# DISTRIBUTED DETECTION AND UNIFORMLY MOST POWERFUL TESTS

*Uri Rogers and Hao Chen*

Department of Electrical and Computer Engineering, Boise State University, Boise, ID 83725 USA

## ABSTRACT

Uniformly Most Powerful (UMP) centralized detection for the composite binary hypothesis problem has been well researched. This paper extends the UMP methodology to the parallel distributed detection problem, when the local observations are independently distributed. A collection of general theorems and corollaries define sufficient conditions for the existence of a UMP parallel Distributed Detection (UMP-DD) under one set of fusion rules. These same conditions under another set of general fusion rules result in at least a Locally Most Powerful Distributed Detection (LMP-DD) rule. The subtleties of these conclusions are explored using informative examples that highlight the strengths of this approach and introduce new groups of UMP-DD tests.

***Index Terms*—Composite Hypothesis Testing; Uniformly Most Powerful Test; Distributed Detection; Log-concave**

## I. INTRODUCTION

Distributed detection has been an active research area with a focus on detection performance analysis and optimization via the design of the local decision and global fusion rules [1]. Optimal local sensor decision rules in a distributed binary hypothesis testing system can be complicated and is a known NP-hard problem [2]. When the sensors are conditionally dependent, the optimal local sensor decision and global fusion rules are coupled and the *form* of the optimal local sensor decision rules is often unknown [3], [4]. Fortunately, when the local sensor observations are conditionally independent where the conditioning is done on each binary hypothesis, the local sensor decision rule can be optimally chosen as likelihood quantizers under many inference regimes [5], [6]. One of these regimes is a fixed binary hypothesis testing problem. Even under this simple model with defined likelihood quantizers, optimizing a distributed sensor network can be extremely difficult, as the solution space is non-linear, can be non-convex, and is coupled to the end fusion rule [1]. Certainly, when the problem is expanded to include composite binary hypothesis testing, the optimization becomes even more challenging.

Under the traditional centralized composite testing framework, *Uniformly Most Powerful* (UMP*)* tests are highly desired because the decision rule is independent of the unknown composite parameter(s) [7]. For reference, a UMP hypothesis test is a test that has the greatest Power (detection probability) among all possible tests of Size (false alarm probability) $\alpha$, where $\alpha \in [0, 1]$ under the Neyman-Pearson (N-P) Lemma [8]. Centralized UMP testing and its associated terminology is common, with excellent examples available in [7]–[9]. This paper extends the UMP concept into the realm of distributed binary hypothesis testing, which we will call *UMP-Distributed Detection* (UMP-DD).

Within distributed detection, UMP-DD tests do exist but the terminology is less common, if not void. This is partially due to the complexity of a distributed detection and the mathematics required to analyze the performance of fusion rules. With that said, the UMP-DD tests explored by this paper are tightly coupled with the concept of equal quantizer thresholds at every sensor in a distributed detection system. This has been studied previously for fixed binary distributed hypothesis testing. When the hypothesis are fixed, Irving and Tsitsiklis showed that for two sensors under *Independent Identically Distributed* (i.i.d.) Gaussian noise that equal quantizers are optimal [10]. Warren and Willett analyzed the multi-sensor fixed and equal observation model to explore sensors with equal quantizer thresholds with “well behaved densities” [6]. That paper generally showed that for those well behaved densities under either *AND* or *OR* fusion, that equal sensor thresholds were optimal, extending the results of Irving and Tsitsiklis beyond two sensors and Gaussian noise. Additionally, Warren and Willett also showed that equal sensor thresholds were at least locally optimal for the general counting (also known as $k$ out of $N$) fusion rule.

Along with introducing the UMP-DD terminology, this paper will establish a generalized framework to both find and prove the existence of UMP-DD tests, including the definition of sufficient conditions. In doing so, new classes of UMP-DD test will be introduced by treating the composite parameter as a random variable versus a constant fixed but unknown parameter. This concept will be applied to a parallel distributed binary hypothesis detection system with conditionally independent sensor observations, where the conditioning is on the respective hypothesis. The composite random variable is assumed to have a defined *a-prior probability density functions* (pdf) that is smooth and log-concave. The conclusions that follow rely on the generalized log-concave efforts of Prékopa [11], [12] and to some extent the log-concave probability work by Bagnoli and Bergstrom [13].

