EXPECTED COMPLEXITY OF SPHERE DECODING FOR SPARSE INTEGER LEAST-SQUARE PROBLEMS

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ABSTRACT

Sparse integer least-squares problems come up in a wide range of applications including wireless communications and genomics. The sphere decoding algorithm can find near-optimal solution to these problems with reduced average complexity if the knowledge of sparsity of the unknown vector is used in decoding. In this paper, we formulate a sphere decoding approach that relies on the ℓ_0-norm constraint on the unknown vector to solve sparse integer least-squares problems. The expected complexity of this algorithm is derived analytically for sparse alphabets associated with common applications such as sparse channel estimation and validated via simulations. The results indicate superior performance and speed compared to the classical sphere decoding algorithm.

Index Terms— Sphere-decoding, sparsity, expected complexity, integer least-squared problems, ℓ_0 norm.

1. INTRODUCTION

Solving integer least-squares problems where the unknown vector is sparse, arises in a broad range of applications including multi-user detection in wireless communication systems [1], sparse array processing [2], collusion-resistant digital fingerprinting [3], and array-comparative genomic hybridization microarrays [4]. Formally, the sparse integer least-squares (ILS) problem can be stated as the cardinality constrained optimization

$$\min_{\mathbf{x}\in\mathcal{D}_L^m} \|\mathbf{y}-\mathbf{H}\mathbf{x}\|_2^2 \qquad (1)$$
$$\text{subject to} \quad \|\mathbf{x}\|_0 \le \ell,$$

where $\mathbf{y} \in \mathbb{R}^{n\times 1}$ and $\mathbf{H} \in \mathbb{R}^{n\times m}$ are the measured vector and coefficient matrix respectively, and $\mathbf{x}$ denotes an m-dimensional unknown vector with entries from a set of integers $\mathcal{D}_L^m \subset \mathcal{Z}^m$ having L elements. ℓ is an upper bound on the estimate of the ℓ_0 norm of the unknown vector. For example, order of a sparse frequency-selective channel [5] can be estimated a priori to serve as the parameter ℓ in (1). All of the aforementioned applications have $n \ge m$. Note that (1) can be interpreted as a search for a point closest to the given point $\mathbf{y}$ in a sparse integer lattice.

In many communication applications, sphere decoding algorithm is capable of solving ILS problems efficiently. The knowledge of sparsity of $\mathbf{x}$ can be exploited in the classical sphere decoding search to improve the accuracy and complexity of the algorithm. The computational complexity of solving (1) is a major concern since the closest lattice point problem is known to be NP hard [6]. In [7] and [8], it was shown that if the sphere radius is chosen according to the perturbation noise $\nu = \mathbf{y} - \mathbf{H}\mathbf{x}$, classical sphere decoding exhibits an expected complexity that is practically feasible over a wide range of signal-to-noise ratios (SNRs) and system dimensions. This served as the motivation for the complexity analysis of sparsity-aware sphere decoder in this paper.

Recently, several variants of sphere decoder which account for sparsity of the unknown vector were proposed [2], [9], [1]. [2] proposed a sphere decoding algorithm using relaxed constraint in the form of ℓ_1 regularizer. However, this scheme works only for non-negative alphabets where $\|\mathbf{x}\|_1$ can be decomposed into sum of components of $\mathbf{x}$. In [9], a generalized sphere decoding approach with ℓ_1 constraint is adopted for sparse ILS with $\{0, 1\}$ alphabet under compressed sensing framework. An ℓ_0-norm regularized sphere decoder has been studied in [1], where the regularizing parameter λ used in the distance metric is a function of the prior probability of the components of the unknown vector. In contrast, in this paper we directly impose ℓ_0-norm constraint on the solution vector and do not require knowledge of its statistical properties. Closest point search in a sparse lattice has been studied in [10] but sparsity there stems from the fact that not all lattice points are valid codewords of linear block codes.

Contributions of this paper are as follows. We propose a sparsity-aware sphere decoding algorithm which enforces ℓ_0-norm constraint on the solution vector. This algorithm is suitable for various alphabets and is applicable to generic problems without any restriction/assumption on the system model. Furthermore, we quantify the complexity of this algorithm by its expected value and perform validation via simulations for commonly occurring sparse alphabets.

