

# NESTED BOUNDS FOR THE CONSTRAINED SENSOR PLACEMENT PROBLEM

*Muhammad Uddin, Anthony Kuh and Aleksandar Kavcic*

University of Hawaii  
Department of Electrical Engineering  
Honolulu, HI 96822

E-mail: uddin@hawaii.edu, kuh@hawaii.edu, kavcic@hawaii.edu

## ABSTRACT

The objective of this paper is to find numerical upper bounds on the optimal solution to the sensor placement problem. Given noisy measurements and knowledge of the state correlation matrix, the sensor placement problem can be formulated as an integer programming problem using a linear minimum mean squared error estimator. Since finding the optimal placements of a fixed number of sensors in a large network is computationally infeasible, finding bounds for the optimal solution is a fundamental task. In this paper we present a family of nested bounds using matrix pencils and their generalized eigenvalues that upper bound the optimal performance. In the analysis we consider nodes that we want to place sensors and other nodes where we cannot or do not want to place sensors. Finally we compare the upper bounds with the optimal solution using simulations on a 5 by 5 grid network.

**Index Terms**— sensor placement, matrix pencils, generalized eigenvalues

## 1. INTRODUCTION

Placement of sensors to monitor and estimate behavior of complex systems has applications ranging from the electric grid to the natural environment to biomedical monitoring. Since only a limited number of sensors can be deployed, selecting the best sensor locations is a fundamental problem. And as such, there has been much work done on the optimal placement of sensors. In [1], the optimal placement of sensors is considered where the probability of sensor detection depends on the distance between the nodes with sensors placed on a two or three dimensional grid. Placement of wireless sensors was studied in [2] where the sensors are placed at nodes such that the network satisfies a predetermined lifetime and coverage requirement. In [3], the authors model spatial phenomena as a Gaussian process and consider placement of sensors using optimal experimental design to maximize mutual information between sensor locations.

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Other research has considered Phasor Measurement Unit (PMU) placement in the power grid [4–6]. In [4] the problem is formulated as a state estimation problem to make the grid observable. More recently, in [5] a greedy approach is proposed for the PMU placement problem. Performance bound for the greedy algorithm is obtained in [3, 5], where the objective functions are submodular. An estimation theoretic approach is proposed in [6]; after posing the optimization problem as a linear regression problem, a convex relaxation is developed to sub-optimally solve the PMU placement problem. In [7], the authors present a unified description of different algorithms proposed to solve the PMU placement problem.

In fact, the different algorithms discussed in the literature give lower bounds to the optimal solution. In this paper we propose a family of upper bounds on the optimal solution to bound the difference between the optimal and suboptimal solutions. Thereby, we significantly extend our prior work in [8] by presenting nested bounds with analysis of the bounds using matrix pencils and their generalized eigenvalues.

The rest of the paper is organized as follows: Section 2 gives the formulation of the problem. Next, in Section 3 we find a family of upper bounds on the optimal solution. In Section 4 we present simulation results. Related works are discussed in Section 5. Finally, Section 6 summarizes the results of this paper and suggests direction of future work.

**Notation** Upper case letters denote random variables; underlined letters stand for vectors; boldface upper case letters denote matrices, and  $\mathbf{I}_n$  denotes the  $n \times n$  identity matrix;  $\langle \mathbf{A}, \mathbf{B} \rangle$  denotes a matrix pencil formed by matrices  $\mathbf{A}$  and  $\mathbf{B}$ ;  $(\cdot)^T$  and  $\mathbf{E}(\cdot)$  stand for transposition and expectation, respectively.

## 2. PROBLEM STATEMENT

Let the state vector be  $\underline{X} \in \mathbb{R}^n$  and the observation vector be  $\underline{Y} \in \mathbb{R}^m$ , where  $m \leq n$ . We assume that the sensors are all identical and make noisy measurements with variance  $\sigma^2$ . We consider a linear model described by

$$\underline{Y} = \mathbf{C}(\underline{X} + \sigma \underline{N}) \quad (1)$$

where  $\underline{X}$  and  $\underline{N}$  are statistically independent zero-mean random vectors with covariance matrices  $\Lambda_{\underline{X}}$  and  $\mathbf{I}_n$ , respectively.  $\mathbf{C}$  is a binary matrix composed of  $m$  rows of the  $n \times n$  identity matrix  $\mathbf{I}_n$ . The positions of ones in the matrix  $\mathbf{C}$  denote the position of the sensors.

