TOPOLOGY OPTIMIZATION FOR A TRADE OFF BETWEEN ENERGY COST AND NETWORK LIFETIME IN AVERAGE CONSENSUS

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ABSTRACT

Consensus algorithms are simple processes that involve repeated communications between the nodes of the network until a consensus is reached with certain accuracy. In this setting, the lifetime of the network and the total required energy not only depend on the number of iterations needed to achieve consensus, but also on the power consumption per node and iteration. In this work, we propose a method to optimize the network topology in order to reduce the total energy required to achieve consensus while increasing the network lifetime. Our solution is based on an optimization technique that performs a tradeoff between these two concepts. Simulation results, under different types of networks, are presented to show clearly the efficiency and validity of our approach.

Index Terms— Complex Networks, Average consensus algorithms, Convex optimization

1. INTRODUCTION

Average consensus algorithms have attracted a great deal of research work in recent years because of their simplicity. These are in-network processes, in the sense that each node is able to obtain global information as a function of some initial data by only exchanging information with its immediate neighbors [1][2]. In spite of the simplicity and decentralized nature of consensus algorithms, since they are inherently iterative it is incurred in a repeated communication cost, whose minimization is crucial.

Although the energy constraints of the nodes forming the network depend on the system being modeled, it is generally of great interest to reduce the energy requirements of the algorithm being executed. In particular, the energy cost of the average consensus algorithm depends on two factors closely related to the topology of the network. On the one hand, the number of iterations needed to reach consensus with certain accuracy is given by the algebraic connectivity of the underlying graph [3]. On the other hand, the power consumption of the nodes in each iteration depends on the number of edges of this underlying graph.

Most of the related work [4-6] only focuses on reducing the convergence time, by increasing the value of the algebraic connectivity of the network. However, besides reducing the number of iterations to reach consensus, it is crucial to decrease the power consumption in each of these iterations. Some relevant works in this direction are [7] and [8], where it is shown how to jointly reduce both terms by redesigning the network topology. Nevertheless, if the energy available at a node is supplied by batteries, it is also important to consider the concept of network lifetime. It indicates the total time that the network is able to correctly operate before a certain number of nodes run out of batteries. This concept is crucial in the execution of an average consensus process because the final achieved value is influenced by the state of all the nodes. Moreover, it is not always true that the network lifetime is maximized when the total energy consumption is minimized. For example, the work in [8] reduces the total energy consumption at the expense of reducing the lifetime of some nodes because these perform many communications.

In this paper we present an optimization problem that presents a tradeoff between the total energy required to achieve consensus, and the maximum power consumption per node. Thus, our problem controls the importance of each of these two parameters in the final solution. It is based on the minimization of a multi-objective function, where the network topology is optimized. Additionally, we present several simulation results showing the differences between the solutions corresponding to different graph models.

The remainder of this paper is structured as follows: some background on consensus problems is presented in Section II. In Section III, we present an optimization method whose solution is a trade off between energy cost and network lifetime. Section IV presents some simulation results to verify and show clearly the efficiency of our approach. Finally, the conclusions of this work are summarized in Section V.

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2. PROBLEM FORMULATION

In this section, we revise some graph theory and consensus related concepts that we use throughout this paper. First, when an information flow exists among the nodes of a network, we can model it as a graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$, consisting of a set $\mathbf{V}$ of N nodes and a set $\mathbf{E}$ of edges. We denote an edge as a pair of nodes (i, j). Given a graph, it can be defined the $N \times N$ adjacency matrix $\mathbf{A}$, where an entry $[\mathbf{A}]_{ij}$ is equal to 1 if $(i, j) \in \mathbf{E}$ and 0 otherwise. The set of neighbors of a node i is defined as $\mathbf{\Omega}_i = \{j \in \mathbf{V} : (i, j) \in \mathbf{E}\}$ and the degree matrix $\mathbf{D}$ is a diagonal matrix, whose entries are given by $d_i = \sum_{j \in \mathbf{\Omega}_i} [\mathbf{A}]_{ij}$. Then, the Laplacian of a graph is a matrix defined as $\mathbf{L} = \mathbf{D} - \mathbf{A}$, whose smallest eigenvalue is equal to zero.

