A FAST WIDELY-LINEAR QR-DECOMPOSITION LEAST-SQUARES (FWL-QRD-RLS) ALGORITHM

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ABSTRACT

This paper considers the development of fast widely-linear (FWL) recursive Least-squares (RLS) algorithm well suited for processing non-circular signals. The proposed algorithm makes use of covariance and modified covariance matrices which take full advantage of second order statistics of non-circular data. Further, the proposed algorithm is based on the fast QR-decomposition recursive least-squares (QRD-RLS) algorithm. Therefore, its computational complexity is of $O(N)$ as compared to $O(N^2)$ of conventional WL-RLS and is numerically more stable in finite precision environment. Simulation results have been presented to test the proposed FWL-QRD-RLS algorithm in two adaptive filtering scenarios: system identification and uniform array beamformer.

Index Terms— QR-decomposition, Complexed-valued signal processing, widely linear, adaptive filtering, Fast algorithms

1. INTRODUCTION

In practice, processing $improper$ (non-circular) data is required in a wide range of applications, such as, wind forecasting using quaternion data for renewable energy generation, multiple access interference mitigation in direct sequence code division multiple access (DS-CDMA) systems, adaptive beamforming for underdetermined scenario where the number of sources is greater than the number of antennas [1–4]. A random signal $\mathbf{x}$ is *proper* if completely described by its covariance matrix $E\{\mathbf{x}\mathbf{x}^{\mathrm{H}}\}$, whereas its complementary covariance $E\{\mathbf{x}\mathbf{x}^{\mathrm{T}}\}$ is zero, otherwise it is *improper*. Standard adaptive filtering techniques have been primarily developed for *proper input* signals, therefore, their performance degrades when processing improper (non-circular) data [5–7]. Widely-linear (WL) adaptive filtering approaches, on the other hand, are now gaining popularity due to their superior performance over the traditional methods in the previously mentioned applications [1–3]. The interest in WL processing has led to the development of new algorithms that are based on least-mean square (LMS) and recursive least-squares (RLS) frameworks [1, 2].

WL adaptive filters are based on the concept of exploiting the original data and its complex conjugate to process non-circular signals. This exploitation has let to performance levels not achievable by traditional adaptive approaches. However, this performance boost comes at cost, which is an additional computational complexity. In LMS type algorithms, this cost is not an issue, as it is in $\mathcal{O}(N)$, N being the number of complex valued filter coefficients. For RLS type algorithms, this is not the case as the computational cost of WL-RLS algorithm is 4 times the traditional algorithm. The RLS type algorithms are often preferred over the LMS type algorithms in processing improper signals since they have superior convergence speed and lower misadjustment [8].

Attempts to reduce the computational complexity of WL-RLS algorithms have been reported in the literature. In [1], a modified WL-RLS algorithm has been proposed for the purpose of reducing the computational cost of the WL-RLS algorithm. The modified WL-RLS algorithm makes use of a unitary transform so that real values are used instead of the complex input data values. Note that even with this transformation, the complexity of the modified WL-RLS is still in $\mathcal{O}(N^2)$. Further, this algorithm suffers from the instability in finite precision environment inherent in traditional RLS algorithms.

In this paper, we reformulate the WL adaptive processing as a multichannel sequential fast QR-decomposition RLS (QRD-RLS) algorithm and exploit the covariance and modified covariance matrices which take full advantage of second order statistics of non-circular data for the development of fast WL-QRD-RLS algorithm. Fast algorithms based on QR-decomposition have been proposed in the literature to reduce the computational burden of RLS algorithms by an order of magnitude (i.e., $\mathcal{O}(N)$ instead of N^2), by taking advantage of the shift structure of the input data matrix which allows vector updates instead of matrix updates [9–11]. Further, these algorithms have better performance when implemented in finite precision which results in a lower system power consumption, which in turn, leads to environmental and human friendly or "Green" solutions.

