STUDENTS' UNDERSTANDING OF CONVOLUTION[1]

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ABSTRACT

This paper examines students' interpretations of open-ended convolution tasks in order to understand typical approaches students use to arrive at solutions. The goal of the study is to determine which solution approaches are more or less effective for students, as well as understand how signals and systems novices interpret open-ended problems in a course.

Index Terms— signals and systems education, convolution, graphing, representations

1. INTRODUCTION

An ongoing challenge in signals and systems instruction is helping students learn to apply and transfer concepts and move beyond rote application of procedures to thinking about when and where to use procedures. Related to this issue of conceptual understanding is the need for understanding multiple representations of problems, including written statements (e.g., explaining the problem), equations, and various forms of graphs. All three representations are important to the conceptual understanding and the flexible use of procedures to solve problems. For example, students need to understand how different graphs illustrate different aspects of a system and which equations may be appropriate or useful for solving a problem. This requires them to transfer and to extend their mathematics knowledge to signal processing contexts and applications. For this study, we closely examined two typical signals and systems problems that incorporated written, symbolic, and graphical representations and which elicited a variety of student responses. We wanted to understand which representations students used and how they utilized graphs, in particular, to solve problems.

The course we are investigating is a junior-level course where students are regularly (at least once per week) given a problem to solve in class. These problems focus on foundational concepts (such as convolution) that the instructor wants to emphasize and ensure that students understand. The in-class problems provide students an opportunity to obtain feedback, as well as indicate to the instructor when students might need additional help. For this study, we have used one in-class problem (solved as a group) and one exam problem (solved individually) to understand students' conceptions of convolution. This work complements existing studies of students' understanding of convolution (e.g., [1]) by focusing on students' use of graphs in solving convolution tasks.

2. RELATION TO PRIOR WORK

This literature review will focus on studies and discussions of graphing rather than representations in general. Cramer [2] discussed a five-part framework of mathematical representations including: symbolic, written, verbal, concrete, and graphical. She described the need for translation between and among these different types of representations as people work on mathematical problems. While signals and systems is a particular application of mathematical activity, the students' interpretations and analyses of graphs is related to other areas of higher mathematics where graphs are used to represent complicated phenomena. Students need to be able to move flexibly between symbolic and graphical representations. In addition, they need to translate between graphical representations. In particular, conceptual understanding of foundational topics such as convolution is difficult for students to develop [3].

A number of authors focus on the concept of graph sense or how users of graphs interpret, analyze and create graphical representations of phenomenon (e.g. [4-7]. There are two aspects of graph sense that are relevant for the current study. First, there is the analysis and interpretation of graphs. Studies have examined how experts and novices read familiar and novel graphs and make sense of the information [8]. In studies of scientific experts who used graphs, Roth and Bowen [6] studied how the experts interpreted graphs both within their own discipline of expertise and graphs from a related, but unfamiliar discipline. The scientific experts use a two-stage process to read graphs. They first identify the salient features and then they use those features to ground the graph within the real context it represents. The meaning of the graph is situated within the scientists' experience making the signs and symbols used in graphing (e.g., standard variable names within the discipline) a critical component of reading

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graphs. The second aspect of graph sense is the development or creation of graphs. Ubuz [7] discussed graphing the derivative in relationship to the function. The study examined how students understood the tangent line and then developed interpretations of the graphs of the function and its derivative. The students' conceptual understanding of a function and its derivative was tightly related to prototypical graphs used in their calculus courses. Similar to Roth and Bowen [6], Ubuz's study underscored the use of graphs as signs or symbols for real phenomenon, concepts, symbolic representations or verbal descriptions. Interpretation of graphs is both an essential part of conceptual understanding of the mathematical phenomena they represent and a significant part of graph sense itself.

In the study presented in this paper, the students were asked to connect graphs to symbols and to other graphs. Prior work has focused on the relationship between graphs and other forms of representation rather than translations between graphs. Within Cramer's [2] five-part framework, there is also room for within-representation translations (e.g., graph to graph, symbols to symbols). While research has examined translation between representations, there is less research about translation within representations. However, within engineering, there are applications (such as signals and systems) where students need to work with multiple representations of the same phenomena that are in the same representational form. For example, one might encounter two different graphs that represent the same system. Students spend significant time in calculus courses using time as the independent variable, but frequency is not used as extensively, nor do students often compare and interpret the same function using different independent variables. In addition, students have limited experience with discrete math in contrast to their experience with continuous problems.