Let us assume that nodes have some initial data at time instant $t = 0$. We collect them in a initial vector $\mathbf{x}(0)$, whose average is $\mathbf{x}_{\text{avg}} = \frac{\mathbf{1}\mathbf{1}^T \mathbf{x}(0)}{N}$, where $\mathbf{1}$ denotes the all ones column vector. We consider the general linear update of the state of each sensor i at time instant t, given by:

$$\dot{\mathbf{x}}(t) = -\mathbf{L}\mathbf{x}(t) \tag{1}$$

If the graph is undirected, the matrix $\mathbf{L}$ is positive semidefinite by construction, and the convergence to the average of the iterative process described in (1) is then ensured. The convergence time T_{conv} of this linear update is defined in [1] as the time required by the slowest mode of (1) to be reduced by a factor $\rho < 1$, and can be expressed as $T_{\text{conv}} = -T_s \frac{\log(\rho)}{\lambda_2(\mathbf{L})}$, where T_s is the duration of a time slot unit and $\lambda_2(\mathbf{L})$ is the so-called algebraic connectivity.

In this work, we propose the problem of optimizing the network topology involving a multi-objective function. First, we focus on minimizing the total energy required by the network to achieve consensus. Considering the factor ρ to be small enough, this energy is the product between the total power consumed per iteration by the whole network P_{iter}, and the convergence time. In a formal way, we have the following:

$$\mathcal{E}_{\text{tot}} = T_{\text{conv}} P_{\text{iter}} = T_{\text{conv}} \sum_{i \in \mathbf{V}} P_i = T_{\text{conv}} \sum_{(i,j) \in \mathbf{E}} p_{ij} \tag{2}$$

where P_i is the power consumption per iteration of node i, and p_{ij} is the power that node i requires to successfully communicate with node j. For simplicity, we assume a generic path loss model $p_{ij} = p_{\min} r_{ij}^{\gamma}$, where $p_{\min}$ is the minimum power required at the receiver to successfully decode the incoming information, r_{ij} is the distance between nodes i and j and γ is the path loss exponent. Specifically, $\gamma = 0$ makes the power distribution among the nodes equal to the degree distribution, $\gamma = 1$ implies a linear cost and $\gamma \geq 2$ represents the signal attenuation associated with wireless communications.

Besides, consensus algorithms require all the nodes to converge to the average of the initial values [1][2]. It implies that the process is ruined as soon as any of the nodes runs out of batteries. Therefore, in order to maximize the lifetime of the network, we are also interested in minimizing the power consumption of the node that consumes more power, that is, $P_{i_{\max}} = \max(P_1, P_2, \ldots, P_N)$. This quantity depends on the network model being considered, as we explain later. In a more formal way, we are interested in maximizing the following expression:

$$\mathcal{L} = \frac{\mathcal{E}_{\text{node}}}{T_{\text{conv}} P_{i_{\max}}} \tag{3}$$

where $\mathcal{E}_{\text{node}}$ is the total energy available at the nodes, assuming that every node present the same energy capabilities. Therefore, this expression determines the number of consensus processes that can be executed in the given network before the first node runs out of batteries. If the network includes nodes having different energy capabilities, the equivalent amount to maximize would be $\min_i(\frac{\mathcal{E}_i}{T_{\text{conv}} P_i})$.

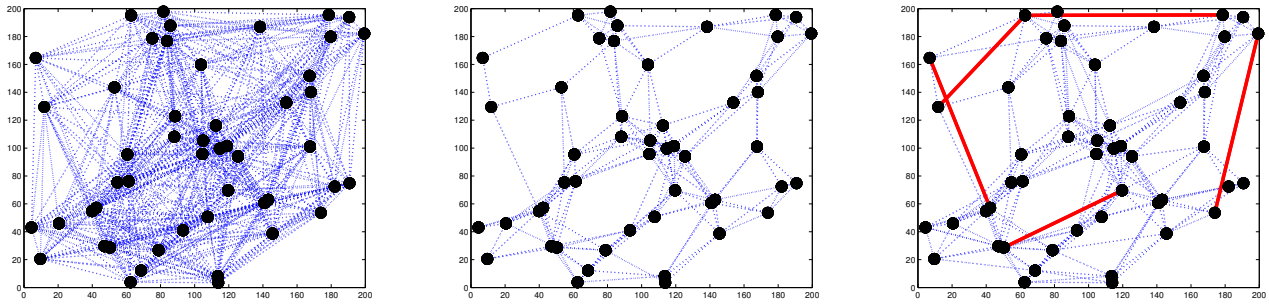
Finally, it is of great interest to ensure that the resulting graph belongs to a specific model, which determines the nodes capabilities and the set of edges to be considered. For example, the nodes in a Wireless Sensor Networks present hard constraints in terms of power, which implies that they can communicate only with their nearest nodes. This type of network has been successfully modeled by using Random Geometric Graphs (RGGs) [11]. Other networks such as Internet can be modeled as Scale Free Graphs (SFGs) [12]. Finally, biology systems exhibit a behavior that can be modeled with Small World Graphs (SWGs) [13]. All these graph particularities should be considered in our problem.

