MATRIX DESIGN FOR OPTIMAL SENSING

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ABSTRACT

We design optimal $2 \times N$ $(2 < N)$ matrices, with unit columns, so that the maximum condition number of all the submatrices comprising 3 columns is minimized. The problem has two applications. When estimating a 2-dimensional signal by using only three of N observations at a given time, this minimizes the worst-case achievable estimation error. It also captures the problem of optimum sensor placement for monitoring a source located in a plane, when only a minimum number of required sensors are active at any given time. For arbitrary $N \geq 3$, we derive the optimal matrices which minimize the maximum condition number of all the submatrices of three columns. Surprisingly, a uniform distribution of the columns is *not* the optimal design for odd $N \geq 7$.

***Index Terms*—** matrix design, sensor network, source localization and monitoring, condition number, singular value

1. INTRODUCTION

We consider the problem of designing sensing schemes to optimize the worst-case estimation performance when only a subset of sensors are operational in sensor networks. Consider a set of N sensors which are used to estimate an M-dimensional signal, where $N \geq M$. In our problem, only K out of these N sensors operate at any instant of time. For example, to maximize the lifetime of a sensor network [1, 3, 9, 11, 12, 14], at any single time instant, only K sensors are turned on to monitor the M-dimensional signal. If we assume that each time these K sensors are uniformly selected from the $\binom{N}{K}$ possible subsets, on average the lifetime of the sensor network is extended by a factor of N/K. As another example, in hostile environments such as battlefields, it is very common that only a limited number of sensors, say K out of N, are able to survive and operate as designed. In these scenarios, while we only have a limited sensing resources at a single time instant, we wish to achieve the best estimation from limited observations. It is thus useful to maximize the worst-case performance of the sensing system, no matter what set of sensors are used or survive. We thus study the design of sensing schemes that optimize worst-case performance. Before a formal mathematical formulation, we review two sensor network applications which relate singular values of certain matrices to estimation performance.

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1.1. Signal Estimation

With $x \in \mathbb{R}^M$ representing the signal, consider a sensing matrix $A \in \mathbb{R}^{M\times N}$. Each of the N sensors generates a real observation represented by an inner product between x and a column of A. Let $KS \subseteq \{1, 2..., N\}$, with cardinality $|KS| = K$, be the subset of sensors that are active at a given time. The measurement matrix of the active sensors is then $A_{KS} \in \mathbb{R}^{M\times K}$ consisting of the K columns of A indexed by KS. With noise w, the measurement $y \in \mathbb{R}^K$ is

$$y = A_{KS}^T x + w, \tag{1.1}$$

Suppose the singular values of A_{KS} are σ_i. Then as long as A_{KS} has full row rank, the estimation error satisfies

$$\|\hat{x} - x\|_2 = \|(A_{KS}A_{KS}^T)^{-1}A_{KS}(w)\|_2 \leq \frac{\|w\|_2}{\sigma_{min}}.$$

To optimize the worst-case performance, we must design A to maximize the smallest singular value among all the $\binom{N}{K}$ possible submatrices A_{KS}. To make the problem meaningful, we assume that each column of A has unit ℓ_2 norm. When $M = 2$, this is equivalent to minimizing the maximum condition number among all $\binom{N}{K}$ submatrices A_{KS}.

1.2. Source Monitoring in the Plane

A second motivating application for this paper is optimum sensor placement for source monitoring in $\mathbb{R}^2$, [5]-[8]. Monitoring is related to the notion of localization, where several sensors collaborate to locate a source, using some relative position information. The latter could be distance, bearing, time of arrival, time difference of arrival or received signal strength (RSS). Monitoring assumes that a hazardous source has already been located at some $z \in \mathbb{R}^2$, and a group of sensors at $x_i \in \mathbb{R}^2$ monitor it by continuously estimating its position from a safe distance. Thus [5]-[8] place sensors, i.e. choose x_i, so that the minimum eigenvalue of the Fisher Information Matrix (FIM) underlying the estimation problem is maximized. This ensures that under continuous monitoring and Maximum Likelihood (ML) estimation, asymptotically, the mean-square error in estimating z, is minimized, [4, 15, 16]. As z is at least roughly known, so also is the FIM.