2. SPHERE DECODING ALGORITHM

The sphere decoding algorithm in an m-dimensional lattice conducts a search within a sphere of radius d, centered at the received point $\mathbf{y}$, and, if no point is found within this sphere, repeats the search over a sphere of larger radius. The search over all lattice points $\mathbf{x}$ satisfying the sphere constraint $d^2 \geq \|\mathbf{y} - \mathbf{Hx}\|_2^2$ is performed by reducing the search over an m-dimensional sphere to a sequence of searches over 1-dimensional intervals. To illustrate the procedure, it is convenient to rewrite the sphere constraint as [7]

$$d^2 - \|Q_2^* \mathbf{y}\|^2 \geq (z_m - r_{m,m} x_m)^2 + (z_{m-1} - r_{m-1,m} x_m - r_{m-1,m-1} x_{m-1})^2 + \dots \quad (2)$$

where $\mathbf{z} = Q_1^* \mathbf{y}$ and $r_{i,j}$ is the (i,j) entry of R, the upper-triangular matrix in the QR-decomposition of $\mathbf{H}$,

$$\mathbf{H} = Q \begin{bmatrix} R \\ 0_{(n-m\times m)} \end{bmatrix} \quad (3)$$

and Q_1 and Q_2 are composed of the first m and last $n-m$ orthonormal columns of Q, respectively. A necessary condition for $\mathbf{Hx}$ to lie inside the sphere is $d^2 - \|Q_2^* \mathbf{y}\|^2 \geq (z_m - r_{m,m} x_m)^2$. For every such x_m, a stronger necessary condition can be found by considering the first two terms on the right-hand side of (2) and imposing a condition on x_{m-1}. One can proceed in a similar way to determine conditions for $x_{m-2}, \dots, x_1$, thus determining all lattice points that satisfy $d^2 \geq \|\mathbf{y} - \mathbf{Hx}\|_2^2$.

The sphere decoding algorithm can be geometrically interpreted as a depth-first search on a tree whose nodes at k^{th} level represent k-dimensional points $[x_{m-k+1}\ x_{m-k+2}\ \dots\ x_m]^T$. The algorithm prunes those nodes at k^{th} level which do not belong to the k-dimensional sphere of radius d. A node at level m which yields lowest objective function value is a solution for the ILS problem.

3. SPARSITY-AWARE SPHERE DECODING

In this paper, the requirement of the solution vector to be sparse is enforced by imposing a constraint on the ℓ_0 norm of lattice points $\mathbf{x}$ during sphere decoding search. As a result, necessary conditions that the components of $\mathbf{x}$ must satisfy in order to qualify for the solution become more restrictive. Consequently, not all lattice points within the search radius now satisfy these new conditions and the search takes lesser time than in the case of having to examine every lattice point satisfying the sphere constraint.

In order to find sparse $\mathbf{x}$ in sphere decoding tree search, it is required to redefine the constraint that is checked for each node of the tree. Let us suppose a node at the k^{th} level of the tree lies within the 1-dimensional interval determined by the nodes in the preceding $k-1$ levels. Note that the number of nonzero symbols along the path from the topmost level leading to this node is a measure of the sparseness of the k-dimensional point under consideration. For this node to lie within the k-dimensional sphere of radius d, this value should not exceed the overall sparsity constraint ℓ. Hence, the ℓ_0 constraint effectively becomes a stricter criterion for pruning nodes from the tree during the depth-first search.

Table 1. *Sparsity-aware Sphere Decoding Algorithm*

Input: $Q = [Q_1\ \ Q_2],\ R, y,\ z = Q_1^* y$, sphere radius d, sparsity constraint ℓ.