3. OPTIMIZATION METHOD

Our problem is to obtain the best topology for a consensus process under some specific graph constraints, such that the weighted sum of the total energy consumption in (2) and the inverse of the network lifetime in (3) is minimized. The corresponding optimization problem can be formulated as follows (**P1**):

$$\begin{array}{ll} \text{minimize}_{\{\mathbf{A}\}} & \beta \frac{P_{i_{\max}}(\mathbf{A})}{\lambda_2(\mathbf{L}(\mathbf{A}))} + (1-\beta) \frac{P_{\text{iter}}(\mathbf{A})}{\lambda_2(\mathbf{L}(\mathbf{A}))} \\ \text{s. t.} & \xi \leq \lambda_2(\mathbf{L}(\mathbf{A})) \\ & \mathbf{A} = \mathbf{A}^T \\ & [\mathbf{A}]_{ij} \in \{0, 1\} \end{array}$$

where ξ is an arbitrary small positive constant to ensure that the resulting value of $\lambda_2(\mathbf{L}(\mathbf{A}))$ is greater than zero and β is a constant between 0 and 1 that controls the trade off between the total energy consumption and the lifetime of the network. The variation of β gives the Pareto-optimal points of **P1**. The objective of **P1** is to find the entries of the adjacency matrix $\mathbf{A}$ that minimize the multi-objective function $\beta \frac{P_{i_{\max}}(\mathbf{A})}{\lambda_2(\mathbf{L}(\mathbf{A}))} + (1-\beta) \frac{P_{\text{iter}}(\mathbf{A})}{\lambda_2(\mathbf{L}(\mathbf{A}))}$. Hence, we have made explicit the dependence of the Laplacian $\mathbf{L}$, $P_{i_{\max}}$ and P_{iter}, on $\mathbf{A}$.



(a) No constraint (b) RGG constraint (c) SWG constraint

Fig. 1. Example of three different graphs generated from the same random deployment and applying Algorithm 1 with $a_{\text{th}} = 0.45$ and $\beta = \frac{1}{2}$. (a) Topology obtained without considering any specific graph model, which yields $P_{i_{\max}} = 0.0136$, $P_{\text{iter}} = 0.5707$ and $\lambda_2(\mathbf{L}(\mathbf{A})) = 9.822$ (b) Random Geometric Graph model, which yields $P_{i_{\max}} = 0.001$, $P_{\text{iter}} = 0.0225$ and $\lambda_2(\mathbf{L}(\mathbf{A})) = 0.6529$. (c) The equivalent Small World Graph model yields $P_{i_{\max}} = 0.0013$, $P_{\text{iter}} = 0.04$ and $\lambda_2(\mathbf{L}(\mathbf{A})) = 0.938$.

The problem **P1** is a combinatorial one because of the binary variables constraint. In order to obtain a polynomial time solvable problem, we introduce a relaxation consisting on assuming the entries of $\mathbf{A}$ to be real variables between 0 and 1, which results in a fractional convex-concave problem. To solve it, we introduce the following function [10]:

$$h(\mu) = \min\{\beta P_{i_{\max}}(\mathbf{A}) + (1-\beta)P_{\text{iter}}(\mathbf{A}) - \mu\lambda_2(\mathbf{L}(\mathbf{A}))\}$$

which allow us to solve the original problem **P1** by applying **Algorithm 1**. This algorithm is based on repeatedly solve the following optimization problem (**P2**):

$$\begin{array}{ll} \min_{\{s,\mathbf{A}\}} & s \\ \text{s. t.} & \beta P_1(\mathbf{A}) + (1-\beta)P_{\text{iter}}(\mathbf{A}) - \mu\lambda_2(\mathbf{L}(\mathbf{A})) \leq s \\ & \vdots \\ & \beta P_N(\mathbf{A}) + (1-\beta)P_{\text{iter}}(\mathbf{A}) - \mu\lambda_2(\mathbf{L}(\mathbf{A})) \leq s \\ & \xi \leq \lambda_2(\mathbf{L}(\mathbf{A})) \\ & \mathbf{A} = \mathbf{A}^T \\ & 0 \leq [\mathbf{A}]_{ij} \leq 1 \end{array}$$

which can be easily obtained from **P1** by applying standard optimization tools [9].

