CONDITIONS FOR TARGET RECOVERY IN SPATIAL COMPRESSIVE SENSING FOR MIMO RADAR

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ABSTRACT

We study compressive sensing in the spatial domain for target localization in terms of direction of arrival (DOA), using multiple-input multiple-output (MIMO) radar. A sparse localization framework is proposed for a MIMO array in which transmit/receive elements are placed at random. This allows to dramatically reduce the number of elements, while still attaining performance comparable to that of a filled (Nyquist) array. Leveraging properties of a (structured) random measurement matrix, we develop a novel bound on the coherence of the measurement matrix, and we obtain conditions under which the measurement matrix satisfies the so-called *isotropy* property. The coherence and isotropy concepts are used to establish respectively uniform and non-uniform recovery guarantees for target localization using spatial compressive sensing. In particular, non-uniform recovery is guaranteed if the number of degrees of freedom (the product of the number of transmit and receive elements MN) scales with $K\left(\log G\right)^2$, where K is the number of targets, and G is proportional to the array aperture and determines the angle resolution. The significance of the logarithmic dependence in G is that the proposed framework enables high resolution with a small number of MIMO radar elements. This is in contrast with a filled virtual MIMO array where the product MN scales linearly with G.

Index Terms— Compressive sensing, MIMO radar, random arrays, direction of arrival estimation.

1. INTRODUCTION

It is well known in array signal processing [1] that resolution improves by increasing the array aperture. A non-ambiguous uniform linear array (ULA) must have its elements spaced at intervals no larger than $\lambda/2$, where λ is the signal wavelength. For a MIMO radar [2], unambiguous direction finding of targets is possible if, N receive elements are spaced $\lambda/2$ and M transmit elements are spaced $N\lambda/2$, a configuration known as virtual array. In compressive sensing parlance, the $\lambda/2$-spaced array and the MIMO virtual array perform spatial sampling at Nyquist rate. The main disadvantage of this Nyquist setup is that the array aperture, and thus resolution, scales only linearly with MN, i.e., the degrees of freedom of the system. In contrast to previous literature on compressive sensing applied to arrays [3] and MIMO radar [4], which discussed the ULA setup, in this work, we are interested in a random array MIMO radar. The goal of spatial compressive sensing, i.e., when spatial sampling is applied at sub-Nyquist rates, is to achieve the same resolution as the filled arrays, but using a significantly reduced number of sensors.

Random array theory can be traced back to the work in [5], where it was shown that, as the number of sensors of an array is increased, the random array beampattern converges to its average. This work was extended to MIMO radar in [6]. While random array theory has been known for a long time, two fundamental questions were left pending: How many sensors are needed for localization as a function of the number of targets, and which method should be used for localization?

The advent of compressed sensing addresses the heart of these questions. In the radar setting, we define a grid of possible target locations, and each column of the matrix $\mathbf{A}$ is the "virtual array" steering vector pointing towards one of the grid points. In this setup, the measurements comply with $\mathbf{y} = \mathbf{A}\mathbf{x}$. The unknown signal $\mathbf{x}$ encodes information about targets locations and gains, and it is *sparse*, i.e., it has only K non-zero elements out of G (with $K \ll G$).

Compressive sensing theory [7] shows that the unknown sparse signal $\mathbf{x}$ can be recovered with high probability by solving a convex problem, whenever the measurement matrix $\mathbf{A}$ satisfies specific properties. There are two kind of recovery guarantees: uniform and non-uniform. Uniform guarantees capture the recovery of the worst-case K-sparse signal for a fixed instantiation of the random measurement matrix $\mathbf{A}$. An important parameter used to obtain uniform recovery guarantee is the coherence μ, defined as the maximum inner product between normalized columns of the matrix $\mathbf{A}$. Despite its widespread use, it is well known that a uniform guarantee based on coherence requires the number of measurements to scale quadratically with the number K of non-zero elements. A more advantageous scaling, e.g. linear, in the sparsity K, can be obtained if we ask for non-uniform recovery, which captures the typical recovery behavior of the random measurement matrix $\mathbf{A}$.