1. Initialize $k \leftarrow m$, $d_m^2 \leftarrow d^2 - \|Q_2^* y\|^2$, $z_{m|m+1} \leftarrow z_m$, $\ell_m \leftarrow 0$.
2. Update Interval $UB(x_k) \leftarrow \lfloor (d_k + z_{k|k+1})/R_{k,k} \rfloor$, $LB(x_k) \leftarrow \lceil (-d_k + z_{k|k+1})/R_{k,k} \rceil$, $x_k \leftarrow LB(x_k) - 1$.
3. Update x_k $x_k \leftarrow x_k + 1$. If $x_k \leq UB(x_k)$, go to 4; else go to 5.
4. Check Sparsity If $\ell_k + \mathbf{1}_{\{x_k \neq 0\}} \leq \ell$, $\ell_k \leftarrow \ell_k + \mathbf{1}_{\{x_k \neq 0\}}$, and go to 6; else go to 3.
5. Increase k $k \leftarrow k + 1$. If $k = m + 1$, stop; else, $\ell_k \leftarrow \ell_k - \mathbf{1}_{\{x_k \neq 0\}}$ and go to 3.
6. Decrease k If $k = 1$, go to 7; else $k \leftarrow k - 1$, $z_{k|k+1} \leftarrow z_k - \sum_{j=k+1}^{m} R_{k,j} x_j$, $d_k^2 \leftarrow d_{k+1}^2 - (z_{k+1|k+2} - R_{k+1,k+1} x_{k+1})^2$, and go to 2.
7. Solution found Save x and its distance from y, $d_m^2 - d_1^2 + (z_1 - R_{1,1}\ x_1)^2,\ \ell_k \leftarrow \ell_k - \mathbf{1}_{\{x_k \neq 0\}}$ and go to 3.

Table 1 presents the sparsity-aware sphere decoding algorithm, which uses the framework provided in [7]. Note that the variable ℓ_k for a node at the k^{th} level of the tree denotes the number of nonzero symbols up to but not including that node. The sparsity criterion is checked in step 4 before proceeding along a node and incurs a nominal increase in the number of computations required per node. Whenever the algorithm goes up a level on the tree, ℓ_k is adjusted to represent the current node.

Remarks 1: $\mathbf{1}$ denotes the indicator function, given by $\mathbf{1}_A$ = 1 if the statement A is true, 0 otherwise. $z_{k|k+1}$ is defined as the received signal z_k adjusted with the already estimated symbol components $x_{k+1}, \dots, x_m$. Also, the algorithm assumes an alphabet with unit minimum spacing and can be generalized easily.

Remarks 2: For nonnegative alphabets, sparsity-aware sphere decoding algorithm can be modified to prune nodes at any level if an earlier node on the same level is found to violate the sparsity constraint. Details are omitted here for brevity.

4. EXPECTED COMPLEXITY

It is mentioned in [7] that the expected complexity of sphere-decoding algorithm depends on the average number of nodes in the tree visited by the algorithm, and, is a random variable if $\mathbf{H}$ and ν are assumed to be random variables. If the perturbation ν follows $\mathcal{N}(0, \sigma^2 I)$ distribution with independent entries, where σ^2 is the component wise variance, then $\frac{1}{\sigma^2}\|\nu\|^2$ is a χ_n^2 random variable. It has been suggested in [7] that given a small $\epsilon > 0$, there is a high probability of finding a lattice point inside a sphere of radius $d = \sigma\sqrt{\alpha n}$, if α satisfies the relation $\gamma(\frac{\alpha n}{2}, \frac{n}{2}) = 1 - \epsilon$, where γ is the normalized incomplete gamma function. It has also been shown in [7] that with this choice of radius, the probability of an arbitrary lattice point $\mathbf{x}_a$ lying inside the sphere of radius d centered at $\mathbf{y}$ is

$$\mathcal{P}(\|\mathbf{y} - \mathbf{H}\mathbf{x}_a\|_2^2 \le d^2) = \gamma\left(\frac{d^2}{2(\sigma^2 + \|\mathbf{x}_a - \mathbf{x}_t\|^2)}, \frac{n}{2}\right) \quad (4)$$

where $\mathbf{x}_t$ denotes the true value of the unknown vector $\mathbf{x}$.

