COORDINATE UPDATE ALGORITHMS FOR ROBUST POWER LOADING FOR THE MISO DOWNLINK WITH OUTAGE CONSTRAINTS AND GAUSSIAN UNCERTAINTIES

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ABSTRACT

We consider the problem of power allocation for the multiple-input single-output (MISO) downlink with uncertain channel state information at the transmitter. The uncertainty is modeled probabilistically and the receivers specify quality-of-service (QoS) constraints in terms of a target signal-to-interference-and-noise ratio that is to be achieved with a given outage probability. The proposed approach is based on a deterministic characterization of the outage probability, and mildly conservative approximations thereof. Although the resulting optimization problems are not convex, the good solutions that we obtain using straightforward coordinate update algorithms provide significantly better performance than the existing convex approaches because the approximations are less conservative.

Index Terms— quality-of-service, uncertainty, robustness

1. INTRODUCTION

It has long been recognized that the provision of multiple antennas at the transmitter of a downlink system has the potential to significantly improve the efficiency with which messages can be communicated; e.g., [1,2]. In the case of fixed-rate traffic, one way in which that potential can be realized is to design a linear transmitter so as to minimize the power that is required to enable reliable communication to each receiver at their specified target rate; e.g., [3,4]. with accurate channel state information (CSI), optimal linear precoders for a variety of such quality-of-service (QoS) problems have been obtained; e.g., [4–8]. In practice, however, the CSI that can be made available at the transmitter is imperfect, due to estimation errors, quantization, feedback delay, feedback errors, and other effects; e.g., [9]. A prudent approach for dealing with the resulting uncertainty in the CSI is to incorporate a model for the uncertainty into the transmitter design. One approach to doing so is to presume a bounded model for the uncertainty and to design a transmitter that satisfies the QoS requirements even for the worst case of the uncertainties admitted by the model; e.g., [10–13]. In this paper we will consider an alternate approach in which the uncertainty is modeled probabilistically, and the QoS requirements are to be satisfied up to a given probability of outage. Several techniques for finding good linear precoders for such problems have been developed [14–16]. In addition, "power loading" techniques have been developed for cases in which the directions of transmission have already been chosen [17, 18].

The principle that underlies the previous approaches to outage-based QoS problems for the downlink [15–18] is to seek a deterministic approximation of the outage constraint that is conservative and can be represented in a form that is convex in design variables. Conservatism means that any feasible point in the resulting restricted optimization problem will satisfy the original outage constraint, and convexity means that a globally optimal solution to the restricted optimization problem can be efficiently found. The principles that underlie the proposed approaches are somewhat different. In Section 3, under the presumption of a Gaussian model for channel uncertainties, we develop a power loading technique for arbitrary beamforming directions that does not involve an approximation of the outage constraint, but employs the precise deterministic representation in [19]. Unlike the previous approaches, the resulting optimization problem is not convex, but we develop a straightforward cyclic coordinate descent algorithm that typically produces good solutions. Indeed, in a number of scenarios our suboptimal solutions to the precise formulation of the problem provide superior performance to that of the globally optimal solutions to the conservative approximation. In Section 4 we use insight from that development to construct a more computationally efficient power-loading technique for the case of nominally "zero-forcing" beamforming directions. While that technique does involve a conservative approximation, the structure of the approximation is quite different from those that have been previously applied, and numerical experience suggests that it can be significantly less conservative. Interestingly, in some important scenarios the lower level of conservatism in the approximation means that the proposed power loading algorithm with nominally zero-forcing directions yields better performance than existing techniques in which the power loading and directions are designed jointly.

