COMPRESSED SENSING OF DIFFUSION FIELDS UNDER HEAT EQUATION CONSTRAINT

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ABSTRACT

Reconstructing a diffusion field from spatiotemporal measurements is an important problem in engineering and physics with applications in temperature flow, pollution dispersion, and disease epidemic dynamics. In such applications, sensor networks are used as spatiotemporal sampling devices and a relatively large number of spatiotemporal measurements may be required for accurate source field reconstruction. Consequently, due to limitations on the number of nodes in the sensor networks as well as hardware limitations of each sensor, situations may arise where the available spatiotemporal sampling density does not allow for recovery of field details. In this paper, the above limitation is resolved by means of using compressed sensing (CS). We propose to exploit the intrinsic property of diffusive fields as side information to improve the reconstruction results of classic CS which we call diffusive compressed sensing (DCS). Experimental results demonstrate the effectiveness and usefulness of the proposed method in substantial data savings while producing estimates of higher accuracy, as compared to classic CS-base estimates.

Index Terms— Diffusion field, spatiotemporal sampling, sensor networks, compressed sensing

1. INTRODUCTION

Many natural phenomenon in physics are governed by diffusion equation, including temperature flow, pollution dispersion, and disease epidemic dynamics. In such applications, sensor networks are used as spatiotemporal sampling devices to sample and reconstruct diffusion fields [1]. In contrast to general multidimensional signals, the effect of temporal and spatial down-sampling are not homogeneous. Generally, it is more expensive to increase the spatial sampling density as more sensors are needed in the network, while temporal sampling density is only limited by each sensor hardware [2]. An efficient sampling scheme will have an impact on real world applications such as pollution detection [3] and plume source detection [4]. In this paper, we try to provide a sampling scheme for such applications in the literature.

Inverse problems of the diffusive fields are generally ill-posed and require a relatively large number of measurements. Typically, such dense data sets are required to allow for accurate reconstruction of fine field details. In such cases, improving the acquisition requirements of the hardware in use through reducing the sampling density would unavoidably produce aliasing artifacts. Fortunately, recent advances in sampling theory offer a means - known as compressed sensing (CS) [5,6] - to overcome this limitation, while allowing for accurate reconstruction of digital sources from sub-Nyquist sampling rates.

In the current note, we consider spatiotemporal sampling and reconstruction of a 1-D diffusive field $u(x,t)$ governed by the heat equation:

$$\begin{aligned} &\frac{\partial u(x,t)}{\partial t} = \gamma \frac{\partial^2 u(x,t)}{\partial x^2}, \quad t \geq 0, \\ &u(x,0) = f(x) \end{aligned} \tag{1}$$

where γ is the diffusion coefficient, x denotes spatial domain variable, t denotes time domain variable, and $f(x)$ represents the initial field value.

If the initial field value is available, we can solve (1) for $u(x,t)$. However, in many situations, initial field value is not available [2], and it is not possible to derive $u(x,t)$ based on solely the partial differential equation constraint, as $u(x,t)$ varies dramatically with different initial condition. In these situations, we can measure spatiotemporal samples and use them to reconstruct $u(x,t)$.

In this paper, we introduce a novel method for reconstruction of diffusion fields governed by heat equation, from the sub-critical (incomplete) measurements of their spatiotemporal samples. Here, we take advantage of CS for efficient field sampling. The classic CS framework does not incorporate arbitrary *a priori* information on the interrogated signals. In particular, at the case at hand, it seems to be natural to reconstruct the source field using the fact that it satisfies the partial differential equation in (1). Specifically, we propose new CS formulation that incorporates the side information derived from (1) to improve the reconstruction quality of the standard CS [5,6], while resulting in substantial reduction in the required sampling density. We show through theory that our CS formulation can reduce the dimension of the feasible region in CS reconstruction, resulting in better reconstruction quality. Experimental results are provided to further demonstrate the effectiveness of the proposed method.