Let $\mathbf{x}^k$ denote the k-dimensional component $[x_{m-k+1} \cdots x_m]^T$ of any vector $\mathbf{x}$. The expected number of points visited at the k^{th} level by the classical sphere decoder is given by [7]

$$E(k, d^2) = \frac{1}{L^k} \sum_{\eta} \sum_{\substack{\mathbf{x}_t^k, \mathbf{x}_a^k \in \mathcal{D}_L^k, \\ \|\mathbf{x}_t^k - \mathbf{x}_a^k\|^2 = \eta}} \gamma\Big(\frac{d^2}{2(\sigma^2 + \eta)}, \frac{n-m+k}{2}\Big).$$

In case of the sparsity-aware sphere decoder, above expression needs to be modified to

$$E'(k, d^2) = \frac{1}{N_L(k,\ell)} \sum_{\eta} \sum_{\substack{\mathbf{x}_t^k, \mathbf{x}_a^k \in \mathcal{D}_L^k, \\ \|\mathbf{x}_t^k - \mathbf{x}_a^k\|^2 = \eta, \\ \|\mathbf{x}_t^k\|_0 \le \ell, \|\mathbf{x}_a^k\|_0 \le \ell}} \gamma\Big(\frac{d^2}{2(\sigma^2 + \eta)}, \frac{n-m+k}{2}\Big).$$

where $N_L(k, \ell)$ is the total number of possible k-dimensional ℓ-sparse vectors, given by $N_L(k,\ell) = \sum_{t=0}^{\min(k,\ell)} \binom{k}{t}(L-1)^t$.

The main challenge in evaluating the above expression is to come up with an efficient enumeration of the symbol space, i.e., to determine the number of pairs of ℓ-sparse vectors $\mathbf{x}_a^k$ and $\mathbf{x}_t^k$ such that $\|\mathbf{x}_t^k - \mathbf{x}_a^k\|^2 = \eta$ holds. While this enumeration appears to be difficult in general, it can be solved for some of the most commonly encountered alphabets in sparse integer least square problems: the binary $\{0, 1\}$ alphabet (relevant in [3], [2], [5]) and the ternary alphabet $\{-1, 0, 1\}$ alphabet (relevant in [4]).

4.1. Binary Alphabet {0,1}

A close inspection of the complexity expression for sparsity-aware sphere decoder reveals that in order to enumerate the symbol space, we need to consider all possible ℓ-sparse vector $\mathbf{x}_a^k \in \mathcal{D}_L^k$ satisfying $\|\mathbf{x}_a^k - \mathbf{x}_t^k\|^2 = \eta$ for each ℓ-sparse vector $\mathbf{x}_t^k \in \mathcal{D}_L^k$. Note that for binary alphabet, η is the number of components of $\mathbf{x}_t^k$ and $\mathbf{x}_a^k$ that differ from each other. Let $k_1 = \|\mathbf{x}_t^k\|_0$. Given $\mathbf{x}_t^k$, the number of lattice points $\mathbf{x}_a^k$ with $\|\mathbf{x}_a^k\|_0 = k_2$ satisfying $\|\mathbf{x}_a^k - \mathbf{x}_t^k\|^2 = \eta$ is given by

$$g(k_1, k_2, k, \eta) = \begin{cases} \binom{k_1}{k_2 - p}\binom{k-k_1}{p}, & \text{if } k_1 \ge k_2 \\ \binom{k_1}{k_1 - p}\binom{k-k_1}{k_2 - k_1 + p}, & \text{if } k_1 < k_2 \end{cases} \quad (5)$$

where $p = (\eta - |k_1 - k_2|)/2$.

Eqn. (5) can be derived by considering all possible ways of combinatorially arranging the alphabets in $\mathbf{x}_a^k$ for a given $\mathbf{x}_t^k$, as described below.

Case 1: $k_1 \ge k_2$

Let p be the number of '1' s in $\mathbf{x}_a^k$ occurring in $k - k_1$ positions corresponding to '0's of $\mathbf{x}_t^k$. The number of ways of arranging these '1's is $\binom{k-k_1}{p}$. For each of the above combinations, remaining $k_2 - p$ '1's of $\mathbf{x}_a^k$ can be arranged in the k_1 positions corresponding to '1's of $\mathbf{x}_t^k$ in $\binom{k_1}{k_2-p}$ ways. The total number of positions where $\mathbf{x}_t^k$ and $\mathbf{x}_a^k$ differ is $\eta = p + k_1 - (k_2 - p)$ or $p = (\eta - (k_1 - k_2))/2$.