To find an estimate of the entire state vector  $\underline{X}$  using the measurement vector  $\underline{Y}$ , we use the linear minimum mean squared error estimator is given by [9]

$$\hat{\underline{X}}(\underline{Y}) = \mathbf{E}(\underline{X}|\underline{Y}) = \mathbf{E}(\underline{X} \underline{Y}^T) \mathbf{E}(\underline{Y} \underline{Y}^T)^{-1} \underline{Y}, \quad (2)$$

where  $\mathbf{E}(\underline{X} \underline{Y}^T) = \Lambda_{\underline{X}} \mathbf{C}^T$  and  $\mathbf{E}(\underline{Y} \underline{Y}^T) = \mathbf{C} \Lambda_{\underline{X}} \mathbf{C}^T + \sigma^2 \mathbf{I}_m$ . The error is defined by  $\underline{\mathcal{E}} = \underline{X} - \hat{\underline{X}}(\underline{Y})$  and the error covariance matrix is given by [9]

$$\mathbf{E}(\underline{\mathcal{E}} \underline{\mathcal{E}}^T) = \Lambda_{\underline{X}} - \mathbf{E}(\underline{X} \underline{Y}^T) \mathbf{E}(\underline{Y} \underline{Y}^T)^{-1} \mathbf{E}(\underline{Y} \underline{X}^T). \quad (3)$$

The optimization problem is to find the matrix  $\mathbf{C}^*$  that minimizes the total estimation error  $\text{tr} \mathbf{E}(\underline{\mathcal{E}} \underline{\mathcal{E}}^T)$ . This can be defined as an integer programming problem of choosing  $m$  rows of  $\mathbf{I}_n$  that minimize  $\text{tr} \mathbf{E}(\underline{\mathcal{E}} \underline{\mathcal{E}}^T)$ .

**Definition.** Let  $\mathcal{C}^{[m \times n]}$  denote the set of all  $m \times n$  matrices composed of  $m$  rows of the identity matrix  $\mathbf{I}_n$ . ■

The optimization problem is then given by

$$\mathbf{C}^* = \arg \min_{\mathbf{C} \in \mathcal{C}^{[m \times n]}} \text{tr} \mathbf{E}(\underline{\mathcal{E}} \underline{\mathcal{E}}^T) = \arg \min_{\mathbf{C} \in \mathcal{C}^{[m \times n]}} \mathbf{E}(\underline{\mathcal{E}}^T \underline{\mathcal{E}}). \quad (4)$$

Since the first term in (3) (i.e.,  $\Lambda_{\underline{X}}$ ) does not depend on the choice of matrix  $\mathbf{C}$ , we can restate the optimization problem as an equivalent maximization problem using the following definition.

**Definition.** Let the efficacy of matrix  $\mathbf{C}$  be defined as

$$J(\mathbf{C}) \triangleq \text{tr} \left\{ \mathbf{E}(\underline{X} \underline{Y}^T) \mathbf{E}(\underline{Y} \underline{Y}^T)^{-1} \mathbf{E}(\underline{Y} \underline{X}^T) \right\} \quad (5)$$

$$= \text{tr} \left\{ [\mathbf{C}(\Lambda_{\underline{X}} + \sigma^2 \mathbf{I}) \mathbf{C}^T]^{-1} \mathbf{C} \Lambda_{\underline{X}}^2 \mathbf{C}^T \right\}. \quad (6)$$

Note that (6) has the form of the generalized Rayleigh quotient. ■

The optimization problem (4) is then equivalent to

$$\mathbf{C}^* = \arg \max_{\mathbf{C} \in \mathcal{C}^{[m \times n]}} J(\mathbf{C}), \quad (7)$$

which is an integer programming problem of choosing  $m$  rows of the identity matrix  $\mathbf{I}_n$  that maximize the efficacy. The optimum solution to (7) requires an exhaustive search by testing all  $\binom{n}{m}$  possible choices of  $m$  rows, which becomes computationally infeasible even for moderately sized  $n$  and  $m$ . In fact, the sensor placement problem is NP-complete [10].