Recent work has shown that, for a sufficient number of independent and identically distributed (i.i.d.) compressive sensing measurements, a non-uniform recovery can be guaranteed if a specific property of the random sensing matrix $\mathbf{A}$, called *isotropy*, holds [8]. This result has been extended in [9] to non-isotropic measurements. Unfortunately, these results cannot be directly used in our localization framework because the MIMO radar MN measurements (rows of $\mathbf{A}$) are not i.i.d., as they conform to the structure of the MIMO random array steering vector. This problem has been addressed in [10], where a non-uniform guarantee is provided for a MIMO radar system with N transceivers. The authors obtain a non-uniform guarantee for a number of measurements proportional to $K\left(\log G\right)^2$ and when a so-called *aperture condition* holds. Interestingly, this condition is equivalent to the *isotropy* condition.

In the present work we provide a novel bound on the coherence of the matrix $\mathbf{A}$, and we determine under which conditions the *isotropy* property holds for MIMO random array radar. Leveraging these results, we develop both uniform and non-uniform recovery guarantees for target localization for MIMO radar systems. In detail, we show that uniform recovery requires the degrees of freedom MN to be proportional to $K^2\left(\log G + \log\left(\log G\right)\right)^2$, while, non-uniform recovery requires MN to scale with $K(\log G)^2$. The proposed random array framework is of practical interest to airborne and other radar applications, where the spacing between antenna elements may vary, or where exact surveying of sensor location is not practical due to natural flexing of the structures involved.

The following notation is used: boldface denotes matrices (uppercase) and vectors (lowercase); for a vector $\mathbf{a}$, $\mathbf{a}_i$ denotes its i-th element, while for a matrix $\mathbf{A}$, $\mathbf{A}(i,j)$ denotes the element at i-th row and j-th column, and $\mathbf{A}(i,:)$ denotes the i-th row; $(\cdot)^*$, $(\cdot)^T$, $(\cdot)^H$, and $(\cdot)^\dagger$ denote the complex conjugate, the transpose, the Hermitian-transpose and the pseudo-inverse operators, respectively. $\mathbb{E}$ denotes expectation and we define $\psi_x(u) \triangleq \mathbb{E}[\exp(jxu)]$ as the characteristic function of the random variable x. The symbol "$\otimes$" denotes the Kronecker product and $\mathbf{x} \sim \mathcal{CN}(\boldsymbol{\mu}, \mathbf{C})$ means that $\mathbf{x}$ has a circular symmetric complex normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\mathbf{C}$.

2. PROBLEM FORMULATION

We model a MIMO radar system (see Fig. 1) in which N sensors collect a coded pulse sent by M transmitters and returned from K stationary targets. We assume that transmitters and receivers each form (possibly overlapping) linear arrays of apertures Z_{TX} and Z_{RX}, respectively. Define $Z \triangleq Z_{TX} + Z_{RX}$ and let the m-th transmitter be at position $Z\xi_m/2$ on the x-axis, while the n-th receiver be at position $Z\zeta_n/2$ (with $\xi_m \in \left[-\frac{Z_{TX}}{Z}, \frac{Z_{TX}}{Z}\right]$, $\forall m$ and $\zeta_n \in \left[-\frac{Z_{RX}}{Z}, \frac{Z_{RX}}{Z}\right]$, $\forall n$). The purpose of the system is to determine the DOA angles to targets in a common range bin. The assumption of a common range bin implies that all waveforms are received with a common time delay after transmission. Targets are assumed in the far-field, meaning that a target's DOA parameter $\theta \triangleq \sin\vartheta$ (where ϑ is the DOA angle) is constant across the array.