2. SYSTEM MODEL AND PROBLEM FORMULATION

We consider a narrowband single-cell downlink scenario in which a base station with N_t antennas sends independent messages to K users, each of which is equipped with a single antenna. The based station employs linear precoding and the transmitted signal is

$$\mathbf{x} = \sum_{k=1}^{K} \mathbf{w}_k s_k = \mathbf{W}\mathbf{s}, \tag{1}$$

where $\mathbf{w}_k \in \mathbb{C}^{N_t}$ is the beamforming vector for the k^{th} user and forms the k^{th} column of the precoding matrix $\mathbf{W}$, and s_k is the symbol to be sent to the k^{th} user, normalized so that $E\{\mathbf{s}\mathbf{s}^H\} = \mathbf{I}$. We store the vector of complex channel gains from the transmitting antennas to the k^{th} receiver in the row vector $\mathbf{h}_k^H$. The transmitter has an estimate $\hat{\mathbf{h}}_k^H$ of $\mathbf{h}_k^H$, and we will model the uncertainty in this estimate additively, as

$$\mathbf{h}_k^H = \hat{\mathbf{h}}_k^H + \mathbf{e}_k^H, \qquad k = 1, \ldots, K. \tag{2}$$

The error $\mathbf{e}_k$ is the result of a number of different effects, and we will model it as a circular complex Gaussian random variable with

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zero mean and covariance matrix $\mathbf{C}_k$; i.e., $\mathbf{e}_k \sim CN(\mathbf{0}, \mathbf{C}_k)$. The received signal at user k is $y_k = \mathbf{h}_k^H \mathbf{W}\mathbf{s} + z_k$, where $z_k \sim CN(0, \sigma_k^2)$ denotes the additive Gaussian noise at that receiver. This signal can be rewritten as

$$y_k = \hat{\mathbf{h}}_k^H \mathbf{w}_k s_k + \big(\hat{\mathbf{h}}_k^H \bar{\mathbf{W}}_k + \mathbf{e}_k^H \mathbf{W}\big)\mathbf{s} + z_k, \tag{3}$$

where $\bar{\mathbf{W}}_k = [\mathbf{w}_1, ..., \mathbf{w}_{k-1}, \mathbf{0}, \mathbf{w}_{k+1}, ..., \mathbf{w}_K]$. The problem of interest in this paper is to minimize the transmission power required to provide each user with a specified quality of service. We specify the QoS in terms of an outage probability for the following measure of the SINR at user k,

$$\mathsf{SINR}_k = \frac{|\hat{\mathbf{h}}_k^H \mathbf{w}_k|^2}{(\hat{\mathbf{h}}_k^H \bar{\mathbf{W}}_k + \mathbf{e}_k^H \mathbf{W})(\bar{\mathbf{W}}_k^H \hat{\mathbf{h}}_k + \mathbf{W}^H \mathbf{e}_k) + \sigma_k^2}. \tag{4}$$

2.1. Chance-constrained robust precoding

For given SINR targets γ_k, our QoS constraint is that the probability that $\mathsf{SINR}_k \geq \gamma_k$ should be greater than $1 - \epsilon_k$, for a pre-specified parameter ϵ_k. Therefore, the problem of interest can be written as

$$\min_{\{\mathbf{w}_k \in \mathbb{C}^{N_t}\}_{k=1}^K} \quad \mathrm{Tr}\big(\mathbf{W}\mathbf{W}^H\big) \tag{5a}$$

$$\text{subject to} \quad \mathrm{Pr}_{\mathbf{e}_k}\big(\mathsf{SINR}_k \geq \gamma_k\big) \geq 1 - \epsilon_k, \qquad \forall k. \tag{5b}$$

The presence of the chance constraints in (5b) makes the problem difficult to tackle directly, especially because the SINR in (4) is the ratio of a quadratic functions of the design variables. One approach is to apply a conservative transformation to the SINR constraint in (5b) to convert the problem in (5) into a chance chance-constrained second-order cone program (SOCP) [15]. By applying various conservative approximations of chance-constrained SOCPs, efficiently-solvable deterministic convex optimization problems are obtained. The conservative nature of the approximations means that when these convex problems are feasible, the solution is guaranteed to satisfy the chance constraints in (5b). A related approach in [16] first applies a semidefinite relaxation to the problem in (5), which yields a semidefinite program (SDP) with chance constraints on quadratic functions of a vector of variables. These chance constraints are then conservatively approximated by deterministic convex constraints leading to an SDP formulation. The solutions to such problems are guaranteed to satisfy the chance constraints in (5b) whenever the semidefinite relaxation is tight.