### 3. UPPER BOUNDS ON THE OPTIMAL EFFICACY

From [3, 7, 8] it is obvious that there exists numerous ways of obtaining lower bounds on the optimal efficacy  $J(\mathbf{C}^*)$ . However, to evaluate the performance of these lower bounds we want to obtain numerically computable upper bounds for the difference  $J(\mathbf{C}^*) - J(\mathbf{C})$ . To achieve this goal, we devote this section to obtain a family of upper bounds by relaxing

the constraints on the optimization problem in (7). And as such, instead of considering matrix  $\mathbf{C}$  whose entries take values in the set  $\{0, 1\}$ , we consider an unconstrained matrix  $\mathbf{F}$  whose entries take values in  $\mathbb{R}$ .

**Definition.** For  $m \leq n$ , let  $\mathcal{F}^{[m \times n]}$  be the set of all  $m \times n$  matrices with rank  $m$ . ■

Similar to (7) we formulate an optimization problem with the relaxed constraints as

$$\begin{aligned} \mathbf{F}^* &= \arg \max_{\mathbf{F} \in \mathcal{F}^{[m \times n]}} J(\mathbf{F}) \\ &= \arg \max_{\mathbf{F} \in \mathcal{F}^{[m \times n]}} \text{tr} \left\{ [\mathbf{F}(\Lambda_{\underline{X}} + \sigma^2 \mathbf{I}) \mathbf{F}^T]^{-1} \mathbf{F} \Lambda_{\underline{X}}^2 \mathbf{F}^T \right\}. \end{aligned} \quad (8)$$

Since  $\mathcal{C}^{[m \times n]} \subset \mathcal{F}^{[m \times n]}$ , we have  $J(\mathbf{F}^*) \geq J(\mathbf{C}^*)$ . The following theorem provides a method for calculating  $J(\mathbf{F}^*)$  in a closed form.

**Theorem 1.** [8] If  $\Lambda_{\underline{X}}$  is a non-negative definite matrix with eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$ , then

$$J(\mathbf{F}^*) = \sum_{j=1}^m \frac{\lambda_j^2}{\lambda_j + \sigma^2}. \quad (9)$$

#### 3.1. Nested bounds

In this section we develop a family of upper bounds to the optimization problem (7). These nested upper bounds are calculated when

1. some  $k \leq m$  optimal sensor locations are available and,
2. some  $\ell \leq n - m$  optimal locations are available where the sensors cannot be placed.

In fact, any of these two smaller optimizations can be replaced by a physical constraint imposed on the sensor network. The nested upper bounds  $\bar{H}_{k,\ell}$  are defined as follows

**Definition.** For any  $k < m$  and  $\ell < n - m$ ,

$$\bar{H}_{k,\ell} \triangleq \max_{\mathbf{C}_1 \in \mathcal{C}^{[k \times n]}} \max_{\substack{\mathbf{C}_2 \in \mathcal{C}^{[k \times n]} \\ \mathbf{C}_2 \mathbf{C}_1^T = \mathbf{0}}} \max_{\substack{\mathbf{F} \in \mathcal{F}^{[(m-k) \times n]} \\ \mathbf{F}[\mathbf{C}_2^T \ \mathbf{C}_1^T] = \mathbf{0}}} J \left( \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{F} \end{bmatrix} \right) \quad (10)$$

**Theorem 2.**

$$\bar{H}_{k,\ell} \geq J(\mathbf{C}^*) \quad (11)$$

*Proof.*

$$\begin{aligned} J(\mathbf{C}^*) &= \max_{\mathbf{C} \in \mathcal{C}^{[m \times n]}} J(\mathbf{C}) \\ &= \max_{\mathbf{C}_1 \in \mathcal{C}^{[k \times n]}} \max_{\substack{\mathbf{C}_2 \in \mathcal{C}^{[k \times n]} \\ \mathbf{C}_2 \mathbf{C}_1^T = \mathbf{0}}} \max_{\substack{\mathbf{C}_3 \in \mathcal{C}^{[(m-k) \times n]} \\ \mathbf{C}_3[\mathbf{C}_2^T \ \mathbf{C}_1^T] = \mathbf{0}}} J \left( \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \\ \mathbf{C}_3 \end{bmatrix} \right) \\ &\leq \max_{\mathbf{C}_1 \in \mathcal{C}^{[k \times n]}} \max_{\substack{\mathbf{C}_2 \in \mathcal{C}^{[k \times n]} \\ \mathbf{C}_2 \mathbf{C}_1^T = \mathbf{0}}} \max_{\substack{\mathbf{F} \in \mathcal{F}^{[(m-k) \times n]} \\ \mathbf{F}[\mathbf{C}_2^T \ \mathbf{C}_1^T] = \mathbf{0}}} J \left( \begin{bmatrix} \mathbf{C}_2 \\ \mathbf{F} \end{bmatrix} \right) = \bar{H}_{k,\ell}, \end{aligned}$$