Students' understanding of functions in general and graphical representations specifically plays a role in their interpretation of the characteristics of the signals and systems they are given in problems. By the junior-level engineering course under investigation, the students have completed the calculus sequence and differential equations so should have a strong sense of variables and the relationships between variables in a function. Thompson [9] presented three interpretations students commonly have of functions: action, process and object. In the action conception, students view functions as a means for calculation. In the process conception, functions shift to representing a process for evaluation, not just the result of the evaluation. Finally, in the object conception, students see functions as objects for manipulation in and of themselves. For signals and systems contexts, all three conceptions are necessary. The objects in question (signals or systems) have real-world meaning, and students particularly need an object conception of function in order to be able to consider symbolic and graphical representations of signals that can be manipulated. A system is an example of a function whose domain and range are also functions. For example, a filter is a type of system where a signal (represented by a function) is passed through a filter (also represented by a function) whose output is another signal. The functions and graphing literature is largely situated within algebra and typically focuses on linear or quadratic functions where students are asked to translate between real-world contexts, graphical and symbolic representations (e.g., [10]).

3. METHODOLOGY

After reviewing two exams and multiple in-class problems from a junior-level signals and systems course, two problems (shown in Figures 1 and 2) were selected for further analysis because of their open-ended nature and the students' use of graphs and other drawings to develop their solutions to convolution problems. Since the in-class problems are completed as a group, we wanted to use the exam problem to analyze individual students' understanding. However, the in-class problem provides understanding of their earlier conceptions in a more informal setting. The goal of the analysis was to describe the types of solutions students employed and the level of their success and efficiency at solving the stated problem. We selected the tasks that were open-ended (in order to have access to which approaches the students used) and that included a graphical component as well as symbolic components in order to understand how students used these two representations. Next, the responses to the tasks were coded by solution strategy and grouped by the correctness of the response. Sixteen student responses to the in-class task were analyzed, and 31 responses to the exam problem were coded.

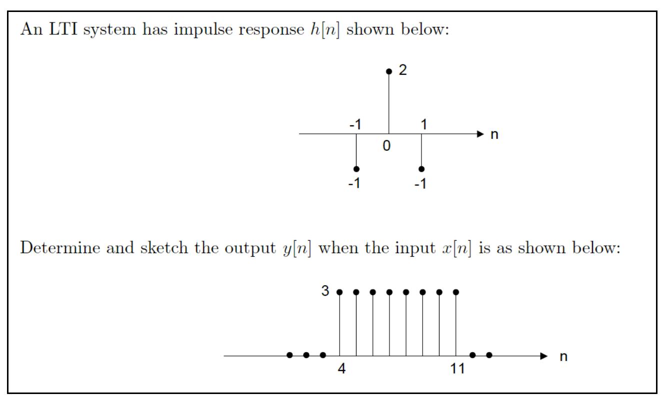

Figure 1. In-class problem

4. RESULTS

We first consider the in-class problem shown in Figure 1. As anticipated, a variety of solution strategies emerged. Most students did draw figures or graphs to support their analysis of the problem. However, they did not all draw the same types of graphs. When the convolution sum was used directly to guide graphical convolution, students flipped and shifted h[n].

Consider a discrete-time LTI system with the impulse response $h[n]$. The system is known to be causal and FIR. Additionally, it is known that when the input to the system is $x[n] = u[n]$, the output is the signal $y[n]$ shown in Figure 7.1 below.

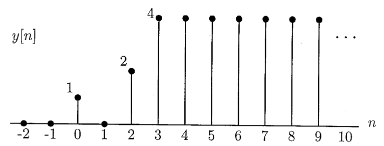

Figure 7.1: $y[n]$, output of the system when the input is $x[n] = u[n]$.

Determine and sketch $h[n]$. Even if you are unable to specify $h[n]$ fully, you may receive partial credit by describing what you know about $h[n]$ given the pieces of information you have about the system.