The rest of the paper is organized as follows. Section 2 provides an introduction to the development of the proposed FWL-QRD-RLS algorithm. A brief derivation of the proposed algorithm is presented in Section 3. Section 4 presents simulation results to show the performance of the proposed FWL-QRD-RLS algorithm as compared to that of RLS and WL-RLS algorithms in two scenarios: system identification and uniform circular array beamformer.

2. FUNDAMENTALS

This section provides basic set of equations that are essential for the derivation of the FWL-QRD-RLS algorithm.

Consider a multichannel adaptive filtering setup with M channels and N coefficients per channel, resulting in a total of MN coefficients. To derive the WL algorithm both the input and its conjugate must be considered as shown in Fig 1 (where the dashed-lines depict the conjugate input). Therefore, WL-QRD-RLS considers $P = 2 \times MN$ coefficients.

The WL-QRD-RLS algorithm minimizes the following objec-

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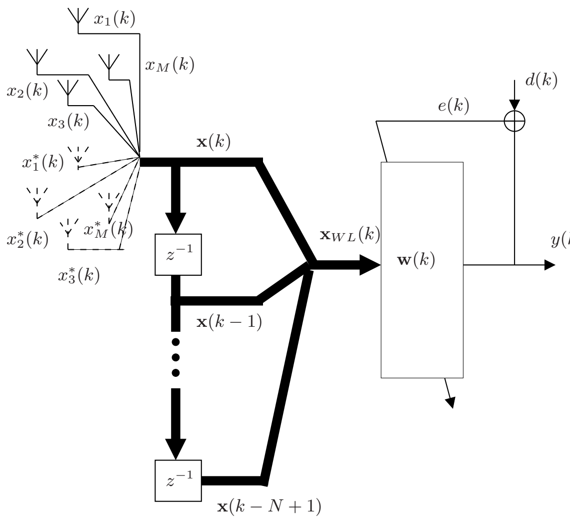

Fig. 1. Multichannel adaptive filter setup.

tive function with respect to the coefficient vector $\mathbf{w}(k) \in \mathbb{C}^{P\times 1}$

$$\xi(k) = \sum_{i=0}^{k} \lambda^{k/2} |d^*(i) - \mathbf{x}_{\mathrm{WL}}^{\mathrm{H}}(i)\mathbf{w}(k)|^2 = \|\mathbf{e}^*(k)\|^2 \quad (1)$$

where λ is the forgetting factor, $\mathbf{e}(k) \in \mathbb{C}^{(k+1)\times 1}$ is the *a priori* error vector given as,

$$\begin{aligned}\mathbf{e}^*(k) &= \begin{bmatrix} d^*(k) \\ \lambda^{1/2} d^*(k-1) \\ \vdots \\ \lambda^{k/2} d^*(0) \end{bmatrix} - \begin{bmatrix} \mathbf{x}_{\mathrm{WL}}^{\mathrm{H}}(k) \\ \lambda^{1/2}\mathbf{x}_{\mathrm{WL}}^{\mathrm{H}}(k-1) \\ \vdots \\ \lambda^{k/2}\mathbf{x}_{\mathrm{WL}}^{\mathrm{H}}(0) \end{bmatrix} \mathbf{w}(k) \\ &= \mathbf{d}^*(k) - \mathbf{X}_{\mathrm{WL}}(k)\mathbf{w}(k), \end{aligned} \quad (2)$$

for the kth time index, $\mathbf{d}(k) \in \mathbb{C}^{(k+1)\times 1}$ is the desired signal vector, $\mathbf{x}(k) \in \mathbb{C}^{P\times 1}$ is the WL input vector

$$\mathbf{x}_{\mathrm{WL}}(k) = \begin{bmatrix} \mathbf{x}^{\mathrm{T}}(k) & \mathbf{x}^{\mathrm{T}}(k-1) & \ldots & \mathbf{x}^{\mathrm{T}}(k-P+1) \end{bmatrix}^{\mathrm{T}}. \quad (3)$$

The input vector for WL process contains both the original complex values and their conjugate pairs augmented into single $2M \times 1$ input vector defined as,

$$\mathbf{x}(k) = \begin{bmatrix} x_1(k) & \ldots & x_M(k) & x_1^*(k) & \ldots & x_M^*(k) \end{bmatrix}^{\mathrm{T}}. \quad (4)$$