Following [11], the DOA estimation problem can be cast in a sparse localization framework. Neglecting the discretization error, it is assumed that the targets' possible locations comply with a grid of G points $\phi_{1:G}$ (with $G \gg K$). We define the $MN \times G$ matrix $\mathbf{A} = [\mathbf{a}(\phi_1), \ldots, \mathbf{a}(\phi_n)]$, where

$$\mathbf{a}(\theta) \triangleq \mathbf{c}(\theta) \otimes \mathbf{b}(\theta) \tag{1}$$

where $\mathbf{b}(\theta) = \left[\exp\left(-j\pi\frac{Z\theta}{\lambda}\zeta_1\right), \ldots, \exp\left(-j\pi\frac{Z\theta}{\lambda}\zeta_N\right)\right]^T$ is the receiver steering vector, while the transmitter steering vector is $\mathbf{c}(\theta) = \left[\exp\left(-j\pi\frac{Z\theta}{\lambda}\xi_1\right), \ldots, \exp\left(-j\pi\frac{Z\theta}{\lambda}\xi_M\right)\right]^T$. We can then express the signal model as

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e} \tag{2}$$

where $\mathbf{e} \in \mathbb{C}^{MN}$ represents the noise, which is assumed to be independent and identically distributed (i.i.d.) complex Gaussian, i.e., $\mathbf{e} \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2\mathbf{I}\right)$. The unknown vector $\mathbf{x} \in \mathbb{C}^G$ contains the targets locations and gains. Zero elements of $\mathbf{x}$ correspond to grid points without a target. The problem (2) is sparse in the sense that $\mathbf{x}$ has only $K \ll G$ non-zero elements.

3. THEORETICAL RESULTS

In this section, we show how to choose the grid-points $\phi_{1:G}$, the number of elements M and N, and the distributions governing the elements' positions $p(\xi)$ and $p(\zeta)$ in order to obtain recovery guarantees for sparse localization with MIMO random arrays. The compressive sensing paradigm implies that a sparse signal $\mathbf{x}$ (encoding

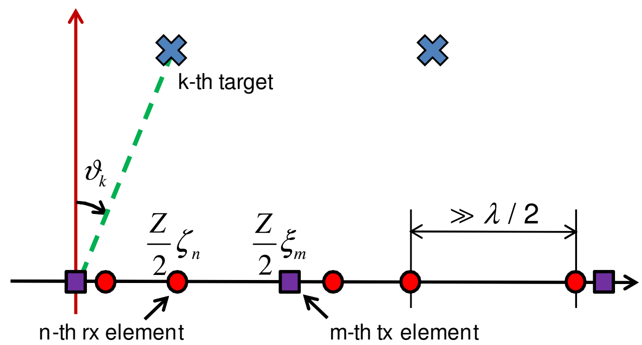

Fig. 1. System model

the targets' locations) can be recovered from a number of observations significantly lower than the Nyquist array (also known as "virtual ULA"). This is possible if random sampling is applied in the measurement process, and if $\mathbf{x}$ is recovered solving either the non-convex combinatorial ℓ_0-norm problem

$$\min_{\mathbf{x}} \|\mathbf{x}\|_0 \;\text{ s.t. }\; \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq \nu \tag{3}$$

or one of its approximations, for example, a greedy method or the constrained complex LASSO:

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 \;\text{ s.t. }\; \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq \upsilon. \tag{4}$$

This section establishes two kinds of recovery guarantees: uniform and non-uniform. A uniform recovery guarantee means that, for a fixed instantiation of the random measurement matrix $\mathbf{A}$, all possible K-sparse signals can be recovered with high probability. In contrast, given an arbitrary K-sparse vector $\mathbf{x}$, and draw $\mathbf{A}$ at random (independently of $\mathbf{x}$), non-uniform recovery details under what conditions an algorithm (e.g., (4)) will recover $\mathbf{x}$ with high probability. Clearly, uniform recovery implies non-uniform recovery, but the converse is not true. Indeed, whereas uniform guarantee focus on the recovery of the worst-case K-sparse signal for a fixed matrix $\mathbf{A}$, a non-uniform recovery result captures the typical recovery behavior for the random measurement matrix $\mathbf{A}$.