2. PROPOSED SAMPLING SCHEME

2.1. Classic Compressive Sensing

Compressed Sensing (CS) is a technique to reconstruct digital signal from sub-Nyquist sampling rates [5,6]. Consider a digital signal $\mathbf{x} \in \mathbb{R}^n$ which can be represented sparsely in a dictionary domain $W \in \mathbb{R}^{n\times n}$, $\mathbf{x} = W\mathbf{c}$. Let $\mathbf{c}$ is k-sparse ($\|\mathbf{c}\|_0 = k$), a classic result of CS states that this signal can be recovered from linear measurements $\mathbf{y} \in \mathbb{R}^m$ with a dimension as low as $m = O(K\log(n/K))$ [5,6], which are assumed to be acquired according to

$$\mathbf{y} = \Psi\mathbf{x} + \mathbf{n}, \tag{2}$$

where $\Psi \in \mathbb{R}^{m\times n}$ is a full-rank sensing matrix (with $n > m$) and $\mathbf{n}$ denotes measurement noise. Due to overcompleteness of Ψ, (2) is clearly ill-posed. However, classic theory of CS states that if $\Phi = \Psi W$ obeys the restricted isometry property (RIP) of order s with parameter δ_s [5,6]

$$\forall \mathbf{c} \in \mathbb{R}^n, \|\mathbf{c}\|_0 = s \to 1 - \delta_s \leq \frac{\|\Phi\mathbf{c}\|_2^2}{\|\mathbf{c}\|_2^2} \leq 1 + \delta_s, \tag{3}$$

then (2) has a unique solution and **c** can be obtained by solving:

$$\mathbf{c} = \arg\min_{\mathbf{c}'} \|\mathbf{c}'\|_1 \quad \text{s.t.} \quad \|\Phi\mathbf{c}' - \mathbf{y}\|_2^2 \leq \epsilon, \tag{4}$$

where $\epsilon > 0$ is a parameter which controls the noise power. Several algorithms, using the convex analysis and optimization, have been developed in the literature for solving (4). Note that the feasible region of the problem (4) is $S_{\Phi,\mathbf{y}} = \{\mathbf{c}|\Phi\mathbf{c} = \mathbf{y}\}$, and is automorphic to null-space of Φ. In other words we search null(Φ) for finding the optimal solution. As the dimension of null(Φ) decreases (e.g., number of measurements m increases), ill-posedness of the problem (2) becomes less severe and the search for the optimal solution would become easier. Instead of increasing the number of measurements, we show that in this paper how to incorporate the underlying property of a diffusive field to reduce the dimension of feasible space.

Note that RIP is not an essential concept for derivation of CS results. A more recent approach for CS provides similar results by studying properties of the null-space of the sensing matrix [7]. A subspace $C \subset \mathbb{R}^n$ with dimension $\dim(C) = n - m$ is said to have spherical section property (SSP) with distortion Δ if:

$$\forall \mathbf{c} \neq 0 \in C \rightarrow \frac{\|\mathbf{c}\|_1}{\|\mathbf{c}\|_2} \geq \sqrt{\frac{m}{\Delta}}. \tag{5}$$

If null(Φ) has Δ-spherical section property then, if the source sparse representation **c** is sparse enough ($\|\mathbf{c}\|_0 \leq \frac{m}{2\Delta} \leq \frac{n}{2}$), then the stated classic results of CS on uniqueness of solution of (2) and solvability by algorithm (4) holds [7].

2.2. Diffusive Compressive Sensing

Let $u(x,t)$ represents an original diffusive field which satisfies (1). For the sake of convenience, $u(x,t)$ is assumed to be defined over a finite-dimensional, uniform, rectangular lattice in $\mathbb{R}^2$. The discretized version of this field can be represented in a matrix $X \in \mathbb{R}^{N\times M}$. We assume that this field is sampled via a sensor network with N_s nodes which are deployed uniformly in the space and each sensor collects N_t uniform samples in time. Clearly, we have $m = N_t N_s$ measurements which can be represented in a matrix $Y \in \mathbb{R}^{N_s\times N_t}$ with $m = N_s N_t \leq NM = n$. X and Y can be concatenated into two column vectors **x** and **y** by means of lexicographic ordering, respectively. It is assumed that the observed version **y** of the vector **x** is obtained as $\mathbf{y} = \Psi\mathbf{x}$, where Ψ is a subsampling matrix which accounts for the effect of uniform downsampling. It is also assumed that **x** admits sparse representations with respect to a linear transformation W, $\mathbf{x} = W\mathbf{c}$. Finally, in order to apply CS to the problem, it is assumed that null(Φ) satisfies SSP by choosing Ψ and W properly.