Consider [5, 10], where no sensor can be closer than D from the source. Each measures the RSS of the signal emanating from the source under log-normal shadowing, i.e. with known positive real scalars A and β, the RSS s_i at the i-th sensor obeys, for mutually independent $w_i \sim \mathcal{N}\left(0, \sigma^2\right)$:

$$\ln s_i = \ln A - \beta \ln ||x_i - z|| + w_i, \tag{1.2}$$

The underlying FIM with N-sensors is, [5]

$$G = \frac{\beta^2}{\sigma^2 (\ln 10)^2} \sum_{i \in N} \frac{(x_i - z)(x_i - z)^T}{||x_i - z||^4}. \tag{1.3}$$

The optimal sensor placement problem then becomes: Given, $z \in \mathbb{R}^2$, and $D > 0$, find $x_i \in \mathbb{R}^2$, $i \in \{1, \cdots, N\}$ so that the minimum eigenvalue of G is maximized, subject to: $||x_i - z|| \geq D$. Because of the denominator in (1.3), the minimum eigenvalue of G is maximized only if for all $i \in \{1, \cdots, N\}$, $\|x_i - z\| = D$. Without loss of generality one can assume $D = 1$ and $z = 0$. Thus effectively one must maximize the minimum eigenvalue of

$$\sum_{i \in N} x_i x_i^T, \tag{1.4}$$

subject to $\|x_i\| = 1$. This is tantamount to minimizing the condition number of F as its trace is constrained to be N.

Now suppose to prolong battery life, only a subset of sensors is activated at a given time, [12, 14]. The logical problem to consider is then for some K, KS as defined above, and

$$F_{KS} = \sum_{i \in KS} x_i x_i^T, \tag{1.5}$$

to minimize the largest condition number of F_{KS}, among all $KS \subseteq \{1, \cdots, N\}$. With A_{KS} having columns x_i, $i \in KS$, we have $F_{KS} = A_{KS} A_{KS}^T$, and a similar setting of Section 1.1. We observe, that the *minimum K needed for source monitoring is three, motivating the rest of this paper where $K = 3$ is considered*. In particular RSS provides a distance estimate. Distances from three non-collinear sources are necessary to localize, [17]. This scenario also applies to the case where only three sensors survive hostilities.

The rest of this paper is organized as follows. Section 2 gives a precise mathematical formulation. Section 3 provides a formula for the minimum condition number of submatrices when $M = 2$. Section 4 characterizes optimal solutions all for $M = 2$, $K = 3$ and arbitrary $N \geq 3$. Section 5 presents simulations.

2. PROBLEM FORMULATION

Let $M \leq N$ be positive integers and $A = [a_1, a_2, ..., a_N]$, where $a_i \in \mathbb{R}^M$ obey $||a_i||_2 = 1$ for $1 \leq i \leq N$. Let $KS \subseteq \{1, 2, ..., N\}$ be a subset with cardinality $|KS| = K$. Now, $A_{KS} \in \mathbb{R}^{M \times K}$ is the submatrix $A_{KS} = [a_{i_1}, a_{i_2}, ..., a_{i_K}]$ with columns indices i_j, $1 \leq j \leq K$, from the set KS. Then our optimal design problem for the parameter set (M, N, K) is:

$$\max_{A \in \mathbb{R}^{M \times N} \text{with unit-normed columns}} \left\{ \min_{KS \subseteq \{1,2,...,N\}} \sigma_{min}(A_{KS}) \right\}.$$

For $M = 2$, this is equivalent to minimizing the condition number:

$$\min_{A \in \mathbb{R}^{M \times N} \text{with unit-normed columns}} \left\{ \max_{KS \subseteq \{1,2,...,N\}} \frac{\sigma_{max}(A_{KS})}{\sigma_{min}(A_{KS})} \right\}.$$

Note the similarity between this problem and the problem of designing compressive sensing matrices [2] satisfying the restricted isometry property (RIP), which also requires the condition numbers for the submatrices be small. As opposed to the design of compressive sensing matrices satisfying RIP [2], in our problem, the submatrices A_{KS} are wide rather than tall. The motivating applications are also different from compressive sensing.