3.1. Uniform recovery

An important quantity for uniform recovery is the coherence of $\mathbf{A}$, defined as the maximum inner product between normalized columns of the matrix $\mathbf{A}$,

$$\mu \triangleq \max_{i \neq l} \frac{\left|\mathbf{a}_i^H \mathbf{a}_l\right|}{\|\mathbf{a}_i\|_2 \|\mathbf{a}_l\|_2}. \tag{5}$$

This parameter is closely related to the *array pattern* (or *beampattern*) [12]. In array processing, the array pattern is the system response of an array beamformed in direction ϕ_l to a unit amplitude target located in direction ϕ_i:

$$\beta(u_{i,l}) \triangleq \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} \exp\left[ju_{i,l}(\zeta_n + \xi_m)\right] \tag{6}$$

where we defined $u_{i,l} \triangleq \pi Z(\phi_l - \phi_i)/\lambda$. The peak of the absolute value of the array pattern for a target colinear with the beamforming direction, i.e., $|\beta(0)|$, is called the *mainlobe*. Peaks of $|\beta(u)|$ for $u \neq 0$, are known as *sidelobes,* and the largest among all the sidelobes is called the *peak sidelobe*. Since the array pattern is the inner product between two normalized columns of the measurement matrix, the coherence μ may be interpreted as the peak sidelobe associated with the dictionary $\mathbf{A}$. While the pattern $\beta(u_{i,l})$ captures both transmit and receive properties of MIMO radar, we also define array patterns addressing the transmit and receive function separately: the transmit array pattern $\beta_\xi(u_{i,l}) \triangleq \frac{1}{M}\sum_{m=1}^{M}\exp(ju_{i,l}\xi_m) = \frac{1}{M}\mathbf{c}_i^H\mathbf{c}_j$, and the receiver array pattern $\beta_\zeta(u_{i,l}) \triangleq \frac{1}{N}\sum_{n=1}^{N}\exp(ju_{i,l}\zeta_n) = \frac{1}{N}\mathbf{b}_i^H\mathbf{b}_j$. The random variables $\beta(u_{i,l})$, $\beta_\xi(u_{i,l})$ and $\beta_\zeta(u_{i,l})$ are related via the Kronecker structure of the each column of $\mathbf{A}$, as per (1). Consequently, the inner product between two columns of $\mathbf{A}$ factors as:

$$\mathbf{a}_i^H\mathbf{a}_j = \mathbf{c}_i^H\mathbf{c}_j\mathbf{b}_i^H\mathbf{b}_j \tag{7}$$

The following theorem exploits (7) to obtain a bound on the coherence of the matrix $\mathbf{A}$:

Theorem 1. *Let the locations ξ of the transmit elements be drawn i.i.d. from a distribution $p(\xi)$, and the locations ζ of the receive elements be drawn i.i.d. from a distribution $p(\zeta)$. Assume that $p(\xi)$, $p(\zeta)$ and the uniform grid $\phi_{1:G}$ are such that the transmitter and receiver array patterns satisfy*

$$\mathbb{E}\big[\beta_\xi(u_{1,i})\big]=\mathbb{E}\big[\beta_\xi(2u_{1,i})\big]=\mathbb{E}\big[\beta_\zeta(u_{1,i})\big]=\mathbb{E}\big[\beta_\zeta(2u_{1,i})\big]=0 \tag{8}$$

for $i = 2$ to G, where $u_{1,i} = \pi Z(\phi_i - \phi_1)/\lambda$. Then the coherence of $\mathbf{A}$ satisfies

$$\Pr(\mu > q) < 1 - \left[1 - 2\sqrt{MN}qK_1\left(2\sqrt{MN}q\right)\right]^{G-1} \tag{9}$$

where $K_1(\cdot)$ is the modified Bessel functions of the second kind.

