DISTRIBUTED STATE ESTIMATION IN MULTI-AGENT NETWORKS

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ABSTRACT

In this paper, we consider the problem of state estimation of a dynamical system in a multi-agent network. The agents are sparsely connected and each of them observes a strict subset of the state vector. The distributed algorithm that we propose enables each agent to estimate any arbitrary linear dynamical system with bounded mean-squared error. To achieve this, the ratio of the algebraic connectivity and the largest eigenvalue of the graph Laplacian has to be larger than a lower bound determined by the spectral radius of the system's dynamics matrix. This extends the notion of Network Tracking Capacity introduced by other authors in prior work. We accomplish this by introducing a new class of estimation algorithm of dynamical systems that, besides a *(consensus + innovations)* term, also includes *consensus on the innovations*.

Index Terms— State estimation, distributed algorithm, multi-agent network, consensus, innovations

1. INTRODUCTION

The problem that is being addressed in this paper is the estimation of potentially unstable linear dynamical systems. The system is being observed by a set of agents, where each agent can observe only a small fraction of the entire state vector. The agents need to obtain an unbiased estimate of the state vector on the basis of its own observations and communications with its neighbors. The connectivity among the agents is very sparse and they can communicate among themselves only once per evolution of the dynamical state. These kinds of problems have applications in smart grid, robotics, power systems and in monitoring physical processes.

In similar lines, single time-scale parameter estimation has been studied in [1]; and [2] analyzed the convergence rate of a mixed time scale parameter estimation algorithm. The authors introduced the notion of (*consensus* + *innovations*) term in distributed estimation of static parameter. Distributed approaches have also been extended to dynamic parameter estimation in [3, 4, 5]. But in these papers the estimation algorithms perform consensus in between each evolution step of the dynamical system.

Recently single time-scale estimation of dynamical systems has been addressed in [6, 7, 8, 9]. In [6] and [7], the authors introduced the concept of Network Tracking Capacity (NTC), which is defined as the largest 2-norm of the system matrix that can be estimated with bounded error. The NTC has been characterized as a function of the observation model and the Laplacian matrix of distributed network. In [8] and [9], the authors employed genericity properties of the dynamical systems from structured systems theory, to derive results on the topology of the agent communication graph and sensor placement such that a stable distributed estimator can be designed.

In this paper, we have introduced a new term called pseudo-innovations which is a modified version of the innovations. The algorithm consists of *consensus on pseudo-innovations* term, in addition to the (*consensus* + *innovations*) term. This kind of algorithms restricts the conventional way of analyzing the stability and performance of the estimator. Hence we have shown that the distributed estimator is asymptotically unbiased to the centralized estimator. The centralized estimator can track any dynamical system with bounded mean-squared error; hence the distributed estimator can be designed such that it can also track any arbitrary system with bounded mean-squared error. In [6] and [7], the ability of the estimator to track a system depends on the NTC, which is a function of both network topology and the observation model. In contrast, our estimator needs the ratio of the algebraic connectivity and the largest eigenvalue of the graph Laplacian to be lower bounded, where the bound is determined by the spectral radius of the system matrix.

We now describe the organization of the rest of the paper. Section 2 introduces the preliminary background on dynamical system estimation. Section 3 includes the problem formulation of the distributed algorithm. In Section 4 the asymptotic unbiasedness of the algorithm is proved. Section 5 deals with the design of the different parameters of the estimator. In Section 6 we provided simulation result, followed by concluding remarks in Section 7.

2. PRELIMINARIES AND BACKGROUND

Consider a linear discrete-time dynamical system given by:

$$\theta(i+1) = \theta(i) + W_{\text{ph}}\theta(i) + v(i) \tag{1}$$

where: $\theta(i) \in \mathbb{R}^M$ is the state vector; i is the discrete time index; the arbitrary system matrix is $W_{\text{ph}} \in \mathbb{R}^{M\times M}$; the initial condition of the state vector is $\theta(0) \sim \mathcal{N}(\bar{\theta}_0, \Sigma_0)$; the noise in the system evolution is $v(i) \sim \mathcal{N}(0, V)$.