This paper will also extend the work of Warren and Willett for those cases where the composite parameter is fixed, but unknown by specifically defining sufficient conditions on the pdfs involved. A result that follows as a natural corollary to the more general framework described in the previous paragraph. Similarly, this framwork can be used to extend the local optimality or LMP-DD of the general counting rule, but a proof that equal sensor thresholds are globally optimal (i.e. UMP-DD) remains an open research topic that will be discussed in detail.

Finally, under the fixed but unknown observation model, we augment the work mentioned previously by presenting a theorem that generalizes the UMP-DD model. This theorem defines the optimal sensor thresholds under monotone fusion rule and monotonic non-decreasing *Likelihood Ratio Test* (LRT) conditions for Neyman-Pearson (N-P) hypothesis testing that do not rely on smooth log-concave pdfs.

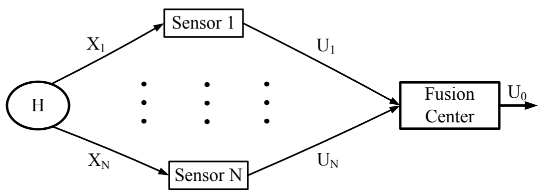


**Fig. 1**. A canonical parallel distributed multiple hypothesis testing system.

## II. PARALLEL DISTRIBUTED DETECTION SYSTEMS

Consider a canonical parallel distributed hypothesis testing system with $N$ sensors and a single Fusion Center (FC), as illustrated in Fig. 1. Within this structure, no inter-sensor feedback or FC to sensor communication is assumed. Under Binary hypothesis conditions the canonical elements can be formalized as follows:

- Hypotheses: $H \in \{H_0, H_1\}$.
- Local sensor observation $X_j$, $j = 1, 2, \cdots, N$.
- Local sensor output $U_j = \gamma_j(X_j) \in \{0, 1\}$, $j = 1, 2, \cdots, N$ and $\gamma_j(X_j)$ the local decision rule.
- Fusion center output $U_0 \in \{0, 1\}$ indicating $H_{U_0}$ hypothesis is accepted.

Boldface capital letters (e.g., $\mathbf{X}$, $\mathbf{U}$) are used to denote vectors of random variables and boldface lower case letters to denote a particular realization of the random vector. Additionally, $p_{\mathbf{Z}}(\mathbf{z})$ denotes the pdf of a continuous random variable, where the shorthand notation of $p\left(\mathbf{z}\right)$will be used to indicate that the pdf is based on the random variable/vector $\boldsymbol{Z}$. Furthermore, all logarithms are base $e$ .

Under this notation, the local sensor observations when conditioned on each hypothesis are

$$p(\mathbf{X}|H_i) = \int_{\mathbf{\Omega}} p\left(\mathbf{X}|\boldsymbol{\theta}, H_i\right) p\left(\boldsymbol{\theta}|H_i\right) d\boldsymbol{\theta}, \tag{1}$$

where $\boldsymbol{\theta}$ is the true signal level at the $N$ sensors with *a-priori* pdf $p\left(\boldsymbol{\theta}|H_i\right)$ having support $\mathbf{\Omega}$, and $p\left(\mathbf{X}|\boldsymbol{\theta}, H_i\right)$ is the probability of observing $\mathbf{X}$ when $\boldsymbol{\theta}$ is received under $H_i$, $i \in \{0, 1\}$. Applications like target detection typically define the $H_0$ hypothesis as "target absent" ($\boldsymbol{\theta} = \mathbf{0}_{N\times 1}$) and $H_1$ denotes "target present" with unknown vector parameter $\boldsymbol{\theta}$; e.g., the target emission power ($\boldsymbol{\theta} > \mathbf{0}_{N\times 1}$). We will use the notation $\boldsymbol{\theta}$ when $H_1$ is active and $\boldsymbol{\theta}'$ when $H_0$ is active to help indicate that the two values explicitly differ.