Proof. An outline of the proof is given in the Appendix. □

Since the coherence μ may be interpreted as the peak sidelobe of the array pattern, (9) characterizes the probability of the peak sidelobe exceeding q. This result is not asymptotic (i.e., it does not need the number of measurements MN to go to infinity), on the contrary, it holds for any values of M, N and G. The bound tightness increase with MN, but it is already tight for $M = N = 15$ elements.

The coherence μ plays a key role in obtaining uniform recovery guarantees of compressive sensing algorithms. For instance, using the coherence μ, it is possible to obtain a bound on the RIP constant, $\delta_K \leq (K-1)\mu$ [13], which ensures stable and robust recovery from noisy measurements using (4). In particular, by building on Theorem 1, it can be shown that if the number of MIMO radar measurements satisfies

$$MN \geq CK^2(\log G + \log(\log G))^2 \tag{10}$$

where C is a constant, uniform recovery of all K-sparse signals is obtained with high probability via (4). The proof of this relation cannot be included here due to space considerations, but will be addressed in a future publication.

The significance of (10) is to indicate the number of elements necessary to control the peak sidelobe. To the authors knowledge Theorem 1 is the first non-asymptotic result to characterize the coherence for the peculiar structure of the measurement matrix $\mathbf{A}$ in MIMO spatial compressive sensing (see (1)).

3.2. Non-uniform recovery

In this section, we investigate non-uniform recovery guarantees. In recent work [8], it has been shown that, for a sufficient number of i.i.d. compressive sensing measurements, performance can be guaranteed if a specific property of the random measurement matrix $\mathbf{A}$, called *isotropy*, holds. The isotropy property states that the components of each row of $\mathbf{A}$ have unit variance and are uncorrelated, i.e., $\mathbb{E}\left[\mathbf{A}^T(t,:)\mathbf{A}^*(t,:)\right] = \mathbf{I}$ for every t. However, this result cannot be directly used in our framework since the MN rows of the matrix $\mathbf{A}$, following (1), are not i.i.d. The structured scenario when the measurements (rows of the matrix $\mathbf{A}$) are not i.i.d. is addressed in [10], where non-uniform recovery is guaranteed for a MIMO radar system with N transceivers if the isotropy property (under the name *aperture condition*) holds. The N transceivers MIMO radar setup is a special case of our M transmitters and N receivers MIMO radar framework, obtained by drawing $\zeta_{1:N}$ at random and deterministically setting $\xi_n = \zeta_n$ for all n. The following theorem derives conditions on grid points $\phi_{1:G}$ and probability distributions $p(\xi)$ and $p(\zeta)$, for the random matrix $\mathbf{A}$ to satisfy the isotropy property:

Theorem 2. *Let the locations of the transmit elements ξ be drawn i.i.d. from a distribution $p(\xi)$, and the locations of the receivers ζ be drawn i.i.d. from a distribution $p(\zeta)$. For every t, the t-th row of $\mathbf{A}$ in (2) satisfies the isotropy property [8], i.e., $\mathbb{E}\left[\mathbf{A}^T(t,:)\mathbf{A}^*(t,:)\right] = \mathbf{I}$, iff $p(\xi)$, $p(\zeta)$ and $\phi_{1:G}$ are such that*

$$\mathbb{E}\left[\exp(jzu_{i,l})\right] = 0 \quad \forall i \neq l \tag{11}$$

where $z \triangleq \zeta + \xi$ and $u_{i,l} \triangleq \pi Z(\phi_l - \phi_i)/\lambda$.

Proof. An outline of the proof is given in the Appendix. □

Theorem 2 links the probability distributions $p(\xi)$ and $p(\zeta)$ (through the characteristic function of z) and the grid-points $\phi_{1:G}$ with the isotropy property of the matrix $\mathbf{A}$. When (11) holds, it can be shown that the aperture condition used in [10] holds too. Therefore, using the same approach as in [10], non-uniform recovery of K targets via (4) can be guaranteed in the proposed spatial compressive sensing framework from

$$MN \geq CK(\log G)^2 \tag{12}$$

MIMO radar noisy measurements, where C is a constant.