where the inequality follows from  $\mathcal{C}^{[(m-k) \times n]} \subset \mathcal{F}^{[(m-k) \times n]}$ . ■

**Corollary 2.1.**  $\bar{H}_{0,0} = J(\mathbf{F}^*)$ .

**Corollary 2.2.**  $\bar{H}_{m,\ell} = \bar{H}_{k,n-m} = J(\mathbf{C}^*)$

We now show that the bounds are nested.

**Theorem 3.** For any  $k \leq m-1$  and  $\ell \leq n-m$ ,

$$\bar{H}_{k,\ell} \geq \bar{H}_{k+1,\ell}, \quad (12)$$

and for any  $k \leq m$  and  $\ell \leq n-m-1$ ,

$$\bar{H}_{k,\ell} \geq \bar{H}_{k,\ell+1}. \quad (13)$$

*Proof.* From (10), we have

$$\begin{aligned} \bar{H}_{k+1,\ell} &= \max_{\mathbf{C}_1 \in \mathcal{C}^{[\ell \times n]}} \max_{\substack{\mathbf{C}_2 \in \mathcal{C}^{[(k+1) \times n]} \\ \mathbf{C}_2 \mathbf{C}_1^T = \mathbf{0}}} \max_{\substack{\mathbf{F} \in \mathcal{F}^{[(m-k-1) \times n]} \\ \mathbf{F} \mathbf{C}_1^T = \mathbf{0} \\ \mathbf{F} \mathbf{C}_2^T = \mathbf{0}}} J\left(\begin{bmatrix} \mathbf{C} \\ \mathbf{F} \end{bmatrix}\right) \\ &= \max_{\mathbf{C}_1 \in \mathcal{C}^{[\ell \times n]}} \max_{\substack{\mathbf{C}_2 \in \mathcal{C}^{[k \times n]} \\ \mathbf{C}_2 \mathbf{C}_1^T = \mathbf{0}}} \max_{\substack{\mathbf{e} \in \mathcal{C}^{[1 \times n]} \\ \mathbf{e} \mathbf{C}_2^T = \mathbf{0}}} \max_{\substack{\mathbf{F} \in \mathcal{F}^{[(m-k-1) \times n]} \\ \mathbf{F}[\mathbf{C}_2^T \ \mathbf{e}^T] = \mathbf{0} \\ \mathbf{F} \mathbf{C}_1^T = \mathbf{0}}} J\left(\begin{bmatrix} \mathbf{C}_2 \\ \mathbf{e} \\ \mathbf{F} \end{bmatrix}\right) \\ &\leq \max_{\mathbf{C}_1 \in \mathcal{C}^{[\ell \times n]}} \max_{\substack{\mathbf{C}_2 \in \mathcal{C}^{[k \times n]} \\ \mathbf{C}_2 \mathbf{C}_1^T = \mathbf{0}}} \max_{\substack{\mathbf{f} \in \mathcal{F}^{[1 \times n]} \\ \mathbf{f} \mathbf{C}_2^T = \mathbf{0}}} \max_{\substack{\mathbf{F} \in \mathcal{F}^{[(m-k-1) \times n]} \\ \mathbf{F}[\mathbf{C}_2^T \ \mathbf{f}^T] = \mathbf{0} \\ \mathbf{F} \mathbf{C}_1^T = \mathbf{0}}} J\left(\begin{bmatrix} \mathbf{C}_2 \\ \mathbf{f} \\ \mathbf{F} \end{bmatrix}\right) \\ &= \max_{\mathbf{C}_1 \in \mathcal{C}^{[\ell \times n]}} \max_{\substack{\mathbf{C}_2 \in \mathcal{C}^{[k \times n]} \\ \mathbf{C}_2 \mathbf{C}_1^T = \mathbf{0}}} \max_{\substack{\mathbf{F}_1 \in \mathcal{F}^{[(m-k) \times n]} \\ \mathbf{F}_1 \mathbf{C}_2^T = \mathbf{0} \\ \mathbf{F}_1 \mathbf{C}_1^T = \mathbf{0}}} J\left(\begin{bmatrix} \mathbf{C}_2 \\ \mathbf{F}_1 \end{bmatrix}\right) = \bar{H}_{k,\ell}, \end{aligned}$$