The WL-QRD-RLS algorithm uses an orthogonal rotation matrix $\mathbf{Q}_{WL}(k) \in \mathbb{C}^{(k+1)\times(k+1)}$ to triangularize $\mathbf{X}_{WL}(k)$ as

$$\begin{bmatrix} \mathbf{0}_{(k+1-P)\times P} \\ \mathbf{U}_{WL}(k) \end{bmatrix} = \mathbf{Q}_{WL}(k)\mathbf{X}_{WL}(k) \quad (5)$$

where $\mathbf{U}_{WL}(k) \in \mathbb{C}^{P\times P}$ is the Cholesky factor of the deterministic autocorrelation matrix $\mathbf{R}_{WL}(k) = \mathbf{X}_{WL}^{\mathrm{H}}(k)\mathbf{X}_{WL}(k)$ which has a block Toeplitz structure. Pre-multiplying (2) with $\mathbf{Q}_{WL}(k)$ gives

$$\mathbf{Q}_{WL}(k)\mathbf{e}^*(k) = \begin{bmatrix} \mathbf{e}_{q1}(k) \\ \mathbf{e}_{q2}(k) \end{bmatrix} = \begin{bmatrix} \mathbf{d}_{q1}(k) \\ \mathbf{d}_{q2}(k) \end{bmatrix} - \begin{bmatrix} \mathbf{0}_{(k+1-P)\times P} \\ \mathbf{U}_{WL}(k) \end{bmatrix} \mathbf{w}(k) \quad (6)$$

We emphasize that $\mathbf{d}_{q1}(k)$ and $\mathbf{d}_{q2}(k)$ are partitions of vector $\mathbf{d}^*(k)$ after rotation, similarly $\mathbf{e}_{q1}(k)$ and $\mathbf{e}_{q2}(k)$ are partitions of vector $\mathbf{e}^*(k)$ after rotation. The cost function in (1) is minimized by choosing $\mathbf{w}(k)$ such that $\mathbf{d}_{q2}(k) - \mathbf{U}_{WL}(k)\mathbf{w}(k)$ is zero, i.e., $\mathbf{w}(k) = \mathbf{U}_{WL}^{-1}(k)\mathbf{d}_{q2}(k)$. The algorithm updates $\mathbf{d}_{q2}(k)$ and $\mathbf{U}_P(k)$ as follows [11]

$$\begin{bmatrix} e_{q1}(k) \\ \mathbf{d}_{q2}(k) \end{bmatrix} = \mathbf{Q}_\theta(k) \begin{bmatrix} d^*(k) \\ \lambda^{1/2}\mathbf{d}_{q2}(k-1) \end{bmatrix} \quad (7)$$

$$\begin{bmatrix} \mathbf{0}_{1\times P} \\ \mathbf{U}_{WL}(k) \end{bmatrix} = \mathbf{Q}_\theta(k) \begin{bmatrix} \mathbf{x}_{WL}^{\mathrm{H}}(k) \\ \lambda^{1/2}\mathbf{U}_{WL}(k-1) \end{bmatrix} \quad (8)$$

where $\mathbf{Q}_\theta(k) \in \mathbb{C}^{(P+1)\times(P+1)}$ is a sequence of Givens rotation matrices which annihilates the input vector $\mathbf{x}(k)$ in (8) and can be partitioned as [11]

$$\mathbf{Q}_\theta(k) = \begin{bmatrix} \gamma(k) & \mathbf{g}^{\mathrm{H}}(k) \\ \mathbf{f}(k) & \mathbf{E}(k) \end{bmatrix} \quad (9)$$

The algorithm is complete with the definition of the *a priori* error value $e(k) = e_{q1}^*(k)$.

3. FAST ALGORITHM

In this section, the WL-approach is incoporated in a sequential multichannel FQRD-RLS algorithm to derive the FWL-QRD-RLS algorithm [9]. Only a brief sketch of derivation is presented.