The dynamical system in (1) is to be estimated by a network of N agents. Each agent makes noise corrupted independent observations of linear functions of the state vector. The observation model at the n^{th} agent:

$$z_n(i) = H_n\theta(i) + r_n(i) \tag{2}$$

where: $z_n(i) \in \mathbb{R}^{M_n}$ is the output vector at agent n; the observation matrix is $H_n \in \mathbb{R}^{M\times M_n}$; the observation noise is $r_n(i) \sim \mathcal{N}(0, R_n)$. The input noise $\{v(i)\}$, the observation noise $\{r_n(i)\}$, and the initial state $\{\theta(0)\}$ are uncorrelated random vectors. Moreover the noise sequences $\{v(i)\}$ and $\{r_n(i)\}$ are statistically independent over time.

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We assume that the system (1) is unobservable at each of the agents, but if we combine the local observation models, then the resultant global observation model (3) is assumed to be observable.

$$z(i) \quad = \quad H\theta(i) + r(i) \tag{3}$$

$$\text{where, } z(i) = \begin{bmatrix} z_1(i) \\ \vdots \\ z_N(i) \end{bmatrix},\ H = \begin{bmatrix} H_1 \\ \vdots \\ H_N \end{bmatrix},\ r(i) = \begin{bmatrix} r_1(i) \\ \vdots \\ r_N(i) \end{bmatrix} \tag{4}$$

the global observation matrix is H and the noise is $r(i) \sim \mathcal{N}(0, R)$, $(R \succ 0)$, where $R = \text{blockdiag}[R_1, \ldots, R_N]$.

The pair $(I+W_{\text{ph}}, H)$ is observable, but the pairs $(I+W_{\text{ph}}, H_n)$ are unobservable $\forall n$. In our model, we have assumed that the system is also distributedly observable, i.e., the following matrix, G, is full rank.

$$G = \frac{1}{N} H^T H = \frac{1}{N} \sum_{n=1}^{N} H_n^T H_n \tag{5}$$

The undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ represent the communications in the multi-agent network. We define the open neighborhood Ω_n and closed neighborhood $\mathcal{N}_n$ of agent n as follows:

$$\Omega_n = \{l | (n,l) \in \mathcal{E}\}, \qquad \mathcal{N}_n = \{n\} \bigcup \{l | (n,l) \in \mathcal{E}\}$$

The multi-agent network is assumed to be connected. Then the eigenvalues of the positive semi-definite Laplacian matrix L of $\mathcal{G}$ are $0 = \lambda_1(L) < \lambda_2(L) \leq \ldots \leq \lambda_N(L)$.

In distributed estimation problems, the challenge lies in estimating unstable systems. We will consider (1) to be a potentially unstable, i.e., the 2-norm of the system matrix $A = (I_M + W_{\text{ph}})$ is $a = \|A\| > 1$. In our model, we make the assumption that

$$\gamma = \frac{\lambda_2(L)}{\lambda_N(L)} \geq \frac{a-1}{a+1} \tag{6}$$

The consensus speed factor γ, defined as the ratio of the algebraic connectivity and the largest eigenvalue of L, needs to be lower bounded, as in (6), to ensure that the agents in the network come to a consensus at a rate faster than the dynamics of the system.

3. PROBLEM FORMULATION

With the assumptions mentioned in Section 2, in this section we formulate the distributed algorithm for the state estimation of (1). In the network of N agents, each agent aims to obtain an estimate of the state vector $\theta(i)$. At time index i, given the observations up to time $i-1$, let the state estimate of agent n be denoted by $x_n(i)$. The algorithm consists of the following three steps:

1. *State Fusion:* First each agent fuse its estimate with the estimates of its neighbors as

$$\overline{x}_n(i) = x_n(i) - \beta \sum_{l \in \Omega_n} (x_n(i) - x_l(i)) \tag{7}$$

where: β is the consensus parameter and the initial condition is $x_n(0) = \bar{\theta}_0, \forall n$.