where the inequality follows from  $\mathcal{C}^{[1 \times n]} \subset \mathcal{F}^{[1 \times n]}$ . Equation (13) can be proved similarly. ■

**Corollary 3.1 (Nesting).**

$$\begin{aligned} \sum_{j=1}^m \frac{\lambda_j^2}{\lambda_j + \sigma^2} &= J(\mathbf{F}^*) = \bar{H}_{0,0} \geq \bar{H}_{1,0} \geq \dots \geq \bar{H}_{m,0} = J(\mathbf{C}^*) \\ \sum_{j=1}^m \frac{\lambda_j^2}{\lambda_j + \sigma^2} &= J(\mathbf{F}^*) = \bar{H}_{0,0} \geq \bar{H}_{0,1} \geq \dots \geq \bar{H}_{0,n-m} = J(\mathbf{C}^*) \end{aligned}$$

### 3.2. Computation of upper bounds

In this section we propose ways to numerically compute the nested bounds in terms of generalized eigenvalues of matrix pencils. To facilitate the computations, we introduce the following definition of efficacy with respect to a general matrix pencil  $\langle \mathbf{A}, \mathbf{B} \rangle$ .

**Definition.** For two  $n \times n$  symmetric matrices  $\mathbf{A}$  and  $\mathbf{B}$ , define the efficacy of a matrix  $\mathbf{F} \in \mathcal{F}^{[m \times n]}$ , with respect to the pencil  $\langle \mathbf{A}, \mathbf{B} \rangle$ , as

$$J_{\langle \mathbf{A}, \mathbf{B} \rangle}(\mathbf{F}) \triangleq \text{tr} \left\{ (\mathbf{F} \mathbf{B} \mathbf{F}^T)^{-1} \mathbf{F} \mathbf{A} \mathbf{F}^T \right\}, \quad (14)$$

where  $\mathbf{A}$  and  $\mathbf{B}$  are two  $n \times n$  matrices. ■

The maximum of the efficacy in (14) can be calculated in closed form in terms of the generalized eigenvalues of the pencil  $\langle \mathbf{A}, \mathbf{B} \rangle$ .

**Theorem 4.** [11] Let  $\mathbf{A}$  and  $\mathbf{B}$  be symmetric and  $\mathbf{B}$  be positive definite. If the generalized eigenvalues of the pencil  $\langle \mathbf{A}, \mathbf{B} \rangle$  are  $\gamma_1 \geq \gamma_2 \geq \dots \geq \gamma_n \geq 0$ , then

$$J_{\langle \mathbf{A}, \mathbf{B} \rangle}(\mathbf{F}^*) = \sum_{j=1}^m \gamma_j. \quad (15)$$

**Definition.** Let  $\bar{\mathbf{C}}$  denote the **complement** of  $\mathbf{C}$ , with constraints  $\bar{\mathbf{C}} \in \mathcal{C}^{[(n-m) \times n]}$  and  $\bar{\mathbf{C}} \mathbf{C}^T = \mathbf{0}$ . ■

Now we propose the following theorem to numerically compute the upper bounds in terms of generalized eigenvalues of matrix pencils.