Within the parallel distributed hypothesis testing framework, the *Chair-Varshney* (C-V) fusion rule defines the optimal fusion rule once the local LRT rules are defined and the observations are conditionally independent [14]. When the observations and local decisions are i.i.d. with a fixed number of sensors, then the general C-V fusion rule can be simplified as

$$U_o = \begin{cases} H_1 & \sum_{j=1}^{N} U_j \geq k \\ H_0 & \sum_{j=1}^{N} U_j < k \end{cases}, \tag{2}$$

which can be read as the $k$ out of $N$ fusion rule. The *OR* rule is defined as $k = 1$, while the *AND* rule defined as $k = N$. All three variations will be explored in the next section.

## III. MAIN RESULTS

Consider the distributed detection problem where both the observations and local decisions are conditionally independent, that is $p\left(\boldsymbol{\theta}|H_i\right) = \prod_{k=1}^{N} p\left(\theta_k|H_i\right)$. Then the general rule (1) can be written as

$$\begin{aligned} p(\mathbf{X}|H_i) &= \prod_{k=1}^{N} \int_{\Omega} p\left(X_k|\theta_k, H_i\right) p\left(\theta_k|H_i\right) d\theta_k \\ &= \prod_{k=1}^{N} p\left(X_k|H_i\right), \end{aligned} \tag{3}$$

where $\theta_k$ is the signal level at sensor $k$ and $p(\mathbf{X}|H_i)$ the resultant joint conditional pdf. Under this framework the LRT for each sensor is $L_k(X_k) = \frac{\int_{\Omega} p(X_k|\theta_k, H_1) p(\theta_k|H_1) d\theta_k}{\int_{\Omega} p\left(X_k|\theta_k', H_0\right) p\left(\theta_k'|H_0\right) d\theta_k'}$, where $\theta_k > \theta_k'$ under a composite hypothesis testing structure for $k = 1, 2, \cdots, N$. The LRT is compared against a quantizer threshold, $\eta_k$, with conditionally independent observations and produces a decision $U_k$, which is an optimal test [1]. With this result, the special $k$ out of $N$ fusion rule in (2) is now applicable.

Before proceeding, it is important to define the term log-concave as it will be critical to our analysis. Using the definition in [15]

**Definition 1.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is *logarithmically concave* if $f(x) > 0 \quad \forall x \in \mathrm{dom} f$ and $\log f$ is concave or $-\log f$ is convex. Defining $\log 0 = -\infty$, then $f$ is log-concave if $f(x) \geq 0$ and the newly extended $\log f$ is concave.

We now explore special cases of (3) that offers sufficient conditions for a UMP-DD test. A key point regarding the following theorem is that the conditions are not overly restrictive, as many problems in distributed detection easily meet these constraints, with examples provided in the sequel.

**Theorem 2.** *If $\int_{\Omega} p\left(X_k|\theta_k, H_i\right) p\left(\theta_k|H_i\right) d\theta_k$ is a smooth function (continuous derivatives), the support $\Omega$ is convex and both $p\left(\theta_k|H_i\right)$ and $p\left(X_k|\theta_k, H_i\right)$ are log-concave for $i \in \{0, 1\}$ with $p\left(X_k|\theta_k, H_i\right)$ conditionally independent. Then equal quantizers, $\eta_1 = \eta_2 = \cdots = \eta_K = \eta$, are a UMP-DD test under the AND or OR fusion rules.*

The following key theorem will be used in the proof of Theorem 2. Prékopa in [12] with theorem VI, showed that when $f\left(\boldsymbol{X}, \boldsymbol{Y}\right)$, $\boldsymbol{X} \in \mathbb{R}^n$, $\boldsymbol{Y} \in \mathbb{R}^m$ is log-concave in $\mathbb{R}^{n+m}$ and with $\boldsymbol{A}$, a convex subset of $\mathbb{R}^m$, then

$$g\left(\boldsymbol{X}\right) = \int_{\boldsymbol{A}} f\left(\boldsymbol{X}, \boldsymbol{Y}\right) d\boldsymbol{y} \tag{4}$$

is log-concave over all of $\mathbb{R}^n$.

