channels with instantaneous channel knowledge at the receiver, while in [5] the topic is readdressed considering statistical knowledge of the channels.

In the current paper we address the particular issue of computing the ZZLB in additive white Gaussian noise (AWGN) plus multiuser interference (MUI) channels for time-hopping pulse position modulation (TH-PPM) signals. To the best of our knowledge, this has not been examined before. However, the bit error rate (BER) analysis of TH-PPM signals in the presence of MUI has been widely studied. Since the ZZLB can be computed by transforming the estimation performance problem into a binary detection one, we capitalize on the existing BER analysis to derive the ZZLB in an AWGN plus MUI environment.

Regarding MUI modeling, the error probability when the interference is assumed to be a Gaussian process is given in [6]. Nevertheless, it is well known that the probability density function (pdf) of a TH-PPM UWB interference signal does not follow a Gaussian distribution [7]. A review of the probability density functions to be considered for modeling MUI can be found in [7]. In [8] and [9] the authors obtain the characteristic function (CF) of the interference in order to derive exact BERs. In [9], the authors also give expressions for the BER performance in MP channels. In the current paper we use both exact CF based BER expressions and Gaussian approximations (GA) to obtain the ZZLB. We also investigated how the single-user maximum likelihood (ML) TOA estimator performs with respect to the bounds.

The paper is organized as follows. In Sec. 2 we describe the system model. In Sec. 3 we review the ZZLB. In Sec. 4 we derive the BERs for the system model described in Sec. 2 and derive the associate ZZLBs. In Sec. 5 numerical results are reported and discussed. Finally, Sec. 6 concludes the paper.

2. THE SYSTEM MODEL

The TH-PPM signal transmitted by user n is defined as in [6] and given by

$$s_n(t) = \sum_{i=-\infty}^{\infty} \sqrt{\frac{E_b}{N_c}} b_n\left(t - iT_p\right) \tag{1}$$

where $b_n(t)$ is the template signal or signature enabling to identify user n, T_p is the length of the period, and E_b is the energy of $s_n(t)$ for a duration T_p. The signature $b_n(t)$ contains N_c pulses inside the period T_p and can be written as

$$b_n(t) = \sum_{j=0}^{N_c-1} p(t - jT_f - c_n(j)T_c) \tag{2}$$

where $p(t)$ is the transmitted pulse, normalized to be unit energy that is $\int_{-\infty}^{\infty} p^2(t) = 1$, T_f is the frame duration or pulse repetition period, c_n is the time hopping code (THC) of user n ($c_n(j) \in \{0, N_h - 1\}$), T_c is the chip duration, and N_c the code length ($T_p = N_c T_f$). N_h represents the number of possible pulse positions per frame ($T_f = N_h T_c$). Note that no data transmission has been considered since we are only interested by the estimation.

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The signal received over an SP AWGN communication channel when N_u users are active can be expressed as

$$r(t) = A_1 s_1 (t - \tau) + \sum_{n=2}^{N_u} A_n s_n (t - \tau_n) + n(t) \quad (3)$$

where $s_1(t)$ is the signal from the user of interest, A_1 its amplitude, and τ its time delay and also the target parameter for the TOA estimation. $\{A_n\}_{n=2}^{N_u}$ and $\{\tau_n\}_{n=2}^{N_u}$ represent the gain and time delay of the interfering users. Signal $n(t)$ is AWGN with double-sided spectral density $N_0/2$. The delays τ and $\{\tau_n\}_{n=2}^{N_u}$ are assumed to be uniformly distributed over one period $[0, T_p]$.