2.2. Chance-constrained robust power loading

In robust precoding the directions of transmission, $\breve{\mathbf{w}}_k = \mathbf{w}_k/||\mathbf{w}_k||_2$, and the power allocated to each direction, $\breve{p}_k = ||\mathbf{w}_k||_2^2$, are found jointly. A potentially simpler approach is to choose the directions $\breve{\mathbf{w}}_k$ based on the transmitters' channel estimates $\hat{\mathbf{h}}_k$ and then to seek solutions to the problem in (5) over the K powers, $\breve{p}_k$. If we allow the specification of the directions using vectors $\mathbf{b}_k$ that are not necessarily normalized, then for powers p_k such that $\mathbf{w}_k = p_k \mathbf{b}_k$, the total power transmitted is $\sum_{k=1}^K p_k ||\mathbf{b}_k||_2^2$. If we define $\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_K]$ and $\mathbf{P} = \mathrm{Diag}(p_1, p_2, ..., p_K)$, the robust power loading problem is

$$\min_{\{p_k \geq 0\}} \mathrm{Tr}\big(\mathbf{B}\mathbf{P}\mathbf{B}^H\big) \tag{6a}$$

$$\text{s.t.} \ \mathrm{Pr}_{\mathbf{e}_k}\Big(\frac{|\hat{\mathbf{h}}_k^H \mathbf{b}_k|^2 p_k}{(\hat{\mathbf{h}}_k^H \bar{\mathbf{B}}_k + \mathbf{e}_k^H \mathbf{B})\mathbf{P}(\bar{\mathbf{B}}_k^H \hat{\mathbf{h}}_k + \mathbf{B}^H \mathbf{e}_k) + \sigma_k^2} \geq \gamma_k\Big) \geq 1 - \epsilon_k, \tag{6b}$$

where $\bar{\mathbf{B}}_k = [\mathbf{b}_1, ..., \mathbf{b}_{k-1}, \mathbf{0}, \mathbf{b}_{k+1}, ..., \mathbf{b}_K]$ and we have left the fact that the constraint applies for all $k = 1, 2, \ldots, K$ implicit. A common choice for the precoding matrix $\mathbf{B}$ is the regularized channel inversion precoder for the estimated channel [20]: Given $\hat{\mathbf{H}} = [\hat{\mathbf{h}}_1, \hat{\mathbf{h}}_2, ..., \hat{\mathbf{h}}_K]^H$ and a non-negative real number α,

$$\mathbf{B}_{\text{RCI}} = \hat{\mathbf{H}}^H\big(\hat{\mathbf{H}}\hat{\mathbf{H}}^H + \alpha\mathbf{I}_K\big)^{-1}. \tag{7}$$

In the development of approaches to solve the problem in (6), it can be helpful to write down the chance constraints in (6b) in the form of chance constraints on quadratic function of a standard complex Gaussian random variable, $\boldsymbol{\delta}_k \sim CN(\mathbf{0}, \mathbf{I})$, namely,

$$\mathrm{Pr}_{\boldsymbol{\delta}_k}\big(\boldsymbol{\delta}_k^H \mathbf{Q}_k \boldsymbol{\delta}_k + 2\,\mathrm{Re}(\boldsymbol{\delta}_k^H \mathbf{r}_k) + v_k \geq 0\big) \geq 1 - \epsilon_k, \tag{8}$$

where $\mathbf{Q}_k = -{\mathbf{C}_k}^{1/2}\mathbf{B}\mathbf{P}\mathbf{B}^H{\mathbf{C}_k}^{1/2}$, $\mathbf{r}_k = -{\mathbf{C}_k}^{1/2}\mathbf{B}\mathbf{P}\bar{\mathbf{B}}_k^H\hat{\mathbf{h}}_k$ and $v_k = -\hat{\mathbf{h}}_k^H\bar{\mathbf{B}}_k\mathbf{P}\bar{\mathbf{B}}_k^H\hat{\mathbf{h}}_k + \frac{1}{\gamma_k}|\hat{\mathbf{h}}_k^H\mathbf{b}_k|^2 p_k - \sigma_k^2$. By writing chance constraints in this form, the conservative approximations that are summarized in [16] can be applied in a straightforward way. For example, given $\mathbf{B}$, the solution to the following SDP yields powers $\{p_k\}$ that satisfy the constraints in (6),