We can now consider the following unitary transformation on the input vector defined in (4),

$$\begin{aligned}\hat{\mathbf{x}}(k) &= \underbrace{\begin{bmatrix} \mathbf{I}(k) & j\mathbf{I}(k) \\ \mathbf{I}(k) & -j\mathbf{I}(k) \end{bmatrix}^{-1}}_{\mathbf{T}_u} \mathbf{x}(k) \\ &= \begin{bmatrix} \Re\{x_1(k) & \ldots & x_M(k)\} & \Im\{x_1(k) & \ldots & x_M(k)\} \end{bmatrix}^{\mathrm{T}} \end{aligned} \quad (10)$$

where $\mathbf{I}(k) \in \mathbb{C}^{M\times M}$ is the identity matrix. It was shown in [1] that if $\hat{\mathbf{x}}(k)$ is used as an input signal to a WL algorithm instead of $\mathbf{x}(k)$, the resulting coefficients $\hat{\mathbf{w}}(k)$ are related to actual coefficients $\mathbf{w}(k)$ by $\mathbf{T}_u$.

In a sequential algorithm, the input vector defined in (3) is updated in $2M$ steps. The first step is given by,

$$\begin{bmatrix} \bar{\mathbf{x}}^{(1)}(k-1) & \Re\{x_1(k-N)\} \end{bmatrix}^{\mathrm{T}} = \mathbf{P}_1 \begin{bmatrix} \Re\{x_1(k)\} & \bar{\mathbf{x}}(k-1) \end{bmatrix}^{\mathrm{T}} \quad (11)$$

where, $\bar{\mathbf{x}}(k)$ is similar to $\mathbf{x}(k)$ in (3) except that its member vectors are now the modified input vectors in (10). $\mathbf{P}_1$ is the permutation matrix that places $x_1(k)$ in the correct position in $\bar{\mathbf{x}}^{(1)}(k-1)$ for the update, and removes $\Re\{x_1(k-N)\}$ from $\bar{\mathbf{x}}(k-1)$. $\Re\{.\}$ denotes the real part of a complex number. Similarly, the $M+1$th step is given as,

$$\begin{bmatrix} \bar{\mathbf{x}}^{(M+1)}(k-1)\, \Im\{x_1(k-N)\} \end{bmatrix}^{\mathrm{T}} = \mathbf{P}_{M+1} \begin{bmatrix} \Im\{x_1(k)\}\, \bar{\mathbf{x}}^{(M)}(k-1) \end{bmatrix}^{\mathrm{T}} \quad (12)$$

where $\Im\{.\}$ denotes the imaginary part of a complex number. After $2M$ iterations, the input vector is updated from index $k-1$ to k,

$$\bar{\mathbf{x}}(k-1) \to \bar{\mathbf{x}}^{(1)}(k-1) \to \ldots \to \bar{\mathbf{x}}^{(2M)}(k-1) = \bar{\mathbf{x}}(k) \quad (13)$$

Let us define the input data matrix of the lth sequential step defined in (13) as,

$$\mathbf{X}_{WL}^{(l)}(k-1) = \mathbf{P}_l \begin{bmatrix} \bar{\mathbf{x}}^{\mathrm{H}(l)}(k-1) \\ \lambda^{1/2}\bar{\mathbf{x}}^{\mathrm{H}(l)}(k-2) \\ \vdots \\ \lambda^{k/2}\bar{\mathbf{x}}^{\mathrm{H}(l)}(-1) \end{bmatrix}. \tag{14}$$

Similar to (5) we can obtain the corresponding Cholesky factor,

$$\begin{bmatrix} \mathbf{0}_{(k-P+1)\times P} \\ \mathbf{U}_{WL}^{(l)}(k-1) \end{bmatrix} = \mathbf{Q}^{(l)}(k)\mathbf{X}_{WL}^{(l)}(k-1). \tag{15}$$