For the pulse $p(t)$, we consider the normalized second derivative of a Gaussian (doublet)

$$p(t) = \frac{1}{\sqrt{3T_m/8}} \left[1 - 4\pi \left(\frac{t}{T_m} \right)^2 \right] e^{-2\pi \left(\frac{t}{T_m} \right)^2} : \quad (4)$$

where T_m is a variable that affects the width of the pulse. The autocorrelation of the normalized doublet is

$$R_p(x) = \left[1 - 4\pi \left(\frac{x}{T_m} \right)^2 + \frac{4\pi^2}{3} \left(\frac{x}{T_m} \right)^4 \right] e^{-\pi \left(\frac{x}{T_m} \right)^2}. \quad (5)$$

3. ZIV ZAKAI LOWER BOUND

In this section we present a short review of the ZZLB. The full derivation of the bound can be found in [10, 11]. We are interested in a lower bound for the mean square estimation error of the time delay $E\{\epsilon^2\} = E\{(\hat{\tau} - \tau)^2\}$, where $E\{\cdot\}$ denotes the expected value. The ZZLB can be obtained from the identity

$$E\{\epsilon^2\} = \frac{1}{2} \int_0^\infty \Pr \left(|\epsilon| \geq \frac{h}{2} \right) h dh \quad (6)$$

and lower bounding $\Pr \left(|\epsilon| \geq \frac{h}{2} \right)$.

Now consider a suboptimal decision scheme where the parameter is first estimated and a nearest-neighbor decision is made afterwards

$$\begin{aligned} H_0 : \tau = a \qquad & \text{if } \hat{\tau} \leq a + \frac{h}{2} \\ H_1 : \tau = a + h \quad & \text{if } \hat{\tau} > a + \frac{h}{2}. \end{aligned} \quad (7)$$

The probability of error for this suboptimum detector can be lower bounded by the minimum error probability $P_e(a, a+h)$ given by the likelihood ratio test . $\Pr \left(|\epsilon| \geq \frac{h}{2} \right)$ can be shown [11] to be greater or equal to

$$\int_{-\infty}^{\infty} \left(p_\tau (a) + p_\tau (a + h) \right) P_e(a, a + h) da \quad (8)$$

where $p_\tau (\tau)$ is the pdf of the TOA. Given that $p_\tau (\tau)$ follows a uniform distribution in the interval $[0, T_p]$, the lower bound on the estimation error can then be expressed as

$$E\{\epsilon^2\} \geq \text{ZZLB} = \frac{1}{T_p} \int_0^{T_p} h \int_0^{T_p - h} P_e(a, a + h) da dh. \quad (9)$$

Moreover, when $P_e(a, a + h)$ is independent of a we can write $P_e(h)$ instead. Assuming this, the ZZLB is given by

$$\text{ZZLB} = \frac{1}{T_p} \int_0^{T_p} h(T_p - h) P_e(h) dh. \quad (10)$$

4. BIT ERROR PROBABILITY AND ZZLB

In the detection scenario associated with the ZZLB , the receiver is required to decide between $s_1(t-\tau)$ or $s_1(t-\tau-h)$. Without loss of generality we assume $\tau = 0$. Therefore, the receiver template for user 1 is $m_1(t) = b_1(t) - b_1(t-h)$. The decision variable at the correlator output is

$$r = \int_{-\infty}^{\infty} r(t) m_1(t) dt = S + I + n \quad (11)$$

where S is the signal component, I is the interference component and n is the noise component. The receiver template $m_1(t)$ consists of N_c waveforms $v(t) = p(t) - p(t-h)$. The correlation of pulse $p(t)$ with waveform $v(t)$ is defined as

$$R_{pv}(x) = \int_{-\infty}^{\infty} p(t-x) v(t) dt = R_p(x) - R_p(x-h) \quad (12)$$

where $R_p(x)$ is the auto-correlation of the waveform $p(t)$. The autocorrelation at the origin is equal to $R_{pv}(0) = 1 - R_p(h)$ since $R_p(0) = 1$ and $R_p(x) = R_p(-x)$.