$$\min_{\{p_k \geq 0\},\, \{t_k \geq 0\}} \quad \mathrm{Tr}\big(\mathbf{B}\mathbf{P}\mathbf{B}^H\big) \tag{9a}$$

$$\text{subject to} \quad \begin{bmatrix} \mathbf{Q}_k + t_k\mathbf{I} & \mathbf{r}_k \\ \mathbf{r}_k^H & v_k - t_k{d_k}^2 \end{bmatrix} \succeq \mathbf{0}, \tag{9b}$$

where $d_k = \sqrt{{\phi^{-1}}_{X^2_{2N_t}}(1 - \epsilon_k)/2}$, where ${\phi^{-1}}_{X^2_{2N_t}}(\cdot)$ is the inverse cumulative distribution function of central Chi-square random variable with $2N_t$ degrees of freedom. In the special case of the "zero-forcing" directions,

$$\mathbf{B} = \mathbf{B}_{\text{ZF}} = \hat{\mathbf{H}}^H\big(\hat{\mathbf{H}}\hat{\mathbf{H}}^H\big)^{-1}, \tag{10}$$

the SDP in (9) simplifies to [17]

$$\min_{\{p_k \geq 0\}} \mathrm{Tr}\big(\mathbf{B}_{\text{ZF}}\mathbf{P}{\mathbf{B}_{\text{ZF}}}^H\big) \tag{11a}$$

$$\text{s.t.} \ \mathbf{C}_k^{1/2}\mathbf{B}_{\text{ZF}}\mathbf{P}\mathbf{B}_{\text{ZF}}^H\mathbf{C}_k^{1/2} + \Big(-\frac{p_k}{\gamma_k {d_k}^2} + \frac{\sigma_k^2}{{d_k}^2}\Big)\mathbf{I} \preceq 0. \tag{11b}$$

3. A NON-CONSERVATIVE APPROACH TO CHANCE-CONSTRAINED ROBUST POWER LOADING

In this section we develop a cyclic coordinate descent algorithm for finding good solutions to (6). Unlike previous approaches, such as those that led to (9) and (11), we do not seek a conservative deterministic approximation of the chance constraint in (8) that results in a tractable convex, but conservative, formulation. Instead, we employ the following closed-form expression for the probability on the left hand side of (8) that was derived in [19].

Lemma 1 ([19]). *Given a deterministic positive-definite Hermitian symmetric matrix* $\mathbf{Q}$ *and a deterministic vector* $\mathbf{a}$*, for the standard circular complex Gaussian random vector* $\mathbf{x} \sim CN(\mathbf{0}, \mathbf{I})$*, the CDF of* $||\mathbf{x} - \mathbf{a}||_{\mathbf{Q}}^2 = (\mathbf{x} - \mathbf{a})^H\mathbf{Q}(\mathbf{x} - \mathbf{a})$, $\mathrm{Pr}\big(||\mathbf{x} - \mathbf{a}||_{\mathbf{Q}}^2 \leq \tau\big)$*, is*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\tau(i\omega+\beta)}}{i\omega + \beta}\, \frac{e^{-c}}{\det(\mathbf{I} + (i\omega + \beta)\mathbf{Q})}\, d\omega, \tag{12}$$

for some $\beta > 0$ *such that* $\mathbf{I} + \beta\mathbf{Q}$ *is positive definite, where* $c = \mathbf{a}^H(\mathbf{I} + \frac{1}{i\omega+\beta}\mathbf{Q}^{-1})^{-1}\mathbf{a}$.