The update equation for the Cholesky factor given in (8) can also be applied to (15), which results in,

$$\begin{bmatrix} \mathbf{0}_{1\times P} \\ \mathbf{U}_{WL}^{(l)}(k-1) \end{bmatrix} = \mathbf{Q}_{\theta}^{(l)}(k) \begin{bmatrix} \bar{\mathbf{x}}^{\mathrm{H}(l)}(k-1) \\ \lambda^{1/2}\mathbf{U}_{WL}^{(l-1)}(k-1) \end{bmatrix}. \tag{16}$$

FWL-QRD-RLS algorithms exploit the shift structure of the input vector, and replace the matrix update in (16) with vector updates of either $\mathbf{f}(k)$ or $\mathbf{g}(k)$ in (9). Here we will adopt the algorithm based on *a posteriori* errors in [9] which updates vector $\mathbf{f}(k)$. In FWL-QRD-RLS algorithm, vector $\mathbf{f}(k)$ is updated in $2M$ successive steps, i.e.,

$$\mathbf{f}(k-1) \rightarrow \mathbf{f}^{(1)}(k-1) \rightarrow \cdots \rightarrow \mathbf{f}^{(2M)}(k-1) = \mathbf{f}(k) \tag{17}$$

where

$$\mathbf{f}^{(l)}(k-1) = \mathbf{U}^{-\mathrm{H}(l)}(k-1)\bar{\mathbf{x}}^{(l)}(k-1). \tag{18}$$

The update equation for vector $\mathbf{f}^{(l-1)}(k-1)$ is given as

$$\begin{bmatrix} \frac{\varepsilon_b^{(l)}(k)}{\|\mathbf{e}_b^{(l)}(k)\|} \\ \mathbf{f}^{(l)}(k-1) \end{bmatrix} = \mathbf{P}_l\mathbf{Q}_{f\theta}^{(l)}(k) \begin{bmatrix} \mathbf{f}^{(l-1)}(k-1) \\ \frac{\varepsilon_f^{(l)}(k)}{\|\mathbf{e}_f^{(l)}(k)\|} \end{bmatrix}, \tag{19}$$

where $\varepsilon_b^{(l)}(k)$ and $\varepsilon_f^{(l)}(k)$ are the *a posteriori* backward and forward prediction errors, $\|\mathbf{e}_b^{(l)}(k)\|$ and $\|\mathbf{e}_f^{(l)}(k)\|$ are the norms of the backward and forward prediction error vectors, and $\mathbf{P}_l$ is the permutation matrix. The resulting FWL-QRD-RLS algorithm is summarized in Table 1. The computational complexity of the proposed algorithm is compared to RLS and WL-RLS algorithms in Table 2.

Table 1. FWL-QRD-RLS algorithm.