**Theorem 5.**

$$\bar{H}_{k,\ell} = \max_{\mathbf{C}_1 \in \mathcal{C}^{[\ell \times n]}} \max_{\mathbf{C} \in \mathcal{C}^{[k \times (n-\ell)]}} \left( \tau_k + \max_{\mathbf{F} \in \mathcal{F}^{[(m-k) \times (n-k-\ell)]}} J_{\langle \tilde{\mathbf{A}}_k, \tilde{\mathbf{B}}_k \rangle}(\mathbf{F}) \right)$$

where the term  $\tau_k$  and the modified pencil  $\langle \tilde{\mathbf{A}}_k, \tilde{\mathbf{B}}_k \rangle$  satisfy

$$\tau_k = \text{tr} \left\{ \tilde{\mathbf{A}} \begin{bmatrix} \mathbf{P}_k^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\} \quad (16)$$

$$\tilde{\mathbf{A}}_k = \begin{bmatrix} \mathbf{P}_k^{-1} \mathbf{Q}_k \\ -\mathbf{I}_{n-k-\ell} \end{bmatrix}^T \tilde{\mathbf{A}} \begin{bmatrix} \mathbf{P}_k^{-1} \mathbf{Q}_k \\ -\mathbf{I}_{n-k-\ell} \end{bmatrix} \quad (17)$$

$$\tilde{\mathbf{B}}_k = \mathbf{R}_k - \mathbf{Q}_k^T \mathbf{P}_k^{-1} \mathbf{Q}_k \quad (18)$$

$$\mathbf{P}_k = \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix}^T \tilde{\mathbf{B}} \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix} \quad (19)$$

$$\mathbf{Q}_k = \begin{bmatrix} \mathbf{I}_k \\ \mathbf{0} \end{bmatrix}^T \tilde{\mathbf{B}} \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n-k-\ell} \end{bmatrix} \quad (20)$$

$$\mathbf{R}_k = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n-k-\ell} \end{bmatrix}^T \tilde{\mathbf{B}} \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n-k-\ell} \end{bmatrix} \quad (21)$$

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{C} \\ \bar{\mathbf{C}} \end{bmatrix} \bar{\mathbf{C}}_1 \Lambda_{\underline{\mathbf{X}}}^2 \bar{\mathbf{C}}_1^T \begin{bmatrix} \mathbf{C} \\ \bar{\mathbf{C}} \end{bmatrix}^T \quad (22)$$

$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{C} \\ \bar{\mathbf{C}} \end{bmatrix} \bar{\mathbf{C}}_1 (\Lambda_{\underline{\mathbf{X}}} + \sigma^2 \mathbf{I}) \bar{\mathbf{C}}_1^T \begin{bmatrix} \mathbf{C} \\ \bar{\mathbf{C}} \end{bmatrix}^T \quad (23)$$

*Proof.* From (10) we have

$$\bar{H}_{k,\ell} = \max_{\mathbf{C}_1 \in \mathcal{C}^{[\ell \times n]}} \max_{\substack{\mathbf{C}_2 \in \mathcal{C}^{[k \times n]} \\ \mathbf{C}_2 \mathbf{C}_1^T = \mathbf{0}}} \max_{\substack{\mathbf{F}_1 \in \mathcal{F}^{[(m-k) \times n]} \\ \mathbf{F}_1[\mathbf{C}_2^T \ \mathbf{C}_1^T] = \mathbf{0}}} J(\Lambda_{\underline{\mathbf{X}}}, \Lambda_{\underline{\mathbf{X}}} + \sigma^2 \mathbf{I}) \left( \begin{bmatrix} \mathbf{C}_2 \\ \mathbf{F}_1 \end{bmatrix} \right) \quad (24)$$

Since  $\mathbf{C}_2 \mathbf{C}_1^T = \mathbf{0}$  and  $\mathbf{F}_1 \mathbf{C}_1^T = \mathbf{0}$ , we can write  $\mathbf{C}_2$  and  $\mathbf{F}_1$  as linear transformations of  $\bar{\mathbf{C}}_1$  such that

$$\mathbf{C}_2 = \mathbf{C} \bar{\mathbf{C}}_1 \quad \text{where } \mathbf{C} \in \mathcal{C}^{[k \times (n-\ell)]}$$

$$\mathbf{F}_1 = \mathbf{F}_2 \bar{\mathbf{C}}_1 \quad \text{where } \mathbf{F}_2 \in \mathcal{F}^{[(m-k) \times (n-\ell)]}.$$