*Proof:* Let $p\left(\theta_k|H_i\right)$ and $p\left(X_k|\theta_k, H_i\right)$ be log-concave and $\Omega$ be convex. Since the product of log-concave functions is log-concave, then $p\left(\theta_k|H_i\right) p\left(X_k|\theta_k, H_i\right)$ is log-concave. Thus $p\left(X_k|H_i\right) = \int_{\Omega} p\left(X_k|\theta_k, H_i\right) p\left(\theta_k|H_i\right) d\theta_k$ is log-concave by the prior Prékopa Theorem. It is worthwhile to note that for nearly all log-concave pdfs of interest, $\Omega$ is either an open or closed connected interval in $\mathbb{R}$, which is both convex and concave.

Consider the probability of detection under the *AND* fusion rule

$$P_D = \prod_{k=1}^{N} \int_{Z_{1_k}} \int_{\Omega} p\left(X_k|\theta_k, H_1\right) p\left(\theta_k|H_1\right) d\theta_k \, dx_k, \tag{5}$$

where $Z_{1_k}$ is the critical region for deciding $H_1$. Similarly, the probability of false detection is

$$P_F = \prod_{k=1}^{N} \int_{Z_{1_k}} \int_{\Omega} p\left(X_k|\theta'_k, H_0\right) p\left(\theta'_k|H_0\right) d\theta'_k \, dx_k. \quad (6)$$

Since $p\left(X_k|H_i\right)$ is log-concave for $i = \{0,1\}$, then $P_D$, $1 - P_D$, $P_F$ and $1 - P_F$ are also log-concave [13]. Therefore, convex minimization can be used to find the optimal $Z_{1_k}$, which is the same as finding the optimal quantizer threshold $\eta_k$ for conditionally independent observations.

Under the N-P framework, our goal is to maximize $P_D$ subject to $P_F \leq \alpha$. This is the same as

$$\begin{aligned} \text{minimize}: &\quad -\log P_D\left(\boldsymbol{\eta}\right) \\ \text{subject to:} &\quad -\log\left(1 - P_F\left(\boldsymbol{\eta}\right)\right) + \log\left(1-\alpha\right) \leq 0, \end{aligned} \quad (7)$$

which is a standard convex minimization problem without equality constraints [15]. Thus the method of Lagrange multipliers can be applied to optimize (7). Let $L\left(\boldsymbol{\eta}, \lambda\right)$ be a Lagrange multiplier function with

$$L\left(\boldsymbol{\eta}, \lambda\right) = \sum_{k=1}^{N} -\log \beta_k + \lambda \left( \sum_{k=1}^{N} -\log\left(1-\alpha_k\right) + C \right), \quad (8)$$

where $\beta_k = \int_{\eta_k}^{\kappa_1} p\left(X_k|H_1\right) dx_k$, $\alpha_k = \int_{\eta_k}^{\kappa_0} p\left(X_k|H_0\right) dx_k$, $C = \log\left(1-\alpha\right)$ and $\lambda$ is a Lagrange multiplier. Here, $\kappa_1$ is the upper support limit of $p\left(X_k|H_1\right)$ and similarly for $\kappa_0$ under $H_0$. Setting the partial derivatives $\frac{\partial}{\partial \eta_k} L\left(\boldsymbol{\eta}, \lambda\right)$ to zero results in

$$\frac{\partial}{\partial \eta_k} L\left(\boldsymbol{\eta}, \lambda\right): \quad \frac{p(\eta_k|H_1) - p(\kappa_1|H_1)}{1 - F(\eta_k|H_1)} \frac{F(\eta_k|H_0)}{p(\eta_k|H_0) - p(\kappa_0|H_0)} = \lambda \quad (9)$$

where $F\left(\cdot|H_i\right)$ is the *Cumulative Distribution Function* (CDF) under hypothesis $i$. Note, $p\left(\kappa_i|H_1\right)$ and $p\left(\kappa_i|H_0\right)$ in (9) are often zero, as $p\left(\kappa_i = \infty|H_i\right) = 0$ for most log-concave distributions of interest. With $\lambda$ a fixed constant across all $N$ sensors, $\eta_1 = \eta_2 = \cdots = \eta_N = \eta$ is a solution to (7). To see this, consider the ratio of $\frac{\partial}{\partial \eta_k} L\left(\boldsymbol{\eta}, \lambda\right)$ to $\frac{\partial}{\partial \eta_j} L\left(\boldsymbol{\eta}, \lambda\right)$ for all $j \neq k$.