As noted earlier, motivated in part by 2-dimensional source monitoring with the minimum number of sensors i.e. $K = 3$, we restrict attention to the case of $K = 3$ and $M = 2$, where closed form expressions are possible and surprising conclusions, that may illuminate the problem solution for higher values of K and M, are obtained.

3. DERIVATION OF THE CONDITION NUMBER FOR $M = 2$

The condition number of $\tilde{A}_{KS} = A_{KS} A_{KS}^T$ is given by

$$\kappa(\tilde{A}_{KS}) = \frac{\max_{||\eta||=1}(\eta^T \tilde{A}_{KS} \eta)}{\min_{||\eta||=1}(\eta^T \tilde{A}_{KS} \eta)} \tag{3.1}$$

Since the columns of A are unit-normed, we can represent $A = [a_1, a_2, ..., a_N]$ with

$$a_i = \begin{pmatrix} \cos\theta_i & \sin\theta_i \end{pmatrix}^T \tag{3.2}$$

for $1 \leq i \leq N$, where $\theta_i \in [0, \pi)$ (we do notice shifting θ_i by π will not change the condition number of any submatrix). Since $||\eta||_2 = 1$ we can choose $\eta = \begin{pmatrix} \cos\alpha & \sin\alpha \end{pmatrix}^T$. Thus

$$\eta^T \tilde{A}_{KS} \eta = \frac{K}{2} + \frac{1}{2} \sum_{j=1}^{K} \cos(2(\alpha - \theta_{i_j})) = J(\alpha).$$

Let us define

$$J(\alpha) = \frac{K}{2} + \frac{1}{2} \sum_{j=1}^{K} \cos(2(\alpha - \theta_{i_j})). \tag{3.3}$$

Then the minimum (maximum) eigenvalue of $\tilde{A}_{KS}$ is achieved when $J'(\alpha) = 0$ and $J''(\alpha) > (<)0$. With

$$\gamma = (\sum_{j=1}^{K} \sin(2\theta_{i_j}))^2 + (\sum_{j=1}^{K} \cos(2\theta_{i_j}))^2 \neq 0,$$

at a minimum or maximum, α satisfies

$$\cos(2\alpha) = \sum_{j=1}^{K} \cos(2\theta_{i_j})/\gamma$$

and

$$\sin(2\alpha) = \sum_{j=1}^{K} \sin(2\theta_{i_j})/\gamma.$$

Thus,

$$J(\alpha) = \frac{K}{2} + \frac{1}{2}\sum_{j=1}^{K} \cos(2\alpha)\cos(2\theta_{i_j}) + \frac{1}{2}\sum_{j=1}^{K} \sin(2\alpha)\sin(2\theta_{i_j}). \tag{3.4}$$

Combining the optimizing α and (3.4), we have

$$J(\alpha) = \frac{K}{2} + \frac{1}{2}\frac{\sum_{j=1}^{K}\sum_{l=1}^{K}(\cos(2\theta_{i_l})\cos(2\theta_{i_j}) + \sin(2\theta_{i_l})\sin(2\theta_{i_j}))}{\sqrt{(\sum_{l=1}^{K}\sin(2(\theta_{i_l})))^2 + (\sum_{l=1}^{K}\cos(2\theta_{i_l}))^2}}.$$

On simplification, the maximum and minimum eigenvalues of $\tilde{A}_{KS}$ are given by

$$J(\alpha_{max}) = \frac{K}{2} + \frac{1}{2}\sqrt{\frac{K}{2} + \sum_{j=1}^{K}\sum_{l=j+1}^{K}\cos 2(\theta_{i_l} - \theta_{i_j})}, \tag{3.5}$$

and

$$J(\alpha_{min}) = \frac{K}{2} - \frac{1}{2}\sqrt{\frac{K}{2} + \sum_{j=1}^{K}\sum_{l=j+1}^{K}\cos 2(\theta_{i_l} - \theta_{i_j})}. \tag{3.6}$$

respectively. Thus minimizing the condition number of $\tilde{A}_{KS}$ for a given set of indices $\{i_1, i_2, .., i_K\}$ is the same as (the equation inside the square root is always nonnegative)

$$\min_{\theta_{i_1},\ldots,\theta_{i_K}} \sum_{j=1}^{K}\sum_{l=j+1}^{K}\cos 2(\theta_{i_l} - \theta_{i_j}). \tag{3.7}$$