2. *Pseudo-innovations Estimate:* We define the pseudo-innovations $\nu_n(i)$ at each agent n:

$$\nu_n(i) = H_n^T z_n(i) - H_n^T H_n \overline{x}_n(i) \tag{8}$$

Each agent aims to obtain an estimate, $\widehat{\nu}_n(i)$, of the global average of the pseudo-innovations. It first fuses its estimate with that of its neighbors,

$$\overline{\widehat{\nu}}_n(i) = \sum_{l \in \mathcal{N}_n} w_{nl} \widehat{\nu}_l(i) \tag{9}$$

and then update the estimate

$$\widehat{\nu}_n(i+1) = \overline{\widehat{\nu}}_n(i) + (\nu_n(i+1) - C_n \overline{\widehat{\nu}}_n(i)) \tag{10}$$

where $W = \{w_{nl}\}$ is the consensus weight matrix with the same sparsity as the graph Laplacian L and C_n is the local pseudo-innovations gain matrix. The initial condition is $\widehat{\nu}_n(0) = \nu_n(0)$.

3. *Predictor:* In this step, an agent combine the outputs of the previous two steps and multiply it with the system matrix to predict the next state estimate as:

$$x_n(i+1) = (I_M + W_{\text{ph}})(\overline{x}_n(i) + K_n(i)\widehat{\nu}_n(i)) \tag{11}$$

where $K_n(i)$ is the local estimator gain at agent n.

In compact notation, equations (7)-(11) can be written as:

$$\begin{aligned}
\overline{x}(i) &= x(i) - \beta(L \otimes I_M)x(i) &(12)\\
\nu(i) &= D_H^T z(i) - \overline{D}_H \overline{x}(i) &(13)\\
\overline{\widehat{\nu}}(i) &= (W \otimes I_M)\widehat{\nu}(i) &(14)\\
\widehat{\nu}(i+1) &= \overline{\widehat{\nu}}(i) + (\nu(i+1) - C\overline{\widehat{\nu}}(i)) &(15)\\
x(i+1) &= (I_N \otimes (I_M + W_{\text{ph}}))(\overline{x}(i) + K(i)\widehat{\nu}(i)) &(16)
\end{aligned}$$

where:

$$x(i) = \begin{bmatrix} x_1(i) \\ \vdots \\ x_N(i) \end{bmatrix},\ \nu(i) = \begin{bmatrix} \nu_1(i) \\ \vdots \\ \nu_N(i) \end{bmatrix},\ \widehat{\nu}(i) = \begin{bmatrix} \widehat{\nu}_1(i) \\ \vdots \\ \widehat{\nu}_N(i) \end{bmatrix}$$

and the blockdiagonal matrices are $D_H = \text{diag}\{H_1, \ldots, H_N\}$, $\overline{D}_H = \text{diag}\{H_1^T H_1, \ldots, H_N H_N^T\}$, $C = \text{diag}\{C_1, \ldots, C_N\}$ and $K(i) = \text{diag}\{K_1(i), \ldots, K_N(i)\}$.

4. RESULTS

First we prove the unbiasedness of the pseudo-innovations estimate. The aim is to estimate the global average of the pseudo-innovations of all the agents denoted by $\nu_{\text{avg}}(i) = \frac{1}{N}(1_N \otimes I_M)^T \nu(i)$. Let $\delta_n(i) = \nu_n(i+1) - \nu_n(i)$ denote the change in pseudo-innovations at agent n. Then we can write

$$\nu_{\text{avg}}(i+1) = \nu_{\text{avg}}(i) + \delta_{\text{avg}}(i) \tag{17}$$

From the concepts of centralized Kalman Filtering, it is evident that $\delta_{\text{avg}}(i)$ is a zero-mean noise process. It can be proved by taking expectation on both sides of equation (17) .

Lemma 1 *The pseudo-innovations at n^{th} agent is a noise corrupted linear function of the average pseudo-innovations, i.e.,*

$$\nu_n(i) = G^{-1} H_n^T H_n \nu_{avg}(i) + \zeta_n(i) \tag{18}$$

$\zeta_n(i)$ is a noise process which is zero-mean over space and time.