Under the above conditions, CS-based reconstruction of the representation coefficients **c** can be performed according to

$$\mathbf{c}^* = \arg\min_{\mathbf{c}'} \left\{ \frac{1}{2}\|\Phi\mathbf{c}' - \mathbf{y}\|_2^2 + \lambda\|\mathbf{c}'\|_1 \right\}. \tag{6}$$

This formulation is equivalent to (4) and hence CS solver algorithms can be used to handle this optimization problem.

Our proposed diffusive CS (DCS) algorithm extends the CS approach by using the fact that $u(\cdot,\cdot)$ satisfies (1). Let $D_x \in \mathbb{R}^{n\times n}$ and $D_t \in \mathbb{R}^{n\times n}$ denote the matrices of discrete partial differences in the spatial and time directions, respectively. Then, the discretized version of the constraint (1) suggests that

$$D_t\mathbf{x} = \gamma D_x D_x \mathbf{x} \rightarrow (D_t - \gamma D_x D_x)W\mathbf{c} = 0. \tag{7}$$

Let $B := (D_t - \gamma D_x D_x)W$, $\Phi' = \begin{bmatrix}\Phi \\ B\end{bmatrix}$, $\mathbf{y}' = \begin{bmatrix}\mathbf{y} \\ 0\end{bmatrix}$, and $\mathbf{n}' = \begin{bmatrix}\mathbf{n} \\ 0\end{bmatrix}$, then:

$$\mathbf{y}' = \Phi'\mathbf{c} + \mathbf{n}'. \tag{8}$$

Algorithm 1: Diffusive Compressive Sampling

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1. *Data:* $\mathbf{y}$, δ, γ and $\lambda > 0$
2. *Initialization:* For a given transform matrix W and matrices/operators Ψ, D_x, D_t, preset the procedures of multiplication by $A = \Psi W$, A^T, B and B^T.
3. *Diffusisive field recovery:* Starting with an arbitrary $\mathbf{c}^{(0)}$ and $p^{(0)} = 0$, iterate (11) until convergence to result in an optimal $\mathbf{c}^*$.
   a. Use CS solver algorithm of [8] for solving the optimization problem in (11).
   b. Update the vector of Bregman variables $p^{(t)}$.
4. *Source recovery:* Use the estimated (full) sparse representation $\mathbf{c}^*$ to recover the values of $\mathbf{x} = W\mathbf{c}^*$.

Note that the problem (8) is an instance of the problem (2) and can be studied in CS framework. Here, the constraint $B\mathbf{c} = 0$ in (8), can be viewed as extra noiseless measurements of the sparse source. Furthermore, we have

$$\text{null}(\Phi') = \text{null}(\Phi) \cap \text{null}(B) \rightarrow \text{null}(\Phi') \subset \text{null}(\Phi), \tag{9}$$

and since null(Φ) satisfies SSP with parameters m and Δ, then null(Φ') as a subset would also satisfy SSP with the same parameters or equivalently from (5) $m + n$ and $\Delta' = \Delta(1 + n/m)$. Consequently (8) share the same unique solution with (1). This result ensures that adding the additional constraint would not violate uniqueness of the solution to (1).