Figure 2. Exam Problem

Table 1 shows the distribution of student solutions across two major coding categories (student strategy and the correctness of the output) for the in-class problem. (Note that a response can receive more than one code). As shown in Table 1, a large percentage of students generated equations for h[n], x[n], or both, even though an equation-based representation of the sequences was not required to solve the problem. Students who used scaling and shifting strategies, whether applied to h[n] or x[n], drew graphs to determine the system output. Students who drew upon the derivation of the convolution sum from linearity and time invariance properties to solve the problem were split among those who summed scaled and shifted copies of x[n] and those who summed scaled and shifted copies of h[n]. Students in the latter group demonstrated an understanding of both the concept of convolution and the commutative nature of the operation. Students who constructed equations for x[n] and/or h[n] often used these equations to determine how copies of the signal or impulse response should be scaled (and shifted) before summing.

In-class Code	**Frequency**
Student Strategy	
Constructs equation for h[n]	6
Constructs equation for x[n]	3
Flips and shifts h[n] to find output at each n	5
Scales and shifts x[n] and sums copies	6
Scales and shifts h[n] and sums copies	3
Correct/Incorrect Output	
Fully correct output	11
Output correct except indices	3
Incorrect output	2

Table 1. In-class problem results

Students' attachment to familiar use of notation was apparent in the solution strategies. Many of those who chose to sum scaled and shifted copies of h[n] exchanged the names of the two sequences (h[n] and x[n]) so that their approach fit the familiar model of summing scaled and shifted copies of the input as dictated by the elements of the impulse response.

Interesting examples of student work are shown in Figures 3 and 4. Figure 3 shows a brute-force solution strategy that is ultimately incorrect. The students attempt to apply the convolution sum brute force, producing flipped and shifted versions of h[n], multiplying h[n-k] by x[k], and summing at each n to obtain y[n]. This is cumbersome, and the students incorrectly address the shifting task. The response illustrates how students need to understand foundational concepts as well as to know which types of graphs or sketches will be useful. In contrast, Figure 4 shows a correct and elegant response where students have drawn directly on the derivation of the convolution sum to produce the system output.

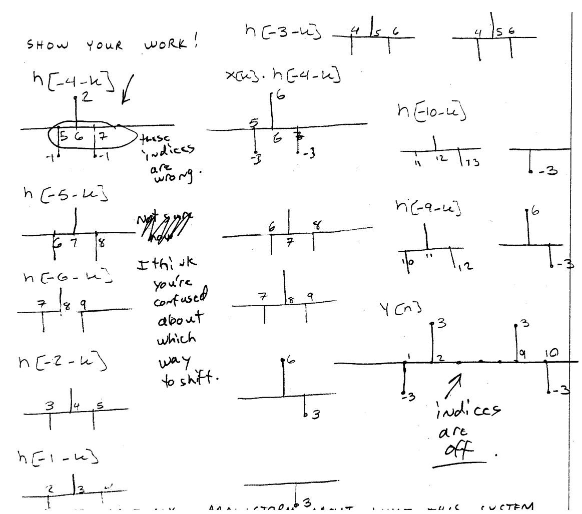

Figure 3. Incorrect student response to the in-class problem

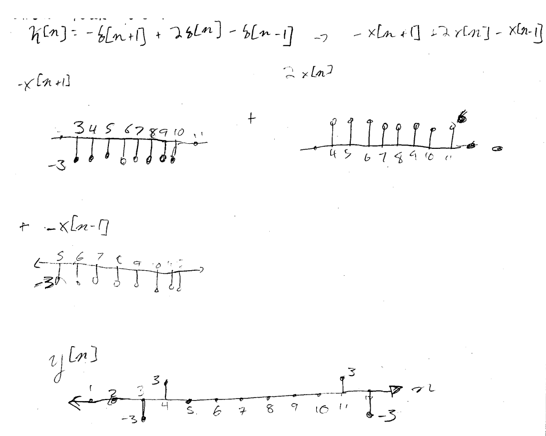

Figure 4. Correct solution to the in-class problem

The 31 responses to the exam problem were coded as shown in Table 2. (Note that a response could receive more than one code). Again, a range of strategies were observed using both sketches and equations with the large majority of students sketching the system input u[n]. However, significantly more responses were incorrect, probably due to the individual nature of the exam and the increased difficulty of the problem. Twelve of the students computed h[n] element by element, noting that starting from n=0, each sample of y[n] is a function of one additional sample of h[n] until h[n] and x[n] fully overlap in the flipping and shifting process. Ten of the twelve students who computed h[n] element by element also had correct solutions, indicating this was a largely successful strategy for solving the problem. Three students used conceptual properties of convolution to find h[n], e.g., using the causality of the system and the value of n at which y[n] becomes constant to infer the length and range of support of h[n]. Also notable is how many students could not articulate an approach to the task. Many of those worked backward through the task but could not express how they were using the definition of convolution to arrive at their solution. Because they were not given an input and an impulse response, students could not directly apply the convolution sum to solve the problem.