If we let $\mathbf{Q} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^H$ denote the eigen-decomposition of $\mathbf{Q}$, with λ_m denoting the eigenvalues arranged in descending order, and define $\tilde{\mathbf{a}} = \mathbf{V}^H\mathbf{a}$, then the constant c in Lemma 1 can be written as $c = -\sum_{m=1}^{M} \frac{|\tilde{a}(m)|^2(i\omega+\beta)\lambda_m}{1+(i\omega+\beta)\lambda_m}$. In order to employ Lemma 1 in our context, we rewrite (8) as $\Pr\big(||\boldsymbol{\delta}_k - \mathbf{a}_k||^2_{(-\mathbf{Q}_k)} \leq \tau_k\big) \geq 1 - \epsilon_k$, where $\mathbf{a}_k = -\mathbf{Q}_k^{-1}\mathbf{r}_k$ and $\tau_k = v_k - \mathbf{a}_k^H\mathbf{Q}_k\mathbf{a}_k = \frac{1}{\gamma_k}|\hat{\mathbf{h}}_k^H\mathbf{b}_k|^2 p_k - \sigma_k^2$, and we recall that $\mathbf{Q}_k$ is negative-definite.

The proposed algorithm starts with a diagonal power allocation matrix $\mathbf{P}^{(0)}$ for which all the SINR constraints in (6b) are satisfied. (We will discuss techniques for finding such a $\mathbf{P}^{(0)}$ below.) The algorithm then seeks a power allocation of lower cost that remains feasible via cyclic coordinate descent; e.g., [21, Section 2-7]. We will describe the steps in each cycle in the natural order, but the principles apply to any ordering of $\{p_k\}$.

At the k^{th} step of the i^{th} cycle of the coordinate descent algorithm we seek to reduce the value of p_k given $p_j = p_j^{(i)}$ for $j = 1, \ldots, k-1$ and $p_j = p_j^{(i-1)}$ for $j = k+1, \ldots, K$. Since a reduction in the value of p_k cannot decrease SINR_j for $j \neq k$, when performing the descent step on p_k we need only consider the constraint on SINR_k. To perform the descent step on p_k at the i^{th} cycle we employ a bisection search on the interval $[0, p_k^{(i-1)}]$ for a value of p_k that provides a significant decrease in the objective and remains feasible. For each postulated value for p_k, the probability that $\mathsf{SINR}_k \geq \gamma_k$ is evaluated using the expression in Lemma 1, and we terminate the bisection search in the i^{th} cycle once that probability lies in $[1-\epsilon_k, 1-\epsilon_k+\Delta_k^{(i)}]$, where $\Delta_k^{(i)}$ is a parameter of the algorithm. The cycles of the algorithm are terminated once we have found a power allocation $\{p_k^{(i)}\}_k$ such that for all k the probability that $\mathsf{SINR}_k \geq \gamma_k$ lies in $[1-\epsilon_k, 1-\epsilon_k+\Delta_k^{(i)}]$. A feature of this algorithm is that at each step in each cycle the allocation $\{p_1^{(i)}, p_2^{(i)}, \ldots, p_k^{(i)}, p_{k+1}^{(i-1)}, \ldots, p_K^{(i-1)}\}$ is feasible and hence whenever the algorithm is terminated, the current power allocation will satisfy the specified QoS constraints. Furthermore, at each step in each cycle, the objective value decreases, or remains the same.

To complete the description of the algorithm, we need to establish a method to determine a feasible starting point. As the feasible set in (6) is not necessarily convex, determining whether or not an instance of the problem in (6) is feasible can be computationally demanding task. Instead, we simply seek an approach that often finds feasible points for reasonable instances of the problem. The proposed approach involves selecting an initial diagonal power allocation matrix and evaluating each of the SINR constraints in (6b) using Lemma 1. If that power allocation is not feasible, the allocation is iteratively doubled until a feasible allocation is found or the power become unreasonably large. In the latter case a new initial power allocation can be selected or the algorithm reports that no feasible point was found. In our implementation we have found that choosing the initial power allocation to be the power allocation that would be chosen if the channel estimates $\hat{\mathbf{h}}_k^H$ were exact and if each SINR_k were set to be equal to its lower bound typically leads to a feasible starting point for the main algorithm after a small number of doubling iterations. By rearranging the terms in the SINR expression, one can obtain the squares of this initial power allocation by solving a set of linear equations.