The optimality of this solution can be verified using the Karush-Kuhn-Tucker (KKT) optimality conditions, which are necessary and sufficient when the objective functions are smooth and convex [15]. With $\boldsymbol{\eta}^*$ and $\lambda^*$ solutions to (8), the KKT conditions are

$$\begin{aligned} \text{primal:} &\ -\log\left(1 - P_F\left(\boldsymbol{\eta}^*\right)\right) + \log\left(1-\alpha\right) \leq 0 \\ \text{dual:} &\ \lambda^* \geq 0 \\ \text{comp. slack.:} &\ \lambda^* \left(-\log\left(1 - P_F\left(\boldsymbol{\eta}^*\right)\right) + \log\left(1-\alpha\right)\right) = 0 \\ \text{vanishing grad.}: &\ \nabla L\left(\boldsymbol{\eta}^*, \lambda^*\right) = 0. \end{aligned}$$

The pdfs ensure $\lambda^* \geq 0$ and $P_F \in [0,1]$ by the smooth function assumption, meeting the complementary slackness constraint and the last two constraints follow by construction. The proof under *OR* fusion is similar. ■

This result is interesting because it establishes a general framework to find and prove the existence of UMP-DD tests. This general framework is applicable to a broad set of problems, including test with composite parameters that are themselves random variables. When the composite parameters are no longer random variables, then the following corollary is applicable.

**Corollary 3.** *When the signal level at each sensor is fixed, but unknown, such that $\theta_k = \theta \; \forall \, k$ so $p\left(X_k|\theta_k, H_i\right) = p\left(X_k|\theta, H_i\right)$ and $p\left(X_k|\theta, H_i\right)$ is smooth log-concave for $i \in \{0,1\}$ on a convex support $\Omega$. Then thresholds $\eta_1 = \eta_2 = \cdots = \eta_N = \eta$ are a UMP-DD test under the AND or OR fusion rules.*

This corollary is similar to the special case formulation proposed by Warren and Willett [6], but here sufficient conditions are defined for when equal thresholds are optimal and the test is UMP-DD. Specifically, $p\left(X_k|\theta, H_i\right)$ must be log-concave and smooth. When one of these conditions is absent, then UMP-DD is not guaranteed. An example is presented later using i.i.d. Laplace noise with $\theta_k = \theta$ for all $k$ to highlight this point, as compared to a Gaussian Noise that is UMP-DD.

We now expand on the previous theorem to address the more general counting $k$ out of $N$ fusion rule.

*Remark* 4. Under the $k$ out of $N$ fusion rule and using the conditions in Theorem 2, then $\eta_1 = \eta_2 = \cdots = \eta_N = \eta$ are at least a local minimum or Locally Most Powerful Distributed Detection (LMP-DD) test. A full proof for this remark follows the same reasoning in [6] as applicable to Corollary 3, but with generalized pdfs and will not be replicated here.

The LMP-DD designation implies that within an $\epsilon-$neighborhood of $\eta_1 = \eta_2 = \cdots = \eta_N = \eta$ for $\left\{\epsilon : \epsilon > 0, \epsilon \in \mathbb{R}^N\right\}$ any other selection of the $\eta'_k$s will result in a lower $P_D$ for a given $P_F \leq \alpha$ N-P constraint. Unfortunately, the LMP-DD label can not be replaced with the more restrictive UMP-DD designation because both the $P_D$ and $P_{FA}$ equations in (7) are a combinatorial summation of probabilities that are not guaranteed convex under the $k$ out of $N$ fusion rule. This results in a non-convex optimization problem. Hence any solution to a Lagrange Optimization method similar to Theorem 2 can only be declared locally optimal (LMP-DD) and not globally optimal (UMP-DD). However, it is likely that this local optimum is in fact the global optimum under the $k$ out of $N$ fusion rule, but a general proof supporting this claim remains an open research topic in distributed detection.

Next, we explore another special UMP-DD case where both the fusion and local LRT are monotone. Let the local likelihood ratio be $L_k(X_k) = \frac{p(X_k|\theta, H_1)}{p(X_k|\theta', H_0)}$ for any $\theta' < \theta$ and $k = 1, 2, \cdots, N$. Many times, a sufficient statistic for the LRT exists, say $T_k\left(X_k\right)$, which is based on sensor $k$'s observation. Using this construct and the widely studied one-sided test, we have the following result.