3.3. Setup for Theorem 1 and 2

We now provide an example of choices of $p(\xi)$, $p(\zeta)$ and $\phi_{1:G}$ that meet the requirements of Theorem 1 and Theorem 2. Let, $Z_{TX} = Z_{RX} = Z/2$, such that the independent random variables ξ and ζ are both confined to the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. For 1, conditions (8) impose constraints on the characteristic functions of ξ and ζ evaluated at $u_{1,i}$ and $2u_{1,i}$. For Theorem 2, condition (11) imposes a constraint on the characteristic function of the random variable $z \triangleq \xi + \zeta$.

The characteristic function of a uniform random variable $\zeta \sim \mathcal{U}\left[-\frac{1}{2}, \frac{1}{2}\right]$ is the sinc function, i.e., $\psi_\zeta(u) = \sin(u/2)/(u/2)$. Therefore, by choosing $\phi_{1:G}$ as a uniform grid of $2\lambda/Z$-spaced points in the range $[-1, 1]$, since $u_{i,l} \triangleq \pi Z(\phi_l - \phi_i)/\lambda = 2\pi|i-l|$, we have that $\psi_\zeta(u_{i,l}) = \psi_\zeta(2u_{i,l}) = 0$ for any $i \neq l$.

It then follows that, by choosing $\phi_{1:G}$ as a uniform grid of $2\lambda/Z$-spaced points in the range $[-1, 1]$: (1) when *both* ζ and ξ are uniformly distributed, relations (8) hold, and we can invoke Theorem 1 (uniform recovery); (2) when either ζ *or* ξ are uniformly distributed, (11) holds (since $\psi_z(u) = \psi_\zeta(u)\psi_\xi(u)$), and we can invoke Theorem 2 (non-uniform recovery).

The number of grid points G is not a free variable, because the grid points $\phi_{1:G}$ must satisfy (8) or (11). For instance, in the example above, $\phi_{1:G}$ must be a uniform grid of $2\lambda/Z$-spaced points between

$[-1, 1]$, and the number of grid points is $G = Z/\lambda + 1$. In other words, the resolution G depends on the "virtual" array aperture Z.

Finally, non-uniform recovery, i.e., (11), requires only one density function, say $p(\zeta)$, to be uniform, while the other distribution, $p(\xi)$, can be arbitrarily chosen, e.g., it can be even deterministically dependent on ζ. For instance, (11) is satisfied in a MIMO radar system with N transceivers, i.e., when $\zeta_{1:N}$ are i.i.d. uniform distributed and we deterministically set $\xi_n = \zeta_n$ for all n.

4. NUMERICAL RESULTS

We present numerical results to illustrate the proposed framework. We follow the setup detailed in the previous section, i.e., $p(\xi)$, $p(\zeta)$ are both uniform distributions, $Z_{TX} = Z_{RX} = Z/2$ and $\phi_{1:G}$ represents a uniform grid of $2\lambda/Z$-spaced points in the interval $[-1, 1]$. This implies that the number of grid points is $G = Z/\lambda+1$. The system transmits M orthogonal spread spectrum waveforms of length M chips each. The waveforms were chosen in discrete form as the rows of the $M \times M$ Fourier matrix. We set the target gains $x_k = \exp(-j\varphi_k)$, with φ_k drawn i.i.d., uniform over $[0, 2\pi)$, for all targets $k = 1, \ldots, K$. We set the noise $\mathbf{e} \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2\mathbf{I}\right)$, and the SNR was defined as $-10\log_{10}\sigma^2$. Monte Carlo simulations were carried out using independent realizations of target gains, targets locations, noise and element positions.