The probability of error for an optimum receiver can be obtained from

$$P_e(h) = \Pr(r < 0 | s_1(t)) = \Pr(S + I + n < 0 | s_1(t)). \quad (13)$$

4.1. Gaussian Approximation

The probability $P_e(h)$ can easily be computed when we use the Gaussian approximation for the interference component, meaning that I is assumed to be a zero-mean Gaussian random process with variance σ_I^2. The noise component n has a variance σ_n^2. The probability of error is then equal to

$$P_e(h) \simeq P_e^{GA}(h) = Q(\sqrt{\text{SINR}}) \quad (14)$$

where $Q(x) = (1/\sqrt{2\pi}) \int_x^\infty \exp(-t^2/2)) dt$ is the Q-function, expressed in terms of the complementary error function as $Q(x) = (1/2)\text{erfc}(x/\sqrt{2})$. SINR is the signal to interference plus noise ratio and can be written as

$$\text{SINR} = \frac{S^2}{\sigma_n^2 + \sigma_I^2} = \left(\text{SNR}^{-1} + \text{SIR}^{-1} \right)^{-1} \quad (15)$$

where SNR and SIR are the signal to noise and signal to MUI ratios, respectively. The useful signal energy S^2 is given by

$$\begin{aligned} S^2 &= \left(\int_{-\infty}^{\infty} A_1 \sqrt{\frac{E_b}{N_c}} b_1(t) m_1(t) dt \right)^2 \\ &= A_1^2 \frac{E_b}{N_c} N_c^2 \left(\int_{-\infty}^{\infty} p(t) v(t) \right)^2 \\ &= A_1^2 E_b N_c (1 - R_p(h))^2. \end{aligned} \quad (16)$$

The noise signal is equal to

$$n = \int_{-\infty}^{\infty} n(t) m_1(t) dt \quad (17)$$

and its variance σ_n^2 is equal to

$$\sigma_n^2 = E\{n^2\} = N_0 N_c (1 - R_p(h)). \quad (18)$$

The multiuser interference term is

$$I=\sum_{n=2}^{N_u} I_n=\sum_{n=2}^{N_u} A_n \int_{-\infty}^{\infty} s_n\left(t-\tau_n\right) m_1(t) dt. \quad (19)$$

In [9], the authors model the interference as being proportional to the number of collisions k multiplied by the cross-correlation $R_{pv}(\alpha)$ of one pulse with the template signal. Depending on the similarity between the THCs c_n and c_1, and the time delay τ_n the interference can be large, low or inexistent. The interference from user n can therefore be expressed as

$$I_n=k A_n \sqrt{\frac{E_b}{N_c}} R_{pv}(\alpha) \quad (20)$$

where $k \leq N_c$ accounts for the number of collisions between codes c_n and c_1, and α is a uniformly distributed variable over $[-\frac{T_c}{2}, \frac{T_c}{2}]$. The collisions k follow a binomial distribution:

$$\begin{aligned} \operatorname{Pr}(k) &= C_k^{N_c}\left(1 / N_h\right)^k\left(1-1 / N_h\right)^{N_c-k} \\ &= \operatorname{Pr}\left(I_n=k A_n \sqrt{\frac{E_b}{N_c}} R_{pb}(\alpha)\right). \end{aligned} \quad (21)$$

The variance of the multiuser interference is then obtained by $\sigma_I^2=E\{I^2\}=\sum_{n=2}^{N_u} E\{I_n^2\}$, given that interferences I_n from different users are independent and identically distributed random variables (i.i.d.) and zero mean. The expectation is taken here over k and α. After averaging one gets

$$\begin{aligned} E\{I_n^2\} &= A_n^2 \frac{E_b}{N_c} \sum_{k=0}^{N_c} k^2 \operatorname{Pr}(k) \frac{1}{T_c} \int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} R_{pv}(\alpha)^2 d\alpha \\ &= A_n^2 \frac{E_b}{N_c} \frac{N_c}{N_h}\left(1+\frac{N_c-1}{N_h}\right) \frac{1}{T_c} \sigma_R^2 \end{aligned} \quad (22)$$

where $\sigma_R^2=\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} R_{pv}(\alpha)^2 d\alpha$ depends on h since $R_{pv}(\alpha)=R_p(\alpha)-R_p(\alpha-h)$. As we can see, all components depend on the time shift parameter h. Finally, the SNR can be expressed as

$$\mathrm{SNR}=\frac{S^2}{\sigma_n^2}=\frac{A_1^2 E_b(1-R_p(h))}{N_0}=\gamma(1-R_p(h)) \quad (23)$$

where $\gamma=A_1^2 E_b / N_o$. The SIR is obtained as

$$\mathrm{SIR}=\frac{S^2}{\sigma_I^2}=\frac{N_c T_f(1-R_p(h))^2}{\left(1+\frac{N_c-1}{N_h}\right) \sigma_R^2 \sum_{k=2}^{N_u} \frac{A_k^2}{A_1^2}}. \quad (24)$$