**Theorem 5.** *Consider a distributed composite hypothesis testing problem with sensor observation model $p(\mathbf{X}|\theta, H_i) = \prod_{k=1}^{N} p(X_k|\theta, H_i)$. To test hypothesis $H_0$: $\theta'$ and $H_1$: $\theta > \theta'$, if*

1) The fusion center employs a *monotone fusion rule* such that the probability of deciding 1 is a monotonic function of $U_k$ and $P(U_0 = 1|U_k = 1, \theta) \geq P(U_0 = 1|U_k = 0, \theta)$ for all $\theta$ and $k$,
2) The local likelihood ratio $L_k(X_k) = \frac{p(X_k|\theta, H_1)}{p(X_k|\theta', H_0)}$ is a monotonic non-decreasing function of $T_k\left(X_k\right)$ for any $\theta' < \theta$,

then, the optimal local sensor decision rule is

$$P(U_k = 1|X_k) = \begin{cases} 0, & T_k\left(X_k\right) < \eta \\ \lambda, & T_k(X_k) = \eta \\ 1, & T_k(X_k) > \eta \end{cases} \quad (10)$$

under the N-P criterion. As is typical, $\lambda$ is a randomization constant and $\eta$ is the quantizer threshold for each $k$'th sensor, with $0 \leq \lambda \leq 1$.

**Table I**. UMP-DD Examples

| Signal Model | Noise | Comment |
|---|---|---|
| $\theta_k = \theta \in \Theta_1$ | $W_k \overset{iid}{\sim} \mathcal{N}(0,1)$ | Corollary 3 and [6] |
| $\theta_k \overset{iid}{\sim}$ Unif $(a,b)$ | $W_k \overset{iid}{\sim} \mathcal{N}(0,1)$ | Equal Uncertainty Model |
| $\theta_k \overset{iid}{\sim}$ Exp $(\lambda)$ | $W_k \overset{iid}{\sim} \mathcal{N}(0,1)$ | Exponential Signal Decay |
| $\theta_k \sim \mathcal{N}(0,\sigma^2)$ | $W_k \overset{iid}{\sim} \mathcal{N}(0,1)$ | Energy Detection Fixed $(\sigma^2 > 1)$ |
| $\theta_k \sim \mathcal{N}(0,\sigma_k^2)$ $\sigma_k^2 \overset{iid}{\sim}$ Unif $(1,b)$ | $W_k \overset{iid}{\sim} \mathcal{N}(0,1)$ | Energy Detection Random$(\sigma_k^2 \geq 1)$ |

*Proof:* From the N-P Lemma, $T_k(X_k)$ is the Uniformly Most Powerful (UMP) test of size $\alpha_k$. With a *monotone fusion rule* and conditional independence, optimization of $P_f = f(\alpha_1|\theta', \cdots, \alpha_K|\theta') \leq \alpha$ and $P_d = f(\beta_1|\theta, \cdots, \beta_K|\theta)$ is the same as separately optimizing $T_k(X_k)$ for all $k$, where $\beta_k|\theta$ is the $k$'th probability of detection given $\theta$ and similarly for $\alpha_k|\theta'$. ■

It is important to note that all UMP-DD under Theorem 2 meet the conditions of Theorem 5, which is more general as it does not specifically rely on smooth log-concave pdfs.

*Remark* 6. As shown in Theorem 5, the optimal local sensor design rule is a single threshold quantizer based on a sufficient statistic, $T_k(X_k)$, similar to the centralized case [9]. Unlike the centralized case, a system-wide Uniformly Most Powerful Distributed Detector (UMP-DD) often does not exist.