With $KS \subseteq \{1, 2, ..., N\}$, the optimal sensing matrix design problem for $M = 2$ can be reformulated as,

$$\min_{\theta_1,\ldots,\theta_N} \max_{KS=\{i_1,i_2,..,i_K\}} \sum_{j=1}^{K}\sum_{l=j+1}^{K}\cos 2(\theta_{i_l} - \theta_{i_j}).$$

In the following sections, we will derive the optimal design for $K = 3$, which has important applications in location monitoring in sensor networks.

4. OPTIMAL PLACEMENT

We now consider solutions for $M = 2$, $K = 3$ and different values of N.

4.1. $K = 3$, N is an even number

For even-numbered N, the optimal design is given as below.

Theorem 4.1 *If $K = 3$ and N is an even number, then the set of angles (a) $\theta_i = \frac{2\pi(i-1)}{N} \mod \pi$, $1 \leq i \leq N$, or (b) $\theta_i = \frac{2\pi(i-1)}{N}$, $1 \leq i \leq N$, minimizes the maximum condition number among all sub-matrices with K columns.*

Observe, (a) actually aligns pairs of angles together (see Fig 1) and is not useful for source monitoring where at least three distinct sensor locations are necessary, [17]. On the other hand (b) leads to distinct locations by separating adjacent sensors $2\pi/N$ radians apart.

4.2. $K = 3$, $N = 3$ or 5

These stand apart from other odd N values:

Theorem 4.2 *Let $K = 3$ and $N = 3$ or 5. Then the set of angles $\theta_i = \frac{\pi(i-1)}{N}$, $1 \leq i \leq N$, minimizes the maximum condition number among all sub-matrices with $K = 3$ columns.*

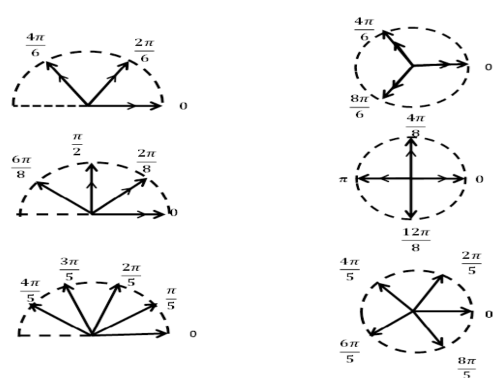

Fig. 1: Illustration of angle arrangements θ_i's for $N = 6$, 7 and 5 respectively, using the 3 rows of figures from top to bottom. The left figures represent the angle (θ_i) for the columns of sensing matrices. Right figures are doubling those angles ($2\theta_i$) as in the objective function in (3.7).

4.3. $K = 3$, $N \geq 7$ is an Odd Number

One might think that the uniform distributed design is optimal for $N \geq 7$. However, this is not true from the following theorem. Instead, the optimal design is to eliminate one angle from the optimal design for $(N+1)$.

Theorem 4.3 *If $K = 3$ and $N \geq 7$ is an odd number, then $\theta_i = \frac{2\pi(i-1)}{N+1} \mod \pi$, $1 \leq i \leq N$, minimizes the maximum condition number among all sub-matrices with $K = 3$ columns.*

5. SIMULATION RESULTS

We now present simulation results.