Case 2: $k_1 < k_2$

Let p be the number of '0's in $\mathbf{x}_a^k$ occurring in k_1 positions corresponding to '1's of $\mathbf{x}_t^k$. The number of ways of arranging these '0's is $\binom{k_1}{p} = \binom{k_1}{k_1-p}$. For each of the above combinations, remaining $(k - k_2 - p)$ '0's of $\mathbf{x}_a^k$ can be arranged in the $k - k_1$ positions corresponding to '0's of $\mathbf{x}_t^k$ in $\binom{k-k_1}{k-k_2-p} = \binom{k-k_1}{k_2-k_1+p}$ ways. The total number of positions where $\mathbf{x}_t^k$ and $\mathbf{x}_a^k$ differ is $\eta = p + k - k_1 - (k - k_2 - p)$ or $p = (\eta - (k_2 - k_1))/2$.

From the above two cases, we obtain (5).

Given $\mathbf{x}_t^k$ and $\mathbf{x}_a^k$, range of η is given by $\mathcal{S} = \{|k_1 - k_2|, |k_1 - k_2| + 2, \ldots, \min\{k_1 + k_2, k\}\}$. Combining all these, the expected number of lattice points visited by the sparsity-aware sphere decoding algorithm in a k-dimensional sphere of radius d is given by

$$\begin{aligned} E'(k, d^2) &= \frac{1}{N_2(k,\ell)} \sum_{k_1=0}^{\min(k,\ell)} \binom{k}{k_1} \sum_{k_2=0}^{\min(k,\ell)} \\ &\times \sum_{\eta \in \mathcal{S}} \gamma\left(\frac{d^2}{2(\sigma^2+\eta)}, \frac{n-m+k}{2}\right) g(k_1, k_2, k, \eta) \end{aligned} \quad (6)$$

Eqn. (6) can be used to identify the complexity for certain special cases of closest lattice point search. For very large radius, the sphere decoding search is based only on the sparsity constraint and (6) reduces to

$$\begin{aligned} E'(k, d^2) &= \frac{1}{N_2(k,\ell)} \sum_{k_1=0}^{\min(k,\ell)} \binom{k}{k_1} \sum_{k_2=0}^{\min(k,\ell)} \binom{k}{k_2} \\ &= \sum_{k_1=0}^{\min(k,\ell)} \binom{k}{k_1} \end{aligned} \quad (7)$$

which corresponds to the worst-case scenario and requires a brute force search over all ℓ-sparse signal, which can lead to a complexity exponential in k.

If the unknown vector $\mathbf{x}_t$ is sparse but the sphere decoder is unaware of this information, then the expected complexity in (6) reduces to

$$E'(k, d^2) \;=\; \frac{1}{N_2(k,\ell)} \sum_{k_1=0}^{\min(k,\ell)} \binom{k}{k_1} \sum_{k_2=0}^{k} \sum_{\eta \in \mathcal{S}} \times\; \gamma\Big(\frac{d^2}{2(\sigma^2+\eta)}, \frac{n-m+k}{2}\Big)\, g(k_1,k_2,k,\eta) \tag{8}$$

Eqn. (8) is clearly an upper bound for eqn. (6), and therefore, sparsity information at the decoder results in reduction in expected complexity.

Lastly, the overall expected complexity is given by

$$C(m, d^2) \;=\; \sum_{k=1}^{m} f_p(k)\; E'(k, d^2) \tag{9}$$

where $f_p(k)$ denotes the flop counts (or the number of operations in the algorithm) in the k^{th} level of the tree. The complexity exponent is defined as $e_c = \log\left(C(m,d^2)\right)/\log m$.

4.2. Ternary Alphabet $\{-1, 0, 1\}$

We present the expected complexity expressions for ternary alphabets in this section and omit the derivation because of space limitation.

The number of all possible vectors $\mathbf{x}_a^k$ with $\|\mathbf{x}_a^k\|_0 = k_2$ for a given $\mathbf{x}_t^k$ ($\|\mathbf{x}_t^k\|_0 = k_1$) and satisfying $\|\mathbf{x}_a^k - \mathbf{x}_t^k\|^2 = \eta$ can be shown to be

$$g(k_1,k_2,k,p,q) = \begin{cases} \binom{k_1}{q}\binom{k_1-q}{k_2-q-p}\binom{k-k_1}{p}\, 2^p, & \text{if } k_1 \geq k_2 \\ \binom{k_1}{p}\binom{k_1-p}{q}\binom{k-k_1}{k_2-k_1+p}\, 2^{k_2-k_1+p}, & \text{if } k_1 < k_2 \end{cases} \tag{10}$$