Multichannel sequential FQR_POS_B algorithm for WL complex signals
Initializations:
$\mathbf{d}_{fq2}^{(i)}(0) = \mathbf{0}_{P\times 1}; \mathbf{f}^{(i)}(0) = \mathbf{0}_{P\times 1}; \mathbf{d}_{q2}(0) = \mathbf{0}_{P\times 1}; \gamma_P^{(0)}(k) = 1; \|\mathbf{e}_{fP}^{(i)}(0)\| = \mu;$
$\mathbf{Q}_{\theta}^{(i)}(k) = \mathbf{I}_{P+1\times P+1}; \mathbf{Q}_{\theta f}^{(i)}(k) = \mathbf{I}_{P+1\times P+1}; i = 1, 2, \ldots, M$
for each k {
$\mathbf{Q}_{\theta f}^{(0)}(k) = \mathbf{Q}_{\theta f}^{(M)}(k-1)$
$\mathbf{f}^{(0)}(k) = \mathbf{f}^{(M)}(k-1)$
for each i 1 to M {
Obtaining $\mathbf{d}_{fq2}^{(i)}(k)$:
$e_{fq1}^{(i)}(k) = x_i(k)$
$\begin{bmatrix} e_{fq1}^{(i)}(k) \\ \mathbf{d}_{fq2}^{(i)}(k) \end{bmatrix} = \mathbf{Q}_{\theta}^{(i-1)}(k) \begin{bmatrix} x_i(k) \\ \lambda^{1/2}\mathbf{d}_{fq2}^{(i)}(k-1) \end{bmatrix}$
Obtaining $\|\mathbf{e}_f^{(i)}(k)\|$:
$\|\mathbf{e}_f^{(i)}(k)\| = \sqrt{(e_{fq1}^{(i)}(k))^2 + \lambda\|\mathbf{e}_f^{(i)}(k-1)\|^2}$
Obtaining $\mathbf{Q}_{\theta f}^{(i)}(k)$:
$\begin{bmatrix} \mathbf{0}_{P\times 1} \\ \|\mathbf{e}_{f0}^{(i)}(k)\| \end{bmatrix} = \mathbf{Q}_{\theta f}^{(i)}(k) \begin{bmatrix} \mathbf{d}_{fq2}^{(i)}(k) \\ \|\mathbf{e}_f^{(i)}(k)\| \end{bmatrix}$
Obtaining $\mathbf{f}^{(i)}(k)$:
$e_f^{(i)}(k) = \gamma^{(i-1)}(k)e_{fq1}^{(i)}(k)$
$\begin{bmatrix} \frac{e_b^{(i)}(k)}{\|\mathbf{e}_b^{(i)}(k-1)\|} \\ \mathbf{f}^{(i)}(k-1) \end{bmatrix} = \mathbf{P}_i\mathbf{Q}_{\theta f}^{(i)}(k) \begin{bmatrix} \mathbf{f}^{(i-1)}(k-1) \\ \frac{e_f^{(i)}(k)}{\|\mathbf{e}_f^{(i)}(k-1)\|} \end{bmatrix}$
Obtaining $\mathbf{Q}_{\theta}^{(i)}(k)$:
$\begin{bmatrix} 1 \\ \mathbf{0}_{P\times 1} \end{bmatrix} = [\mathbf{Q}_{\theta}^{(i)}(k)]^{\mathrm{T}} \begin{bmatrix} \gamma^{(i)}(k) \\ \mathbf{f}^{(i)}(k-1) \end{bmatrix}$
}
Joint Process Estimation:
$\begin{bmatrix} e_{q1}(k) \\ \mathbf{d}_{q2}(k) \end{bmatrix} = \mathbf{Q}_{\theta}(k) \begin{bmatrix} d(k) \\ \lambda^{1/2}\mathbf{d}_{q2}(k-1) \end{bmatrix}$
$e(k) = e_{q1}(k)/\gamma^{(M)}(k)$
}

4. SIMULATION RESULTS

This section investigates the equivalence of FWL-QRD-RLS algorithm with a WL-RLS algorithms in two applications. A system identification of an unknown complex valued linear-system when only real valued desired signal is available. Such a situation occurs in multiple access interference mitigation for DS-CDMA systems [4]. An adaptive beamforming for interference cancellation in a uniform circular antenna array is also presented.

4.1. System identification

The setup consists of an unknown system the output of which is defined as $d(k) = \Re\{\mathbf{w}^{\mathrm{H}}\mathbf{x}(k)\} + \mathbf{n}(k)$, where $\mathbf{w} = [w_1\ w_2\ \ldots\ w_P]$, $w_p = \beta(1 + \cos(2\pi(p-3)/5) - j(1 + \cos(2\pi(p-3)/10)$, $\beta = 0.432$, $p = 1, 2, \ldots, P$ and $\mathbf{n}(k)$ is the Gaussian noise a variance of -20dB [1]. And the input signal vector $\mathbf{x}(k)$ is defined using two uncorrelated real-valued Gaussian processes, $x_{\mathrm{R}}(k)$ and $x_{\mathrm{I}}(k)$, with zero mean and $1/\sqrt{(2)}$ variance. That is, $\mathbf{x}(k) = \sqrt{1 = \alpha^2}x_{\mathrm{R}}(k) + j\alpha x_{\mathrm{I}}(k)$, where $\alpha = 0.2$. The forgetting factor λ for RLS, WL-RLS and FWL-QRD-RLS algorithms is set to 0.999. Fig. 2 shows the learning curves for the three algorithms. It is observed that WL algorithms match each other while outperforming the conventional RLS algorithm.