Then from (14), we have,

$$J_{\langle \Lambda_{\underline{\mathbf{X}}}, \Lambda_{\underline{\mathbf{X}}} + \sigma^2 \mathbf{I} \rangle} \left( \begin{bmatrix} \mathbf{C}_2 \\ \mathbf{F}_1 \end{bmatrix} \right) = J_{\langle \bar{\mathbf{C}}_1 \Lambda_{\underline{\mathbf{X}}}^2 \bar{\mathbf{C}}_1^T, \bar{\mathbf{C}}_1 (\Lambda_{\underline{\mathbf{X}}} + \sigma^2 \mathbf{I}) \bar{\mathbf{C}}_1^T \rangle} \left( \begin{bmatrix} \mathbf{C} \\ \mathbf{F}_2 \end{bmatrix} \right) \quad (25)$$

Because  $\text{rank}(\mathbf{F}_1) = \text{rank}(\mathbf{F}_2) = m-k$  and  $\mathbf{F}_1 \mathbf{C}_2^T = \mathbf{0}$ , we have  $\mathbf{F}_2 \mathbf{C}^T = \mathbf{0}$ . Let  $\mathbf{F} \in \mathcal{F}^{[(m-k) \times (n-k-\ell)]}$  and let

$\mathbf{F}_2 = \mathbf{F}\bar{\mathbf{C}}$ . Then from (14), (22) and (23) it follows that

$$J_{\langle\bar{\mathbf{A}},\bar{\mathbf{B}}\rangle}\left(\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix}\right) = J_{\langle\bar{\mathbf{C}}_1\Lambda_{\underline{\mathbf{x}}}^2\bar{\mathbf{C}}_1^T, \bar{\mathbf{C}}_1(\Lambda_{\underline{\mathbf{x}}}+\sigma^2\mathbf{I})\bar{\mathbf{C}}_1^T\rangle}\left(\begin{bmatrix} \mathbf{C} \\ \mathbf{F}_2 \end{bmatrix}\right) \quad (26)$$

Now we can split the efficacy

$$J_{\langle\bar{\mathbf{A}},\bar{\mathbf{B}}\rangle}\left(\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix}\right)$$

into two terms such that

1. the first term does not depend on  $\mathbf{F}$
2. the second term equals to the efficacy of  $\mathbf{F}$  with respect to a modified matrix pencil of smaller dimension.

Using (19)–(21), we express  $\tilde{\mathbf{B}}$  in terms of  $\mathbf{P}_k$ ,  $\mathbf{Q}_k$  and  $\mathbf{R}_k$  as,

$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{P}_k & \mathbf{Q}_k \\ \mathbf{Q}_k^T & \mathbf{R}_k \end{bmatrix}$$

Then using the partitioned matrix inversion lemma [12], we have

$$\begin{aligned} J_{\langle\bar{\mathbf{A}},\bar{\mathbf{B}}\rangle}\left(\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix}\right) &= \text{tr} \left\{ \tilde{\mathbf{A}} \begin{bmatrix} \mathbf{P}_k^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\} \\ &+ \text{tr} \left\{ \begin{bmatrix} \mathbf{P}_k^{-1}\mathbf{Q}_k\mathbf{F}^T \\ -\mathbf{I}_{m-k} \end{bmatrix} \left\{ \mathbf{F}(\mathbf{R}_k - \mathbf{Q}_k^T\mathbf{P}_k^{-1}\mathbf{Q}_k)\mathbf{F}^T \right\}^{-1} \right. \\ &\quad \left. \times \begin{bmatrix} \mathbf{P}_k^{-1}\mathbf{Q}_k\mathbf{F}^T \\ -\mathbf{I}_{m-k} \end{bmatrix}^T \begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix} \tilde{\mathbf{A}} \begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix}^T \right\}. \end{aligned}$$

Using (16)–(18), we simplify the above equation as

$$\begin{aligned} J_{\langle\bar{\mathbf{A}},\bar{\mathbf{B}}\rangle}\left(\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix}\right) &= \tau_k + \text{tr} \left\{ \left( \mathbf{F}\tilde{\mathbf{B}}_k\mathbf{F}^T \right)^{-1} \mathbf{F}\tilde{\mathbf{A}}_k\mathbf{F}^T \right\} \\ &= \tau_k + J_{\langle\bar{\mathbf{A}}_k,\bar{\mathbf{B}}_k\rangle}(\mathbf{F}) \end{aligned} \quad (27)$$