As the problem in (6) is not convex, the proposed algorithm is not guaranteed to find the globally optimal solution. Indeed, it is not even guaranteed to find a feasible point when one exists. However, we will demonstrate in Section 5 that by tackling the problem directly, without a conservative approximation, the proposed approach often provides better performance than the existing conservative approaches. Having said that, the repeated requirement to compute an integral of the form in (12) imposes a significant computational burden. (The SDPs that must be solved in the existing approaches also impose a significant computational burden.) To address this issue, in the following sections we will develop customized variants of the algorithm for the case of the zero-forcing directions $\mathbf{B}_{\text{ZF}}$.

4. EFFICIENT CONSERVATIVE ALGORITHMS FOR THE ZERO-FORCING CASE

In this section we focus on the case of the zero-forcing beamforming direction, $\mathbf{B} = \mathbf{B}_{\text{ZF}}$. In this case, $\bar{\mathbf{B}}_k^H\hat{\mathbf{h}}_k = \mathbf{0}$ and $|\hat{\mathbf{h}}_k^H\mathbf{b}_k|^2 = 1$. These simplifications mean that we have $\mathbf{r}_k = \mathbf{0}$, $v_k = p_k/\gamma_k - \sigma_k^2$, $\tau_k = p_k/\gamma_k - \sigma_k^2$, and $\mathbf{a}_k = \mathbf{0}$, and hence for each k, the term c in (12) is zero. As we will outline below, these simplifications enable the application of residue theory to show how integral in (12) can be computed as a summation of $N_t + 1$ terms. (Residue theory was employed in a related context in [22].) By subsequently applying a conservative approximation to that summation we obtain efficient power loading algorithm. To simplify the development, we will first consider the case of equal power loading, $\mathbf{P} = p\mathbf{I}$, in a scenario that is homogenous in the sense that all users have the same noise variance, σ, the same QoS requirement, γ, the same error covariance, $\mathbf{C}$, and the same outage probability, ϵ. The insight extracted from that case is then extended to the case of full power loading.

4.1. Equal power loading for the homogenous scenario

For a homogeneous scenario with equal power loading, $\mathbf{P} = p\mathbf{I}$, we have $\mathbf{Q}_k = p\tilde{\mathbf{Q}}$, where $\tilde{\mathbf{Q}} = -\mathbf{C}^{1/2}\mathbf{B}_{\text{ZF}}\mathbf{B}_{\text{ZF}}^H\mathbf{C}^{1/2}$. We will denote the eigenvalues of $\tilde{\mathbf{Q}}$, arranged in descending order, by $\tilde{\lambda}_m$ and will assume that they are distinct. In this case, the probability to be evaluated is $\Pr\big(||\boldsymbol{\delta}||^2_{(-\tilde{\mathbf{Q}})} \leq 1/\gamma - \sigma^2/p\big)$. If we convert the integral in (12) to an appropriate contour integral, then we can apply residue theory to that contour integral and show that $\Pr\big(||\boldsymbol{\delta}||^2_{(-\tilde{\mathbf{Q}})} \leq 1/\gamma - \sigma^2/p\big) = 1 + \sum_{\ell=1}^{N_t} f_\ell(p)$, where

$$f_\ell(p) = -\exp\Big(-\frac{1}{\gamma\tilde{\lambda}_\ell} + \frac{\sigma^2}{p\tilde{\lambda}_\ell}\Big)\frac{1}{\prod_{j\neq\ell}(1-\tilde{\lambda}_j/\tilde{\lambda}_\ell)}. \qquad (13)$$

Using (13), the design problem becomes $\min_{p\geq 0} p$ subject to $1 + \sum_{\ell=1}^{N_t} f_\ell(p) \geq 1-\epsilon$. The optimal solution is simply the smallest non-negative root of $\sum_{\ell=1}^{N_t} f_\ell(p) + \epsilon = 0$. Since $f_\ell(p)$ is smooth, a number of root finding algorithms could be considered. Instead of doing that, we will employ a conservative approximation of the constraint $1 + \sum_{\ell=1}^{N_t} f_\ell(p) \geq 1-\epsilon$ and show that the resulting problem has a closed-form solution. To develop the conservative approximation we note that since $-\tilde{\mathbf{Q}}$ is positive-definite, it only makes sense to consider cases where $1/\gamma - \sigma^2/p \geq 0$. In those cases, as ℓ increases the argument of the exponential in (13) becomes more negative, and hence the magnitude of $f_\ell(p)$ decreases. Furthermore, for odd ℓ, $f_\ell(p) < 0$, whereas for even ℓ, $f_\ell(p) > 0$. As a result we have that $\sum_{\ell=2}^{N_t} f_\ell(p) \geq 0$ and that this term will typically be small in comparison to $|f_1(p)|$. Therefore, if p is chosen such that $1 + f_1(p) \geq 1-\epsilon$, then the outage constraint is guaranteed to hold. More explicitly, if p is chosen such that