Proof It can be proved by summing both sides of equation (18) from $n = 1$ to N and premultiplying the sum with $(1/N)$. Then both sides of (18) turns out to be $\nu_{\text{avg}}(i)$. ∎

Theorem 1 *The estimate of pseudo-innovations $\{\widehat{\nu}_n(i)\}_{i\geq 0}$, at agent n, is asymptotically unbiased*

$$\lim_{i\to\infty} \mathbb{E}[\widehat{\nu}_n(i) - \nu_{avg}(i)] = 0, \quad 1 \leq n \leq N \tag{19}$$

Proof Taking expectations on both sides of (15),

$$\mathbb{E}[\widehat{\nu}(i+1)] = \mathbb{E}[\overline{\widehat{\nu}}(i)] + (\mathbb{E}[\nu(i+1)] - C\mathbb{E}[\overline{\widehat{\nu}}(i)]) \tag{20}$$

Taking expectations on both sides of (18) and using (17)

$$\mathbb{E}[\nu(i+1)] = (I_N \otimes G^{-1})\overline{D}_H \mathbb{E}[1_N \otimes \nu_{\text{avg}}(i)] \tag{21}$$

Note that $(W \otimes I_M)(1_N \otimes \nu_{\text{avg}}(i)) = 1_N \otimes \nu_{\text{avg}}(i)$. We choose $C = (I_N \otimes G^{-1})\overline{D}_H$. Subtracting $\mathbb{E}[1_N \otimes \nu_{\text{avg}}(i+1)]$ from both sides of (20) and using (14) and (21),

$$\begin{aligned}&\mathbb{E}[\widehat{\nu}(i+1) - 1_N \otimes \nu_{\text{avg}}(i+1)]\\ &= (I_{MN} - C)\,(W \otimes I_M)\,\mathbb{E}[\widehat{\nu}(i) - 1_N \otimes \nu_{\text{avg}}(i)]\end{aligned}$$

Since C is a positive semi-definite matrix, $\|I_{MN} - C\| \leq 1$ and since W is a stochastic matrix, $\|W \otimes I_M\| = \|W\| < 1$.
Taking norm and continuing recursion in the previous equation,

$$\begin{aligned}&\|\mathbb{E}[\widehat{\nu}(i+1) - 1_N \otimes \nu_{\text{avg}}(i+1)]\|\\ &\leq \left(\prod_{j=i_0}^{i} \|I_{MN} - C\|\|(W \otimes I_M)\|\right) \|\mathbb{E}[\widehat{\nu}(i_0) - 1_N \otimes \nu_{\text{avg}}(i_0)]\|\end{aligned}$$

Taking limit, $\lim_{i\to\infty} \|\mathbb{E}[\widehat{\nu}(i+1) - 1_N \otimes \nu_{\text{avg}}(i+1)]\| = 0.$ ∎

We have proved that the pseudo-innovations estimate at each agent is asymptotically unbiased to the average of pseudo-innovations of all the agents, which will be used to prove that the distributed estimator tracks the centralized estimator. Before going to those results, we define the averaged estimator $\{x_{\text{avg}}(i)\}$ in (22) and the centralized estimator $\{u(i)\}$ in (23).

$$x_{\text{avg}}(i) = \frac{1}{N}\sum_{n=1}^{N} x_n(i) = \frac{1}{N}(1_N \otimes I_M)^T x(i) \tag{22}$$

$$u(i+1) = A\left(u(i) + \frac{1}{N}K_c \sum_{n=1}^{N}\left(H_n^T z_n(i) - H_n^T H_n u(i)\right)\right) \tag{23}$$

We first derive the conditions under which the estimates at each agent $\{x_n(i)\}$ converge to the averaged estimate $\{x_{\text{avg}}(i)\}$. Then we will show that the averaged estimate converges to the centralized estimate $\{u(i)\}$. The study of these two convergence properties results in the following two theorems.