Combining (24)–(27), we have

$$\bar{H}_{k,\ell} = \max_{\mathbf{C}_1 \in \mathcal{C}^{[\ell \times n]}} \max_{\mathbf{C} \in \mathcal{C}^{[k \times (n-\ell)]}} \left( \tau_k + \max_{\mathbf{F} \in \mathcal{F}^{[(m-k) \times (n-k-\ell)]}} J_{\langle\bar{\mathbf{A}}_k,\bar{\mathbf{B}}_k\rangle}(\mathbf{F}) \right). \quad \blacksquare$$

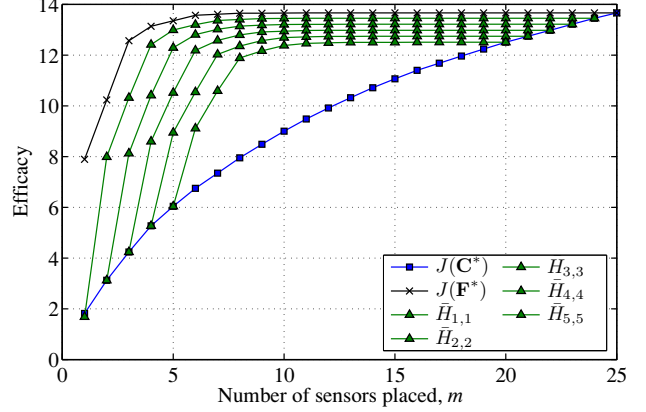
#### 4. SIMULATION RESULTS

We performed simulations on 5 by 5 grid with unit distance between neighboring points, i.e., each point is located at unit distance from its horizontal and vertical neighbors, to evaluate the performances of the upper bounds  $\bar{H}_{k,\ell}$  (for  $k = 0$  to 5 and  $\ell = 0$  to 5). We generated the covariance matrix using a Gaussian distribution, where the variance between the points, considered as vectors  $\underline{x}_1$  and  $\underline{x}_2$  is given by [3]:

$$\Lambda(\underline{x}_1, \underline{x}_2) = \exp \left( -\|\underline{x}_1 - \underline{x}_2\|_2^2 / (2\pi) \right). \quad (28)$$

Figure 1 shows the simulation results. It is observed from the simulation that the upper bounds get tighter as  $k$  and  $\ell$  are increased, which confirms our results in Section 3.

We also performed simulations for different simulation scenarios, such as sensor placement in IEEE test bus system [8], data generated at random etc. (which could not be



**Fig. 1.** Performances of the upper bounds  $\bar{H}_{k,\ell}$  compared to the optimal solution for a  $5 \times 5$  grid.

presented here due to space limitations). From these simulations we observe that, for any  $k$  and  $\ell$ , Theorem 3 holds, i.e.,  $\bar{H}_{k,\ell} \geq \bar{H}_{k+1,\ell}$  and  $\bar{H}_{k+1,\ell-1} \geq \bar{H}_{k+1,\ell}$ . However, we observe that the apriori comparison between the bounds  $\bar{H}_{k,\ell}$  and  $\bar{H}_{k+1,\ell-1}$  is not possible without computing the bounds explicitly.

#### 5. RELATION TO PRIOR WORK

This work significantly extends previous work [8] by considering a family of nested bounds to upper bound the performance of the efficacy. The computation of these bounds makes a tradeoff between complexity and the tightness of the bounds. The bounds not only consider placement of sensors, but places where we do not want to place sensors. This has not been considered in previous literature [1–5] and has applicability so that we can analyze cases where there are certain nodes where we want information, but can not place sensors in these locations.

#### 6. CONCLUSION

This paper opens up new directions for studying the sensor placement problem. We developed analytical upper bounds for the sensor placement problem constraining where we can put and not put sensors at different nodes using matrix pencils and generalized eigenvalues. It is also easy to formulate a class of greedy realizable algorithms extending previous work where we are constrained to both add and delete nodes where we can place sensors. This also leads to more generalized sensor placement problems where placement is constrained due to the topology of the network (allowing for distributed algorithms to place sensors in certain neighborhoods).

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