With the SINR term defined in (15) , the expression of GA based ZZLB is readily obtained from (10) using $P_e^{GA}(h)$ defined in (14) instead of $P_e(h)$

$$\mathrm{ZZLB}^{GA}=\frac{1}{T_p} \int_0^{T_p} h(T_p-h) P_e^{GA}(h) dh. \quad (25)$$

4.2. CF Method

In this subsection we remind the reader of the exact BER expression obtained in [9] using hte CF of the interference. We denote the ber as $P_e^{CF}(h)=P_e(h)$. Note that now we are computing the exact probability of error. The computation of $\operatorname{Pr}(S+I+n<0 | s_1(t))$ can be computed as a function of the CF as follows

$$P_e^{CF}(h)=\frac{1}{2}-\frac{1}{\pi} \int_0^{\infty} \frac{\sin(S w)}{w} \Phi_n(w) \Phi_I(w) dw \quad (26)$$

where $\Phi_n(w)$ and $\Phi_I(w)$ are the CFs of the noise and the interference, respectively. The CF of the noise is $\Phi_n(w)=\exp\left(\frac{-\sigma_n^2 w^2}{2}\right)$. With the change of variable $w_o=w A_1 \sqrt{\frac{E_b}{N_c}}$, the error probability becomes

$$\begin{aligned} P_e^{CF}(h)=\frac{1}{2}-\frac{1}{\pi} \int_0^{\infty} & \frac{\sin(N_c(1-R_p(h)) w_o)}{w_o} \\ & \times \exp\left(\frac{-N_c^2(1-R_p(h)) w_o^2}{2 \gamma}\right) \Phi_I(w_o) dw_o \end{aligned} \quad (27)$$

where $\Phi_I(w_o)$ is the CF of the normalized interference $\overline{I}=I /(A_1 \sqrt{E_b / N_c})$. Since the interfering terms of the different users n are i.i.d., the CF of $\overline{I}$ can be computed as follows

$$\Phi_I(w_o)=\prod_{n=2}^{N_u} \Phi_{I_n}(w_o) \quad (28)$$

where $\Phi_{I_n}(w_o)$ is the CF of the normalized interference from user n, $\overline{I}_n$. The CF is defined as

$$\Phi_{I_n}(w_o)=E\left\{\exp\left(j w_o\left(\overline{I}_n\right)\right)\right\} \quad (29)$$

where $\overline{I}_n=k A_n R_{pv}(\alpha) / A_1$ and the expectation is taken over k and α. Thanks to the symmetry properties of $p(t)$, the CF of $\overline{I}_n$ can be further expressed as

$$\Phi_{I_n}(w_o)=\int_{-\frac{T_c}{2}}^{\frac{T_c}{2}} \sum_{k=0}^{N_c} \frac{\operatorname{Pr}(k)}{T_c} \cos\left(w_o k \frac{A_n}{A_1} R_{pv}(\alpha)\right) d\alpha. \quad (30)$$

Finally, the CF based ZZLB is obtained by inserting in (10) the expression of $P_e^{CF}(h)$ given in (27).

$$\mathrm{ZZLB}^{CF}=\frac{1}{T_p} \int_0^{T_p} h(T_p-h) P_e^{CF}(h) dh. \quad (31)$$

5. RESULTS AND DISCUSSION

In this section we present numerical and simulation results. For all results we use the Gaussian monocycle described in (4). The TH-PPM parameters used are listed in Table 1. The $c_n(i)$ are integer values uniformly distributed between $[0, N_h-1]$. Perfect power control is always implemented, meaning that all A_n are equal. All results are root mean square errors (RMSE) reported as a function of $\gamma=A_1^2 E_b / N_o$.