5.1. Worst Case Condition Number vs N

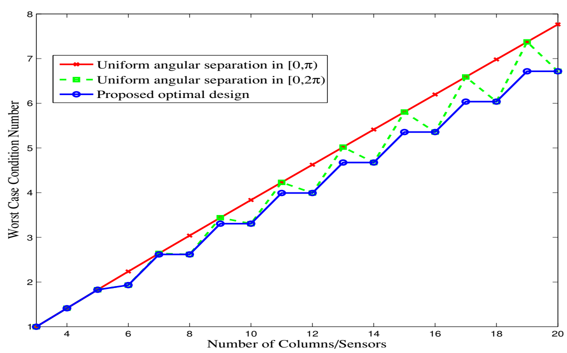

Fig. 2: Worst case condition number versus N

We compare the maximum condition number among all the possible 2×3 submatrices in three different cases shown in Fig.2. The cases are, (i) when successive sensors are placed in a semicircle π/N apart, namely $\theta_i = 0, \frac{\pi}{N}, ..., \frac{\pi(N-1)}{N}$, (ii) they are placed $2\pi/N$ apart, namely $\theta_i = 0, \frac{2\pi}{N}, ..., \frac{2\pi(N-1)}{N}$, and (iii) they are placed in a manner specified by our theorems. That the performance of (ii) matches (iii) for even N conforms with earlier observations.

5.2. Worst Mean Square Signal Estimation Error vs N

Consider the setting of Section 1.1. We compare in Fig. 3 the mean square error (MSE) for worst-case submatrices yielded by (i) above with that yielded by the postulated optimum for sensors ranging in number from 3 to 15. The signal x in (1.1) is $[9, 9]^T$. The noise in each measurement is $\mathcal{N} \sim (0, 1)$. For each value N, the estimation error $||\hat{x} - x||^2$ for worst-case submatrices was averaged over 2000 instances. Again the predicted optimal placement is superior.

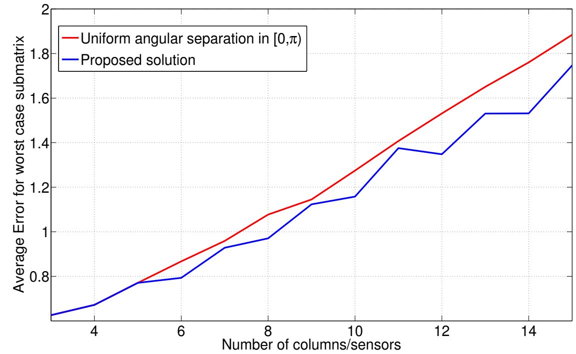

Fig. 3: Worst case estimation error versus the number of columns in the sensing matrix.

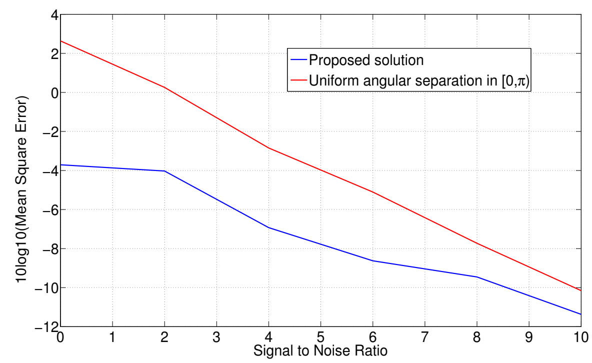

Fig. 4: Mean square error(dB) in the source location estimate when the worst performing subset of sensors are active versus the Signal to Noise Ratio(dB).

5.3. Monitoring Error vs SNR

Fig. 4 compares the ML estimation of a source at the origin with $N = 10$, from RSS under log-normal shadowing in the case where the sensors are placed as in (i) against optimal placement. The latter's superiority is evident.

6. CONCLUSION AND FUTURE WORK

We propose the problem designing optimal $M \times N$ ($M \leq N$) sensing matrices which minimize the maximum condition number of all the submatrices of K columns. Such matrices minimize the worst-case estimation errors when only K sensors out of N sensors are available for sensing at a given time. When $M = 2$ and $K = 3$, for an arbitrary $N \geq 3$, we derive the optimal matrices which minimize the maximum condition number of all the submatrices of K columns. It is interesting that minimizing the maximum coherence between columns does not always guarantee minimizing the maximum condition number.

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