## IV. ILLUSTRATIVE EXAMPLES

This section will highlight the sufficient conditions of log-concave and smooth probability distributions to guarantee UMP-DD. The general observation model

$$\begin{array}{ll} H_1(\theta > 0), & X_k = \theta_k + W_k, \\ H_o(\theta' = 0), & X_k = W_k, \end{array} \tag{11}$$

where $\theta_k$ is the signal and $W_k$ is the noise received by the $k'$th sensor. With different $\theta_k$, this model is common for problems in distributed detection systems, including Collaborative Cognitive Radio Spectrum Sensing. With the theorems and corollary of the previous section, it is easy to show that the $\theta_k$ models appearing in Table I are UMP-DD. As discussed during the introduction, the first model with a fixed but unknown signal was shown optimal in [6]. The next four UMP-DD models are new and describe cases where the signal level at each sensor are randomly distributed. The last two are particularly interesting as they describe UMP-DD Energy Detectors for both fixed and random energy levels. While all cases presented are based on zero mean Gaussian noise, any other smooth log-concave noise distribution with appropriate support would also be UMP-DD. Note, under *AND* fusion, Gaussian noise, and $P_{FA} \leq \alpha$ constraint for $N$ sensors, the local sensor threshold is determined using, $\eta = Q^{-1}\left(\alpha^{1/N}\right)$, where $Q^{-1}(\cdot)$ is the inverse standard normal complementary distribution function.

The last point regarding smooth log-concave pdfs is important enough to present an instructive counter example that appears UMP-DD, but is not. Specifically, when the noise follows a Laplace distribution versus a Gaussian pdf for the same observation model, then the test is no longer UMP-DD.

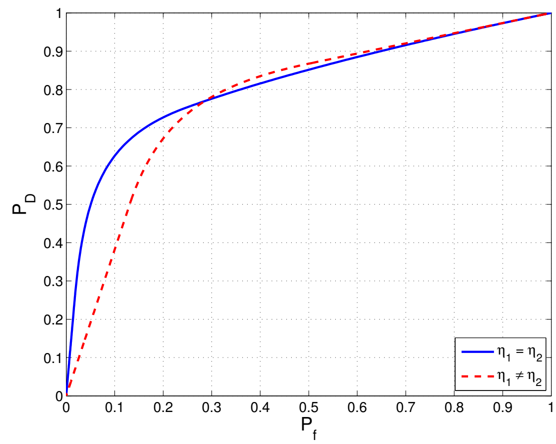


**Fig. 2**. ROC Curves under Laplace Noise

Consider the fixed but unknown observation model with two sensors, $N = 2$, and the noise, $W_k$, following a zero mean i.i.d. Laplace distribution. Since the Laplace distribution is not a smooth function at the mean, we expect that even with equal observations, $\theta_1 = \theta_2$, that a UMP-DD is not possible under *AND* or *OR* fusion. The i.i.d. Laplace noise distribution is

$$p(W|\mu, b) = \frac{1}{2b} e^{-\frac{|w-\mu|}{b}}, \tag{12}$$

with mean $\mu = 0$ and width parameter $b = 0.75$. We now consider the Receiver Operating Characteristic (ROC) curves based on this noise distribution. The ROC curves will show that asymmetric quantizer thresholds are optimal for some false alarm $(\alpha)$ levels, thus the UMP-DD does not exist under the *AND* fusion rule. These results appear graphically in Fig. 2 for $\eta_1 = \eta_2$, $\eta_1 \neq \eta_2$, and $\theta_1 = \theta_2 = 1$, where the latter $\eta_2$ is set such that the second sensor generates a constant detection rate of 99.9% for all $\eta_1$. These data clearly indicate that better detection performance for some false alarm rates can be achieved using asymmetric quantizer thresholds when $0.28 < P_{FA} < 0.80$. Thus, the UMP-DD does not exist for a fixed $\theta = 1$, so it clearly does not exist for all $\theta$.

## V. CONCLUSION

Two theorems defining when a UMP-DD exists has been presented. These theorems generalize and extend the current research in the area of distributed detection. The first theorem generally defines when a UMP-DD exists under *AND* and *OR* fusion rules. The critical constraint is that the conditionally independent observations be smooth, log-concave pdfs even if the observations model is random. Using this result, an UMP-DD energy detector for both fixed and random energy levels was also introduced. Next, the same theorem was extended to show that at least a locally most powerful (LMP-DD) exist for the $k$ out of $N$ fusion rule, where we conjecture that this LMP-DD is actually UMP-DD. The second theorem defines that a single threshold quantizer is optimal under the N-P criterion under appropriate conditions and that these tests are also UMP-DD.

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