Equivalently, the DCS approach recovers the optimal $\mathbf{c}^*$ as a solution to the following constrained problem

$$\mathbf{c}^* = \arg\min_{\mathbf{c}'} \left\{ \frac{1}{2}\|\Phi\mathbf{c}' - \mathbf{y}\|_2^2 + \lambda\|\mathbf{c}'\|_1 \right\}, \quad \text{s.t. } B\mathbf{c}' = 0. \tag{10}$$

A solution to (10) lays in intersection of null(B) and $S_{\Phi,\mathbf{y}}$. Thus the feasible region of (10) is a subset of the feasible region of (4) and clearly with smaller dimension. Intuitively, one can expect that solving (10) will result in a more accurate approximation for **c** as compared to (6) which motivated us to consider solving (10) instead of (6).

A solution to (10) can be found by means of the Bregman algorithm [9], in which case $\mathbf{c}^*$ is computed iteratively as given by

$$\begin{cases} \mathbf{c}^{(t+1)} = \arg\min_{\mathbf{c}'} \left\{ \frac{1}{2}\|\Phi\mathbf{c}' - \mathbf{b}\|_2^2 + \right. \\ \qquad \left. +\lambda\|\mathbf{c}'\|_1 + \frac{\delta}{2}\|B\mathbf{c}' + p^{(t)}\|_2^2 \right\} \\ p^{(t+1)} = p^{(t)} + \delta B\mathbf{c}^{(t+1)}, \end{cases} \tag{11}$$

We used the FISTA algorithm [8] to solve the **c**-update step, due to the simplicity of its implementation. Given the optimal solution $\mathbf{c}^*$, the *dense* source signal can be recovered as $\mathbf{x} = W\mathbf{c}$. Algorithm 1 summarizes all the DCS algorithmic steps.

Table 1. PSNR comparisons of diffusion field recovery results for noise level of 10 dB

d_s	1	2	2	1	4	4	1	8	8	1	16	16
d_t	2	1	2	4	1	4	8	1	8	16	1	16
PoS	50%	50%	25%	25%	25%	6.25%	12.5%	12.5%	1.56%	6.25%	6.25%	0.39%
	PSNR comparison (in dB) for $u_1(\cdot,\cdot)$											
CS	12.50	13.36	11.67	3.13	3.21	2.25	0.32	0.31	-0.01	-0.23	-0.24	-0.03
DCS	19.33	18.79	18.39	18.31	17.23	16.64	15.46	16.64	14.85	13.79	15.57	9.34
	PSNR comparison (in dB) for $u_2(\cdot,\cdot)$											
CS	12.60	12.81	11.47	3.09	3.13	2.18	0.31	0.31	-0.01	-0.24	-0.23	-0.02
DCS	22.44	23.48	22.17	20.72	22.64	21.09	18.76	21.401	16.65	16.73	17.67	9.31
	PSNR comparison (in dB) for $u_3(\cdot,\cdot)$											
CS	12.43	12.93	11.12	3.01	3.01	2.15	0.31	0.32	0.01	-0.24	-0.24	-0.03
DCS	20.98	21.69	20.47	19.43	20.94	18.65	17.78	19.64	16.16	16.04	16.70	9.64

3. EXPERIMENAL RESULTS

The proposed algorithm is tested over three different solutions of the heat equation (1) as the source field, denoted by $u_1(\cdot,\cdot), u_2(\cdot,\cdot)$, and $u_3(\cdot,\cdot)$ for different boundary and initial conditions. The fields are assumed to be defined over the lattice $[0, 2\pi] \times [0, 1] \subset \mathbb{R}^2$, uniformly discretized with $M = N = 128 \rightarrow n = 16384$. We set the boundary conditions to be non-homogeneous for $u_1(\cdot,\cdot)$ and $u_2(\cdot,\cdot)$, and homogenous Neumann condition for $u_3(\cdot,\cdot)$. The initial conditions are chosen to be $f_1(x) = \Pi(0, \pi)$, $f_2(x) = \delta(x - \pi)$ (local point source), and $f_3(x) = x$ for each case, respectively. The subsampling matrix Ψ is assumed to downsample the source field uniformly with downsampling d_t and d_s factor in time and spatial domains, respectively:

$$Y(i, j) = X(d_s i, d_t j), \quad 1 \leq i \leq N_s, 1 \leq j \leq N_t \tag{12}$$

For sparse representation basis, an over complete W was derived from a four-level orthogonal wavelet transform using the nearly symmetric wavelets of Daubechies with five vanishing moments and $\delta = 0.5, \lambda = 0.001, \gamma = 1$.