$$p \geq \frac{\sigma^2}{1/\gamma + \tilde{\lambda}_1 \ln\big(\epsilon\prod_{k\neq 1}(1-\tilde{\lambda}_k/\tilde{\lambda}_1)\big)} = \frac{\sigma^2}{\nu}, \qquad (14)$$

then the SINR constraints in (6b) are guaranteed to be satisfied. Since the objective is to minimize transmitted power, if $\nu > 0$ we choose $p = \sigma^2/\nu$. If $\nu \leq 0$ there is no p that satisfies the conservative approximation of the constraints.

4.2. General power loading

We now use insight from the above analysis to develop a method for scenarios that are heterogenous in the sense that $\mathbf{C}$, γ, σ^2 and ϵ can differ for each user. In this setting, by applying residue theory to the integral expression (cf. (12)) of the chance constraint we obtain $\Pr(||\boldsymbol{\delta}_k||^2_{-\mathbf{Q}_k} \leq p_k/\gamma_k - \sigma_k^2) = 1 + \sum_{\ell=1}^{N_t} f_{\ell_k}(\mathbf{P})$, where

$$f_{\ell_k}(\mathbf{P}) = -\exp\Big(\big(\frac{1}{\gamma_k}p_k - \sigma_k^2\big)\frac{-1}{\lambda_{\ell k}}\Big)\frac{1}{\prod_{j\neq\ell}(1-\lambda_{jk}/\lambda_{\ell k})} \quad (15)$$

and λ_{mk} is the m^{th} largest eigenvalue of $\mathbf{Q}_k$. Unlike the homogenous case, $f_{\ell_k}(\mathbf{P})$ is a function of all the powers, through the eigenvalues of $\mathbf{Q}_k$. This complicates the development of a cyclic coordinate update algorithm. To address that difficulty, at the k^{th} step of the i^{th} cycle we will construct an approximation of $\mathbf{Q}_k$ as $\hat{\mathbf{Q}}_k^{(i)} = -\mathbf{C}_k{}^{1/2}\mathbf{B}_{\text{ZF}}\mathbf{P}^{(i-1)}\mathbf{B}_{\text{ZF}}^H\mathbf{C}_k{}^{1/2}$. The resulting approximation of $f_{\ell_k}(\mathbf{P})$ depends only on $p_k^{(i)}$, which simplifies the coordinate update step. That step can be further simplified using a conservative approximation of form that led to (14) to obtain

$$p_k^{(i)} \geq \gamma_k\sigma_k^2 - \gamma_k\hat{\lambda}_{1k}^{(i)}\ln\Big(\epsilon_k\prod_{j\neq 1}(1-\hat{\lambda}_{jk}^{(i)}/\hat{\lambda}_{1k}^{(i)})\Big) \quad (16)$$

where $\hat{\lambda}_{mk}^{(i)}$ is the m^{th} largest eigenvalue of $\hat{\mathbf{Q}}_k^{(i)}$. Since we seek small powers, $p_k^{(i)}$ is chosen such that equality holds in (16). To initialize the algorithm, we chose $p_k^{(0)}$ as if the system were homogenous with the k^{th} user's parameters, cf. (14). That is, we choose

$$p_k^{(0)} = \frac{\sigma_k^2}{1/\gamma_k + \tilde{\lambda}_{1k}\ln\big(\epsilon_k\prod_{j\neq 1}(1-\tilde{\lambda}_{jk}/\tilde{\lambda}_{1k})\big)}, \quad (17)$$

where $\tilde{\lambda}_{mk}$ is the m^{th} largest eigenvalue of the corresponding $\tilde{\mathbf{Q}}_k$. Unlike the algorithm in Section 3, this initial power allocation is not necessarily feasible, but as the coordinates are updated the power allocation tends to move toward the feasible set. The cyclic updates are terminated once a feasible point is found or if no feasible point is found in a reasonable time. (Feasibility is evaluated using the expression that precedes in (15).) Our numerical experience suggests that the starting point in (17) is particularly effective in that the level of conservatism in the first feasible point tends to be low. In structures where that is not the case, one can use this feasible point to initialize an algorithm analogous to that in Section 3.