In order to compare the ZZLBs with the performance of a practical estimator, simulations have also been conducted for the single-user maximum likelihood estimator (MLE), provided hereafter:

$$\hat{\tau}=\arg \max_\tau \int_{-\infty}^{\infty} r(t) b_1(t-\tau) dt. \quad (32)$$

Figure 1 reports the RMSE values achieved with the two bounds and with the single-user MLE. Figure 2 shows the same results but with a zoom on the high SNRv zone only. From Figure 1 it appears that the three different SNR regions [12] can easily be identified. The high SNR region corresponds to situations where the estimates are close to the main peak of the auto-correlation function, and therefore close to correct values. In the region of intermediate SNRs, the estimates are impacted by the secondary lobes of the auto-correlation function and also by the cross-correlation with the codes of the other users. This creates the so-called ambiguity effect, meaning that the

Table 1. Parameters of TH-PPM System

Parameter	Notation	Value
Pulse Width Modifier	T_m	1 ns
Chip Width	T_c	2.5 ns
Frame Width	T_f	37.5 ns
Code Length	N_c	15
Number of Chips per Frame	N_h	15
Number of Users	N_u	1, 2, 4, 8

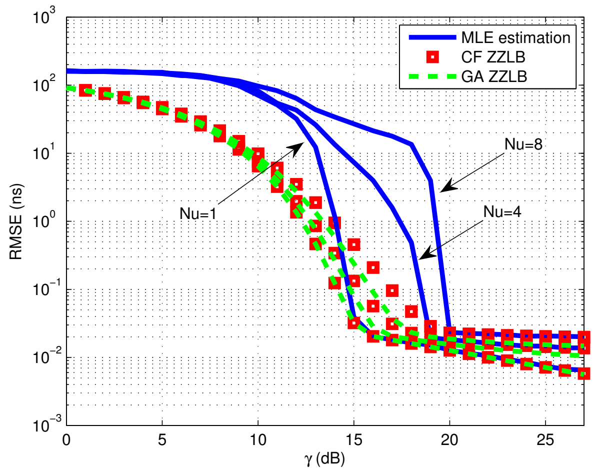

Fig. 1. RMSE with respect to γ. $N_u = 1, 4, 8$.

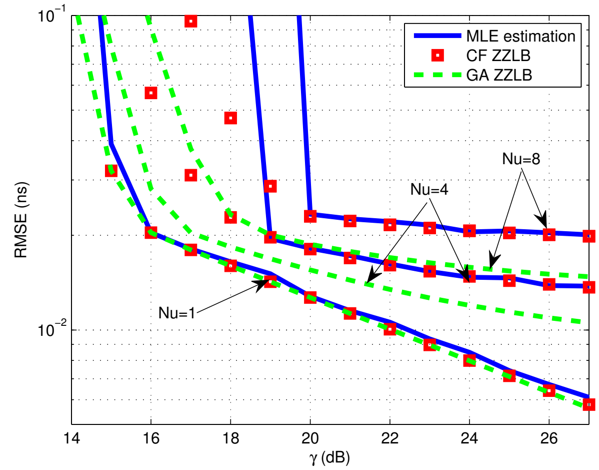

Fig. 2. RMSE with respect to γ. $N_u = 1, 4, 8$.

estimates can be associated with a peak which is not the main or the correct one. In the low SNR zone, the estimates are spread all over the observation interval.

From the two figures it appears that the GA based ZZLB turns out to be more optimistic than the CF based ZZLB. As reported in [11], the GA tends to underestimate the value of the error probability, which leads to a looser bound.

From the two figures it can also be noticed that the MLE performs close to the bounds at high SNR. This behaviour has been noticed in [8]. It can be concluded that the bounds nicely predict the performance of the MLE at high SNR.

6. CONCLUSIONS AND FUTURE WORK

This paper has investigated the ZZLBs for the TOA estimation in SP AWGN channels and in the presence of MUI. TH-PPM UWB signals have been considered. The ZZLB has been obtained by using the known method which consists in converting the related estimation problem into a equivalent binary detection one. BERs have been obtained both by handling correctly the MUI and by modeling the MUI by means of the Gaussian approximation. The bounds have been illustrated by numerical results and compared with the performance of the single-user MLE.

In the future MP channels will also be considered.

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