For the purpose of comparison, we have compared our algorithm with classic CS approach in terms of reconstruction SNR. The results of this comparison are summarized in Table 1 and Table 2 for different levels of noise and different percentage of the samples (PoS) in each table. In each table results for downsamling factors of $2, 4, 8, 16$ in different directions are provided. As expected, for all cases one can see that DCS results in substantially high values of output SNR as compared to classic CS, which implies a higher accuracy of field reconstruction. A close look on both tables reveals interesting results of the proposed algorithm. Note if we downsample a source field in one direction with the same downsampling factor, regardless of the direction, the resulting number of measurements are the same. Now consider those columns of tables with the same downsampling factor but different direction, e.g. third and fourth column, while for the case of classic CS the reconstruction quality differs in both tables, the quality of reconstruction for the case of DCS is similar. This can be explained through different correlations of the samples in different dimensions. From (7) one concludes that a field sample $X(i, j)$ is correlated with $X(i+1, j)$ and $X(i+2, j)$ in spatial domain while it is only correlated with $X(i, j+1)$ in time domain. In other words dependency of the samples are not the same in time and spatial domain and it is harder to reconstruct the field when we lack time samples which is reflected in CS reconstruction results. In contrast, when we apply DCS these dependencies are considered as an additional data and thus the reconstruction quality is similar and is independent of downsampling direction. Generally, when we encounter insufficient spatial samples, oversampling in time domain is used to compensate [2]. Our result indicates that DCS can recover the source with less time samples which can be translated as energy saving in sensor nodes.

Another important result is on robustness of the proposed scheme towards insufficient samples. Consider a row in Table 1 or Table 2, it can be seen that as the downsampling factor increases the reconstruction quality for classic CS degrades severely and for downsampling factors of 8 and 16 almost no information is recovered. While for the case of DCS, the algorithm is robust and even when we downsample a field with factor of 16 in both directions, using almost 0.4% of the samples, it still can recover some information. For better comparison Fig. 2 depicts performances of CS and DCS algorithm for a range of downsampling factors with $d_t = 1$, $SNR = 40dB$, and $u_2(\cdot,\cdot)$. It can be seen that for the case of CS, the reconstruction quality degrades sharply for downsampling factors greater than 4 while DCS is robust towards downsampling. This can be explained by the constraint exploited by DCS. The constraint $B\mathbf{c}' = 0$ in (10), can be considered as extra measurements of the sparse source which can compensate for insufficient real measurements. This can explain while the difference between CS and DCS is negligible for small downsampling factors, why it becomes considerable as the scaling factor increases. When we have enough information to recover the source then constraint (7) does not provide considerable information but when we lack enough information, this constraint becomes more important.

A comparison between the result of Table 1 and Table 2 also reveals that although reconstruction quality degrades as the additive noise power of measurements increases but DCS seems more robust towards the noise. To investigate the robustness of the proposed algorithms towards measurement noises, its performances has been compared for a range of SNR values (as a measure for noise power) with classic CS for the case $d_t = 2, d_s = 2$ and $u_3(\cdot,\cdot)$. The results of this comparison are summarized in Fig. 3. As expected in both cases the reconstruction quality degrades by decreasing SNR, but this dependency is more critical for classic CS, which results in steeper graph in Fig. 3. Again, this can be explained by the constraint exploited by DCS which restricts the feasibility region for an optimal solution. Moreover, as explained the constraint $B\mathbf{c}' = 0$ in (10), can be considered as extra measurements of the sparse source. These measurements are noise free and consequently one concludes that if we use this constraint, the reconstruction algorithm will become more robust towards the noise power. Intuitively one can say that since $\mathbf{n} \in \mathbb{R}^m$, $\mathbf{n}' \in \mathbb{R}^{n+m}$, and $\|\mathbf{n}'\|_2 = \|\mathbf{n}\|_2$, the noise power has been multiplied by $\frac{m}{n+m} < 1$.This fact represents another ad-