where η is given by $\eta = |k_1 - k_2| + 2p + 4q$, and ranges of p and q are defined by the sets $\mathcal{S}_p = \{0, \ldots, \min(k_1, k_2, k-k_1, k-k_2)\}$ and $\mathcal{S}_q = \{0, \ldots, (\min(k_1,k_2) - p)\}$ respectively. Then, the expected number of lattice points visited by the sparsity-aware sphere decoder in a k-dimensional sphere of radius d is given by

$$E'(k,d^2) = \frac{1}{N_3(k,\ell)} \sum_{k_1=0}^{\min(k,\ell)} \binom{k}{k_1} 2^{k_1} \times \Bigg(\sum_{k_2=0}^{\min(k,\ell)} \sum_{p\,\in\,\mathcal{S}_p} \sum_{q\,\in\,\mathcal{S}_q} g(k_1,k_2,k,p,q) \times\; \gamma\Big(\frac{d^2}{2(\sigma^2+|k_1-k_2|+2p+4q)}, \frac{n-m+k}{2}\Big) \Bigg)$$

5. SIMULATIONS

We present complexity exponent versus SNR plots to demonstrate the expected complexity of the sparsity-aware sphere decoder in Figure 1(a), (b). Figure 1(a) shows the comparison of theoretical complexity of sparsity-aware SD with that of actual simulations for binary alphabet $\{0, 1\}$ for $n = m = 20, \ell = 5$. This figure also shows the expected complexity of classical SD which is sparsity-unaware, indicating that the proposed algorithm performs faster. Figure 1(b) compares the theoretical expression of expected complexity with simulations for ternary alphabet $\{-1, 0, 1\}$ for $n = m = 10, \ell = 1$. Figure 1(c) shows the comparison of the error rate performance of sparsity-aware SD with the classical SD as a function of the parameter ℓ at SNR = 10 dB. It is clear from this plot that the sparsity-aware SD has significantly better error rate for low sparsity levels.

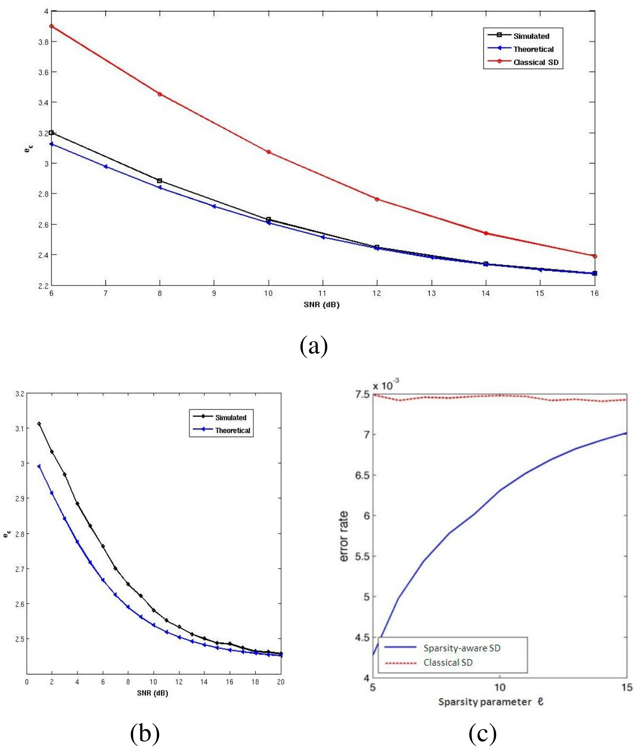

Fig. 1. *Simulation results for expected complexity and error rate of sparsity-aware sphere decoder.*

6. CONCLUSION

We have proposed a sparsity-aware sphere decoding approach based on ℓ_0 norm constraint and analyzed its expected complexity for sparse binary and ternary alphabets. Simulations show that the expected complexity of the sparsity-aware sphere decoder is practically more feasible than classical sphere decoder for the considered system parameters.

As part of the future work, we will study higher-order moments of the algorithm complexity as well as the performance of the proposed algorithm in various applications.

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