Exam Problem Code	Frequency
Sketch/Equation	
Sketches x[n]	19
Generates equation for y[n]	5
Approach	
Computes h[n] element-by-element	12
Scales and shifts copies of x[n]	2
Uses properties of convolution procedure	3
Assumes h[n] has same shape as y[n]	7
Applies other misconception	2
No approach provided	5
Correctness	
Correct	16
Incorrect	12
No answer	3

Table 2. Exam problem coding

5. CONCLUSIONS AND DISCUSSION

An ongoing question related to students' conceptions of convolution and their approaches to solving convolution tasks is how to help them find effective and efficient strategies. In addition, students need to be able to think conceptually about a problem rather than simply applying equations or procedures. Graphing is often a more conceptual approach, and students seem to struggle more to construct appropriate graphs and employ them correctly. While there might be multiple approaches to a problem, there are strategies that are more or less efficient. The students in this sample were also challenged by their ability to use graphs and equations.

For teaching introductory signals and systems courses, it is important for instructors to understand how novices interpret problem situations and may rely on a few, rote procedures when solving tasks rather than approaching problems conceptually. This means that changes that may appear trivial to an instructor (expert) are not necessarily trivial to the learner (novice). The problem responses also continue to illustrate students' reliance on equations even when graphs or other diagrams may be more effective and efficient at leading to a solution and illustrating salient characteristics of the problem situation. In addition, instructors need to ensure that assessments and homework assignments include a wide variety of problem types in order to challenge students to analyze the approaches they are selecting to solve problems. There are assessments available such as the signals and systems concept inventory [11] that include tasks that assess students' conceptual understanding. For example, one strategy is to remove numeric values from tasks or focus students on graphical responses to require them to consider the concepts rather than relying on rote calculations and procedures.

6. REFERENCES

[1] Nasr, R., Hall, S. R., & Garik, P. (2009). Understanding naïve reasonings in signals and systems: A foundation for designing effective instructional material. Proceedings of the IEEE Digital Signal Processing Workshop, pp. 720-725.

[2] Cramer, K. (2003). Using a translation model for curriculum development and classroom instruction. In R. Lesh & H. Doerr (Eds.), Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning and teaching (pp. 449-464). Mahwah, NJ: Lawrence Erlbaum.

[3] Buck, J. R., Wage, K. E., Hjalmarson, M. A., & Nelson, J. K. (2007). Comparing student understanding of signals and systems using a concept inventory, a traditional exam and interviews. Proceedings of the ASEE/IEEE Frontiers in Education Conference, pp. S1G-1 – S1G-6.

[4] Friel, S. N., Curcio, F. R., & Bright, G. W. (2001). Making sense of graphs: Critical factors influencing comprehension and instructional implications. Journal for Research in Mathematics Education, 32(2), 124-158.

[5] Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. Review of Educational Research, 60(1), 1-64.

[6] Roth, W.-M., & Bowen, G. M. (2001). Professionals read graphs: A semiotic analysis. Journal for Research in Mathematics Education, 32(2), 159-194.

[7] Ubuz, B. (2007). Interpreting a graph and constructing its derivative: Stability and change in students' conceptions. International Journal of Mathematical Education in Science and Technology, 38(5), 609-637.

[8] Roth, W., & Lee, Y. J. (2004). Interpreting unfamiliar graphs: A generative, activity theoretic model. Educational Studies in Mathematics, 57(2), 265-290.

[9] Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), Research in Collegiate Mathematics Education, 1 (Issues in Mathematics Education Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.

[10] Moschkovich, J. N. (1999). Students' use of the x-intercept and an instance of a transitional conception. Educational Studies in Mathematics, 37(2), 169-197.

[11] Wage, K. E., Buck, J. R., Wright, H. G., & Welch, T. B. (2005). The signals and systems concept inventory. IEEE Transactions on Education, 48(3), 448-461.