Table 2. PSNR comparisons of diffusion field recovery results for noise level of 40 dB

d_s	1	2	2	1	4	4	1	8	8	1	16	16
d_t	2	1	2	4	1	4	8	1	8	16	1	16
PoS	50%	50%	25%	25%	25%	6.25%	12.5%	12.5%	1.56%	6.25%	6.25%	0.39%
	PSNR comparison (in dB) for $u_1(\cdot,\cdot)$											
CS	12.86	13.49	11.83	3.11	3.25	2.24	0.31	0.32	-0.00	-0.24	-0.23	-0.02
DCS	18.32	19.96	18.77	17.40	18.77	16.98	15.51	17.43	15.19	13.87	15.91	9.52
	PSNR comparison (in dB) for $u_2(\cdot,\cdot)$											
CS	12.74	13.12	11.32	3.07	3.13	2.16	0.33	0.33	-0.01	-0.24	-0.24	-0.03
DCS	22.75	23.88	22.35	20.84	22.95	20.34	18.87	21.57	16.95	16.81	17.63	9.36
	PSNR comparison (in dB) for $u_3(\cdot,\cdot)$											
CS	12.27	13.09	11.08	3.05	3.08	2.14	0.30	0.31	0.01	-0.24	-0.24	-0.02
DCS	21.27	22.34	20.82	19.60	21.26	18.89	17.91	19.85	16.45	16.24	16.72	9.55

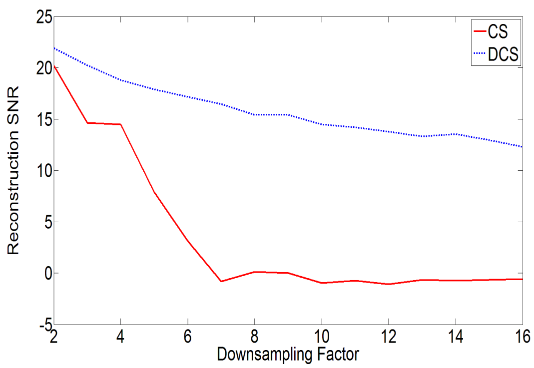

Fig. 1. SNR of field reconstruction as a function of spatial downsampling factor. Here, the solid and dashed lines correspond to classic CS and DCS, respectively, and $d_t = 1$.

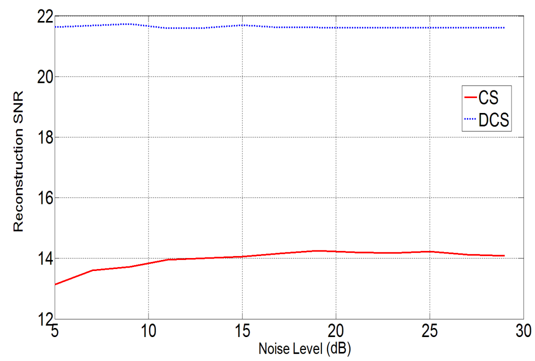

Fig. 2. SNR of field reconstruction as a function of noise SNR. Here, the solid and dashed lines correspond to classic CS and DCS, respectively, and $d_t = 2, d_s = 2$.

vantage of incorporating the diffusive field constraints in the process of field recovery.

4. CONCLUSION

In this paper, the problem of diffusive field reconstruction using sub-Nyquist sampling rates is studied. A CS-based approach has been proposed to simplify the measuring devices and improve the device resolution. The proposed method applies CS for field reconstruction subject to an additional constraint, which stems from the intrinsic property of a diffusive field. Experiments confirm the source estimates by DCS have better quality as compared to the case of classic CS and comparable as to the case of dense sampling. One direction for future work is applying the algorithm in designing the sampling devices for diffusive field reconstruction. Applying the algorithm in the sampling device structure will improve the capability of reconstructing diffusive field details in the presence of low density measurements. Another direction is to understand the performance under partial model knowledge.

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