Additionally, in order to ensure that the resulting graph obtained by using **Algorithm 1** belongs to a particular graph model: RGG, SWG, etc., we introduce additional topological constraints to the problem **P2**. For example, in the particular case of considering a RGG, the transmission range of the nodes is limited to a maximum distance R, so that, several of the entries of the resulting matrix $\mathbf{A}$ are fixed to zero, that is, $\mathbf{A} \in \mathcal{A}_{\text{RGG}} = \{\mathbf{M} \in \mathbb{R}^{N\times N} : [\mathbf{M}]_{ij} = 0 \text{ if } r_{ij} > R\}$. Equivalently, a SWG can be obtained in a similar way than a RGG, by allowing a small set of nodes to establish extra long links (shortcuts). Similar constraints can be obtained for other graph models. Fig. 1 shows an example of the effect of these additional constraints on the solution graph.

Finally, the problem **P2** can be shown to be convex, which implies that it can be efficiently solved by using numerical tools. Thus, applying the Dinkelbachs's algorithm [10], we are able to obtain the optimal value of the parameter μ, which is equivalent to the optimal value of the relaxation of the original problem **P1**, as described by **Algorithm 1**.

The result that is obtained from **Algorithm 1** are the entries $[\mathbf{A}]_{ij}$ of the adjacency matrix, which define the network topology. The value of the parameter ϵ in **Algorithm 1** determines the precision of the algorithm, that is, how close its solution is from the optimal one of the problem **P1**. However, due to the relaxation procedure, these matrix coefficients are real variables belonging to the interval $[0, 1]$, instead of binary values that determine the presence or absence of a specific link. Thus, in order to obtain a real network topology, entries with smaller values than a predefined threshold a_{th} are removed. For a given solution, the choice of a_{th} determines the final topology. As a_{th} approaches 0, total connectivity is obtained. On the other hand, for values of a_{th} above certain value, the network becomes disconnected. This choice affects both the total required energy and the network lifetime, as it is shown in Fig. 2.

Algorithm 1

Require: ϵ

Ensure: $\beta P_{i_{\max}}(\mathbf{A}) + (1-\beta)P_{\text{iter}}(\mathbf{A}) - \mu\lambda_2(\mathbf{L}(\mathbf{A})) \leq \epsilon$

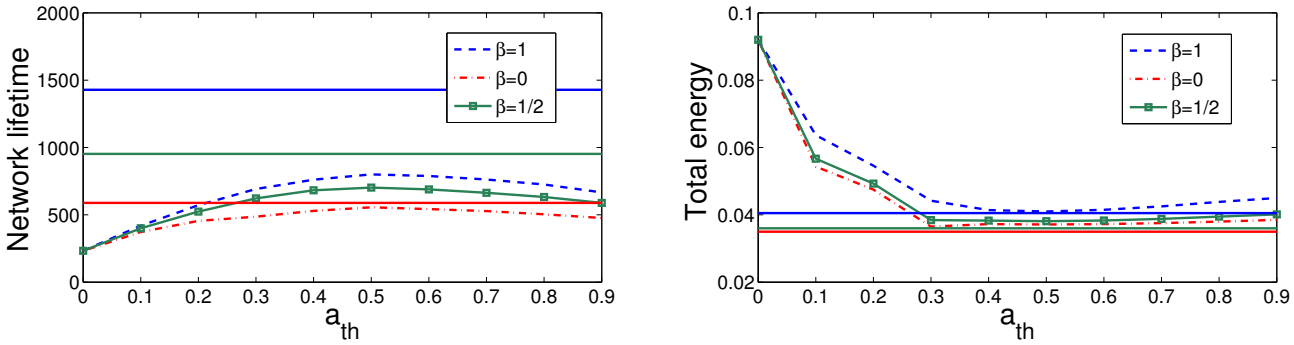
Set matrix $\mathbf{A}$ as a feasible solution

while $\beta P_{i_{\max}}(\mathbf{A}) + (1-\beta)P_{\text{iter}}(\mathbf{A}) - \mu\lambda_2(\mathbf{L}(\mathbf{A})) > \epsilon$ **do**

set μ as $\frac{\beta P_{i_{\max}}(\mathbf{A}) + (1-\beta)P_{\text{iter}}(\mathbf{A})}{\lambda_2(\mathbf{L}(\mathbf{A}))}$

Solve **P2** with the current μ

end while



(a) Lifetime of the network

(b) Total required energy

Fig. 2. Lifetime of the network and total required energy as a function of the applied threshold. The results of different values of the parameter β are compared ($\beta = 1$, $\beta = 0$ and $\beta = \frac{1}{2}$). The solid lines represent the corresponding relaxed solutions.