Theorem 2 *The estimates of each agent $\{x_n(i)\}$ is asymptotically unbiased to the averaged estimates $\{x_{avg}(i)\}$,*

$$\lim_{i\to\infty} \mathbb{E}[x_n(i) - x_{avg}(i)] = 0, \quad 1 \leq n \leq N \tag{24}$$

Proof Let $P_{NM} = \frac{1}{N}(1_N \otimes I_M)(1_N \otimes I_M)^T$ and note that $P_{NM}\, x(i) = 1_N \otimes x_{\text{avg}}(i)$. Define the process $\{y_{\text{avg}}(i)\}$:

$$y_{\text{avg}}(i) = x(i) - 1_N \otimes x_{\text{avg}}(i) \tag{25}$$

$$\begin{aligned}y_{\text{avg}}(i+1) &= (I_N \otimes A)\Big(\Big(I_{NM} - \beta(L \otimes I_M)\Big)\Big(I_{NM}\\ &\quad - P_{NM}\Big)x(i) + K(i)\Big(I_{NM} - P_{NM}\Big)\widehat{\nu}(i)\Big)\\ &= (I_N \otimes A)\Big(\Big(I_{NM} - \beta(L \otimes I_M)\Big)y_{\text{avg}}(i)\\ &\quad + K(i)\Big(\widehat{\nu}(i) - 1_N \otimes \nu_{\text{avg}}(i)\Big)\Big)\end{aligned}$$

Taking expectations on both sides,
$\mathbb{E}[y_{\text{avg}}(i+1)] = (I_N \otimes A)\Big(I_{NM} - \beta(L \otimes I_M)\Big)\mathbb{E}[y_{\text{avg}}(i)]$
. $+ \ (I_N \otimes A)K(i)\mathbb{E}[\widehat{\nu}(i) - 1_N \otimes \nu_{\text{avg}}(i)]$.
From (19), it is evident that there exists a sufficiently large $i_0 > 0$, such that $\forall i > i_0$, $\mathbb{E}[\widehat{\nu}(i) - 1_N \otimes \nu_{\text{avg}}(i)] = 0$ and
$\mathbb{E}[y_{\text{avg}}(i+1)] = (I_N \otimes A)\Big(I_{NM} - \beta(L \otimes I_M)\Big)\mathbb{E}[y_{\text{avg}}(i)]$.
Taking norm on both sides and continuing the recursion, we have

$$\|\mathbb{E}[y_{\text{avg}}(i+1)]\| \leq a^{i-i_0}\|I_{NM} - \beta(L \otimes I_M)\|^{i-i_0}\|\mathbb{E}[y_{\text{avg}}(i_0)]\|$$

By assumption (6), there exist β such that $a\|I_{NM} - \beta(L \otimes I_M)\| < 1$. Optimal choice of β is discussed in Section 5. Taking limit in the above inequality, $\lim_{i\to\infty} \|\mathbb{E}[y_{\text{avg}}(i+1)]\| = 0$. ∎

Theorem 3 *The averaged estimate sequence $\{x_{avg}(i)\}$ is asymptotically unbiased to the centralized estimate sequence $\{u(i)\}$,*

$$\lim_{i\to\infty} \mathbb{E}[u(i) - x_{avg}(i)] = 0 \tag{26}$$

Proof From the definition of $x_{\text{avg}}(i)$,
$1_N \otimes x_{\text{avg}}(i+1) = P_{NM}x(i+1)$
$= (I_N \otimes A)\Big((W \otimes I_M)\big(1_N \otimes x_{\text{avg}}(i)\big) + P_{NM}K(i)\widehat{\nu}(i)\Big)$
Define the process $\{y_c(i)\}$: $y_c(i) = 1_N \otimes x_{\text{avg}}(i) - 1_N \otimes u(i)$.
Note that, $(W \otimes I_M)(1_N \otimes u(i)) = 1_N \otimes u(i)$.
$y_c(i+1) = (I_N \otimes A)\Big((W \otimes I_M)\Big(1_N \otimes \big(x_{\text{avg}}(i) - u(i)\big)\Big) + K(i)$
$\times P_{NM}\widehat{\nu}(i) - \frac{1}{N}(I_N \otimes K_c)\Big(1_N \otimes \sum_{n=1}^{N}\big(H_n^T z_n(i) - H_n^T H_n u(i)\big)\Big)\Big)$
The estimator gain matrix is designed such that as $i \to \infty$, $K_n(i) \to K_c\ \forall n$. Taking expectation on both sides