5. SIMULATION

We now compare the proposed power loading algorithms with the power loading algorithm for zero-forcing directions in [17] (cf. (11)) and the algorithm in (9) for the case of RCI directions (which is based on [16]). We will also compare with the robust precoding method in [16]. We consider an environment with $N_t = 3$ transmit antennas, $K = 3$ users, and i.i.d. Rayleigh fading channels. The uncertainty in the channel estimation is modeled by Gaussian random variables with zero mean and covariance $\mathbf{C}_k = 0.002\mathbf{I}$. The probability of outage is set to be $\epsilon_k = 0.05$ for all users. We fix the QoS requirement for users 1 and 2 to be 3 dB. We generated 10,000 realizations of the set of channel estimates $\{\hat{\mathbf{h}}_k^H\}_{k=1}^K$ and examined

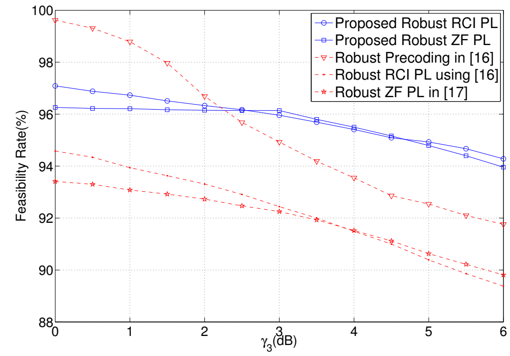

Fig. 1. Percentage of channel realizations for which the probabilistic SINR guarantee can be made, against γ_3.

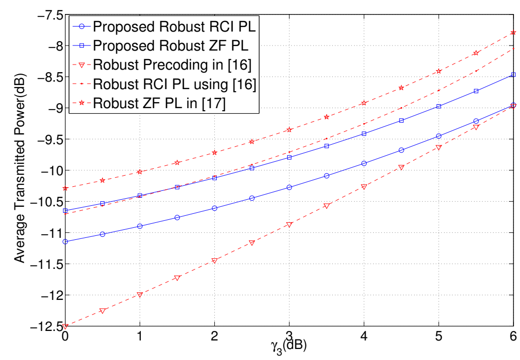

Fig. 2. Average transmitted power against γ_3.

the performance of each method as the target SINR of user 3, γ_3, increases from 0 to 6 dB. For each set of channel estimates and for each value of γ_3 we determine whether each design formulation generates a beamformer that guarantees that the probabilistic SINR constraints are satisfied. In Fig. 1, we plot the percentage of channel realizations for which each design generated a feasible beamformer. Then, we subsequently selected all the realization from the set of 10,000 for which all methods provide a feasible solution for $\gamma_3 = 6$ dB. In Fig. 2, we plot the average transmission power over the 8,938 such channels against γ_3.

From Fig. 1 it can be seen that by tackling the power loading problem directly (or closely), even with suboptimal algorithms, the proposed power loading methods are able to satisfy the QoS constraints more often than the existing power loading methods which are based on optimal solutions to tractable conservative approximations of the problem. This is because the approximations made in those methods are quite conservative. What is perhaps more interesting is that for higher SINR targets, the proposed power loading methods provide better performance than the robust precoding method in [16], despite the fact that that method has many more degrees of design freedom. Once again, this is due to the fact that the approximation used in the proposed method for zero-forcing directions is much less conservative. In our implementations, the method proposed in Section 3 incurred similar computational costs to the existing SDP-based methods, whereas the method proposed in Section 4.2 is significantly cheaper.

6. REFERENCES

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