4. NUMERICAL RESULTS

Our simulation setup includes $N = 50$ nodes randomly placed in a square area of $L = 200$ meters side. The power that each node requires to reach other node is computed by applying a simplified path loss model, with a path loss exponent $\gamma = 3$, a signal to noise threshold $\Phi = 10$, and a background noise $N_0 = 10^{-10}$ mW., such that $p_{ij} = \Phi N_0 r_{ij}^{\gamma}$, expressed in mW. Finally, we assume $\mathcal{E}_{\text{node}} = 1$ and $T_s = 1$ ms.

Fig. 1 shows three topologies resulting from applying **Algorithm 1** with a) no topological constraints, b) the RGG constraint and c) the SWG constraint. It is interesting to see how the network model considered affects the solution graph. The RGG in Fig. 1 b) has been generated by constraining the connectivity range to 80 meters, while some random nodes are allowed to establish extra large links (shortcuts) in the SWG shown in Fig. 1 c). The size of these shortcuts depend on the value of the path loss exponent. As it increases, the size of these shortcuts decreases. Furthermore, since the nodes involved in the shortcuts are incurring in an extra energy consumption that affects the value of $P_{i_{\max}}$, other nodes establish extra links, increasing P_{iter} as a consequence. In spite of that, since the value of $\lambda_2(\mathbf{L}(\mathbf{A}))$ is significantly increased, the total required energy $\mathcal{E}_{\text{tot}}$ and the network lifetime $\mathcal{L}$ are both improved with respect to Fig. 1 b).

Fig. 2 shows the averaged results of applying **Algorithm 1** (no particular graph model considered), where the x-axis represents the applied threshold. Fig. 2 a) shows the lifetime of the network, expressed in number of consensus processes that can be carried out before the first node runs out of batteries. The dashed blue curve corresponds to the case of $\beta = 1$, that is, the lifetime is maximized and the total required energy is not considered. It can be seen that a maximum is obtained for threshold values around $a_{\text{th}} = 0.45$. The blue solid line represents the initial solution without thresholding, giving an upper bound as a result. On the other hand, the red dashed and pointed curve corresponds to the case $\beta = 0$, such that the only condition is the minimization of the total required energy. For this case, the upper bound is given by the red straight line. The gap between both curves corresponds to the different Pareto optimal solutions of the problem, e.g. the green curve in the middle shows the case of $\beta = 1/2$.

Fig. 2 b) shows the total energy required to reach consensus, expressed in μJules. This quantity is minimized for $\beta = 0$, represented by the dashed red curve, and the rest of Pareto solutions lie between this curve and the dashed blue one. It is interesting to point out that the value $\beta = 1/2$, represented by the green curve, yields competitive lifetime values and quasi minimum energy consumption.

Finally, Table 1 summarizes the simulation results for the different models of graphs considered in this paper.

Table 1. Simulation Results with $a_{\text{th}} = 0.45$

Graph	$\mathcal{L}^{\beta=1}$	$\mathcal{E}_{\text{tot}}^{\beta=1}$	$\mathcal{L}^{\beta=0}$	$\mathcal{E}_{\text{tot}}^{\beta=0}$	$\mathcal{L}^{\beta=\frac{1}{2}}$	$\mathcal{E}_{\text{tot}}^{\beta=\frac{1}{2}}$
none	795	0.037	475	0.035	734	0.036
RGG	1000	0.038	600	0.027	750	0.027
SWG	1100	0.031	440	0.024	850	0.025

5. CONCLUSIONS

In this paper, we present a multi-objective problem to optimize the network topology. It presents a trade-off between the minimization of the total energy required to achieve consensus and the maximization of the network lifetime. Moreover, the solution graph is also influenced by the topological constraints imposed to obtain a specific graph model. Simulation results are presented to show the difference between the pareto-optimal solutions for different graph models.

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