$$\begin{aligned}\mathbb{E}[y_c(i+1)] &= (I_N \otimes A)\Bigg((W \otimes I_M)\mathbb{E}[y_c(i)] + K(i)P_{NM}\mathbb{E}[\widehat{\nu}(i)]\\ &\quad - \frac{1}{N}K(i)\Big(1_N \otimes \mathbb{E}\Big[\sum_{n=1}^{N}\big(H_n^T z_n(i) - H_n^T H_n u(i)\big)\Big]\Big)\Bigg)\end{aligned}$$

There exists some sufficiently large $i_0 > 0$ such that $\forall i > i_0$, $\mathbb{E}[\widehat{\nu}(i)] = \mathbb{E}[1_N \otimes \nu_{\text{avg}}(i)]$ by using (19)

$$\begin{aligned}&\mathbb{E}[y_c(i+1)]\\ &= (I_N \otimes A)\Bigg((W \otimes I_M)\mathbb{E}[y_c(i)] + K(i)\Bigg(P_{NM}\mathbb{E}[1_N \otimes \nu_{\text{avg}}(i)]\\ &\quad - 1_N \otimes \mathbb{E}\Big[\frac{1}{N}\sum_{n=1}^{N}\big(H_n^T z_n(i) - H_n^T H_n u(i)\big)\Big]\Bigg)\Bigg)\\ &= (I_N \otimes A)(W \otimes I_M)\Big(I_{NM} - K(i)(I_N \otimes G)\Big)\mathbb{E}[y_c(i)]\end{aligned}$$

Taking norm on both sides and continuing the recursion,

$$\|\mathbb{E}[y_c(i+1)]\| \leq \left(\prod_{i=i_1}^{i} a.\|W\| \|I_{NM} - K(i)(I_N \otimes G)\| \right) \|\mathbb{E}[y_c(i_1)]\|$$

We know that $\|W\| < 1$. There exists some $i_1 \geq i_0$ such that $\forall i > i_1, K(i)$ can be designed such that $\|I_{NM} - K(i)(1_N \otimes G)\| < \frac{1}{a\|W\|}$.
Thus taking limit, $\lim_{i\to\infty} \|\mathbb{E}[y_c(i+1)]\| = 0$. ∎

We have shown that the distributed estimator is asymptotically unbiased to the averaged estimator, which in turn asymptotically unbiased to the centralized estimator. In [6], it is shown that given any arbitrary linear dynamical system, the centralized estimator can be designed such that it can track the system with bounded mean-squared error. For fastest convergence as well as for large value of a, the optimal value of the gain matrix is $K_c^* = G^{-1}$. Hence if we choose $K_n(i)$ such that as $i \to \infty$, $K_n(i) \to G^{-1}$, then that will be the optimal gain matrix for (16), which can track any arbitrary dynamical system with bounded mean-squared error.

5. ESTIMATOR DESIGN

The topology of the network plays a significant role in the convergence of the distributed estimation algorithm. The convergence speed of the network is dependent on the parameter γ as defined in (6) and the dynamics of the system (1) is dependent on a. The network topology should be designed in such a way that there exists β such that

$$a\|I_{MN} - \beta(L \otimes I_M)\| = a\|I_N - \beta L\| < 1 \tag{27}$$

$$\text{where, } \|I_N - \beta L\| = \max\{1 - \beta\lambda_2(L), \beta\lambda_N(L) - 1\} \tag{28}$$

The parameter β that satisfies both (27) and (28) should lie in the range:

$$\frac{a-1}{a.\lambda_2(L)} \leq \beta \leq \frac{a+1}{a.\lambda_N(L)} \tag{29}$$

Thus there exists a β satisfying (27) if and only if (6) is satisfied. Since we have assumed that the network topology satisfies (6), therefore in the distributed estimation algorithm any β from the range (29) will serve the purpose. The optimal β is $\beta^* = 2/(\lambda_2(L) + \lambda_N(L))$.