4.2. Uniform circular array broadband beamforming

A uniform circular array with $M = 4$ antenna elements with spacing equal to half wavelength is used in a beamforming system in persence of 4 interferering signals. The desired signal is at 45^o, while the interferers are at $90^o, 1450^o, 180^o$, and 270^o. The signal to interference ratio -40dB for desired signal. Fig. 3 shows the performance of the WL algorithms against conventional RLS algorithm. Whileas, Fig. 4 shows the resulting beam pattern of the array. Both WL algorithms show same performance.

5. CONCLUSIONS

This paper has presented a new fast adaptive algorithm, which is a WL version of multichannel sequential fast QRD-RLS algorithm. The proposed algorithm exploits the covariance and modified covariance matrices which take full advantage of second order statistics of improper (non-circular) data. The resulting algorithm has computational complexity, which is in an order of magnitude lower than that of the WL-RLS algorithm. Simulation results show that the learning curve of the proposed FWL-QRD-RLS algorithm exactly matches that of the WL-RLS algorithm. That is, the proposed algorithm maintains the same performance as that of the WL-RLS algorithm but with lower computational cost and greatly enhanced numerical

Table 2. Computational complexities of FWL-QRD-RLS, WL-RLS and RLS.

Algorithm	$+$	$\times$	$\div$
FWL-QRD-RLS	$12P + 11M + 4N - 7\sum_{i=1}^{M} p_i$	$14P + 13M + 5N - 9\sum_{i=1}^{M} p_i$	$3P + 5M + 3\sum_{i=1}^{M} p_i$
WL-RLS	$6P^2 + 11P$	$8P^2 + 14P + 1$	1
RLS	$6P^2 + 14P - 1$	$7P^2 + 21P + 1$	1

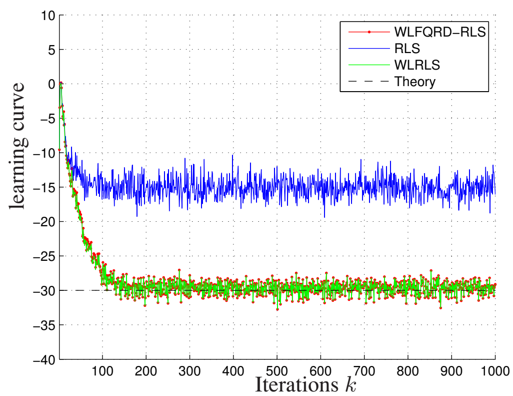

Fig. 2. Learning curves of RLS, WL-RLS, and FWL-QRD-RLS algorithms.

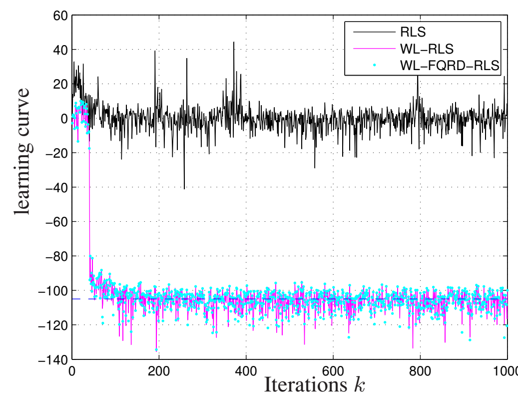

Fig. 3. Learning curves of RLS, WL-RLS, and FWL-QRD-RLS algorithms.

stability due to the numerical properties of the QRD based RLS algorithms.

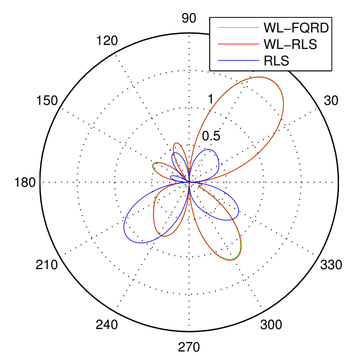

Fig. 4. Beam patterns of RLS, WL-RLS, and FWL-QRD-RLS algorithms.

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