Now we will discuss about the design of the estimator gain matrix $K_n(i)$ at each agent n. The gain matrix at each agent n can be initialized as follows:

$$\overline{K}_n(0) = H_n^T H_n, \quad K_n(0) = \overline{K}_n^+(0) = \left(H_n^T H_n\right)^+ \tag{30}$$

where F^+ is the Moore-Penrose pseudo-inverse of the matrix F as defined in [10]. The gain at each time index is updated as follows:

$$\overline{K}_n(i+1) = \sum_{l \in \mathcal{N}_n} w_{nl}\overline{K}_l(i); \quad K_n(i+1) = \overline{K}_n^+(i+1) \tag{31}$$

Under the assumptions of the network topology, from equations (30) and (31), it is evident that $\overline{K}_n(i)$ asymptotically converges to G. Hence the estimator gain converges to:

$$\lim_{i\to\infty} K_n(i) = G^{-1}, \ \ \forall n \tag{32}$$

The gain matrix thus designed is optimal for the distributed estimator. It is to be noted that all the gain terms can be pre-computed and saved at each agent. This will reduce the computational complexity while running the distributed algorithm.

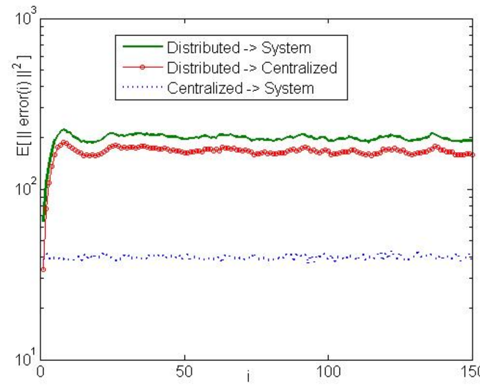

Fig. 1. The expected value of norm-square of error is plotted with time index i.

6. SIMULATION RESULTS

We simulated an unstable dynamical system of dimension $M = 10$. The system matrix W_{ph} is randomly generated. The norm of the overall system matrix is 1.1667. The system is observed by a network of $N = 10$ agents. We have considered $M_n = 2, \forall n$. The agent network is taken to be a regular lattice graph with number of nodes $= 10$ and neighborhood size $= 2$. The eigen-ratio of the graph is $\gamma = 0.0814$. The values of γ and a conforms with the condition (6) . In this simulation, we have chosen $\beta = \beta^* = 0.3135$. The fusion weight matrix is chosen to be $W = I - \beta^* L$.The estimator gain matrix is pre-computed and saved at each agent following (30) and (31).The Monte Carlo simulation plot is shown in Fig.1.

In Fig.1, the expected value of norm-square of error, or in other terms, the trace of the error covariance matrix is plotted with time for three different cases. The first one is the error between the distributed estimate $x(i)$ and the system state vector $\theta(i)$. The second one is the error between the distributed estimate $x(i)$ and the centralized estimate $u(i)$. The third one is the error between the centralized estimate $u(i)$ and the system state vector $\theta(i)$. From the plot, it is evident that the distributed estimates converge to the centralized estimates with bounded mean-squared error. Also, both the distributed and the centralized estimates converge to the system state vector with bounded mean-squared error but, as expected, the performance of the centralized estimator is better than the distributed estimator.

7. CONCLUSIONS

In this paper, we proposed a new class of distributed algorithm for the estimation of a linear discrete-time dynamical system. We have considered single time-scale update algorithm, i.e., between successive evolution of the system dynamics, the agents in the distributed network can communicate among themselves only once. Here we introduced a new term called pseudo-innovations. Our distributed algorithm consists of two parts - consensus and estimate of global average of pseudo-innovations. We have shown that the estimator can track any arbitrary dynamical system with bounded mean-squared error. Our algorithm requires the eigen-ratio of the graph Laplacian to be lower bounded by a function of spectral radius of system matrix, where as in prior work the ability of designed estimators depend on Network Tracking Capacity, which is a function of the graph Laplacian and the system observation model.

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