Target localization was implemented using the Complex Approximate Message Passing (CAMP) proposed in [14] to solve (4). In addition, we also simulated the Multi-Branch Match Pursuit (MBMP) algorithm, a greedy method proposed in [11]. For the CAMP algorithm, we set the parameter $\tau = 2$, as suggested in [14], and, if not converged, the algorithm was terminated after 1000 iterations. The support was then estimated as the K largest modulo entries of the product of CAMP (see [14] for more details). Concerning MBMP, it requires as input a K length branch vector, set to $\mathbf{d} = [1, \ldots, 1]$ (see [11] for details on setting parameters for MBMP). The output of MBMP is the estimated support.

Fig. 2 illustrates the probability of support recovery error (defined as the event that at least one index of the support is estimated incorrectly) as a function of the number of measurements MN. The virtual aperture was $Z = 250\lambda$ (thus $G = 251$), the SNR was 15 dB, and tests were carried out for $K = 5$, 10 and 20 targets. In addition to the superior performance of MBMP, which may be partially explained by a slightly higher complexity, but merits further investigation, it can be seen that, as dictated by (12), the number of sensors MN needed is approximately linear in K.

5. CONCLUSIONS

We propose a spatial compressive sensing framework to address the DOA localization problem for a random array MIMO radar system. We link system design quantities, such as the probability distributions $p(\xi)$ and $p(\zeta)$ of the sensors locations and the sparse localization grid points $\phi_{1:G}$, to the coherence μ and the isotropy property of the measurement matrix $\mathbf{A}$. Based on these two results, we establish uniform and non-uniform recovery guarantees. In particular, the proposed framework supports non-uniform recovery of K targets with $MN = K(\log G)^2$ MIMO radar elements, where G is proportional to the resolution. The significance of the logarithmic dependence in G is that the proposed framework enables high resolution with a small number of MIMO radar elements. This is in contrast with a filled virtual MIMO array where MN scales linearly with G.

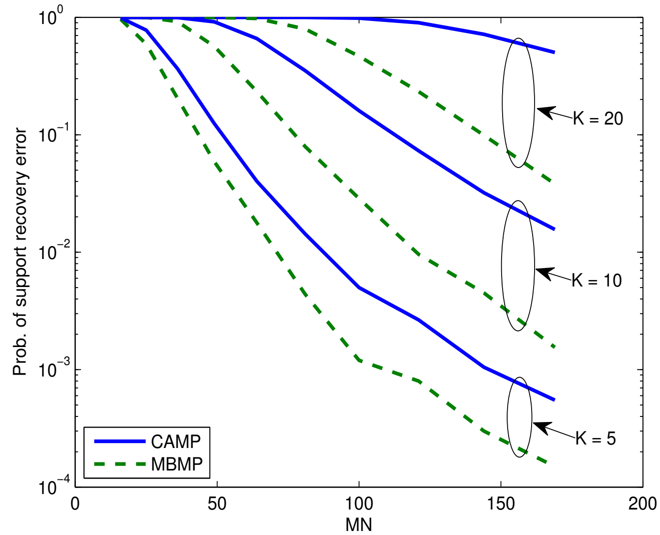

Fig. 2. Probability of support recovery error as a function of the number of rows MN of the measurement matrix $\mathbf{A}$.

6. APPENDIX

6.1. Theorem 1: Outline of proof

Here we outline the steps to obtain (9). For a uniform grid $\phi_{1:G}$, the Hermitian matrix $\mathbf{A}^H\mathbf{A}$ has a Toeplitz structure. As such, the coherence of $\mathbf{A}$ is the maximum among the elements of the first row of $\mathbf{A}^H\mathbf{A}$, i.e., $\mu = \max_{i>1} \frac{1}{MN}\left|\mathbf{a}_1^H\mathbf{a}_i\right|$. Consider the term $\frac{1}{MN}\left|\mathbf{a}_1^H\mathbf{a}_i\right|$. Using (7), $\frac{1}{MN}\left|\mathbf{a}_1^H\mathbf{a}_i\right| = \left(\frac{1}{N}\left|\mathbf{b}_1^H\mathbf{b}_i\right|\right)\left(\frac{1}{M}\left|\mathbf{c}_1^H\mathbf{c}_i\right|\right)$. In [5], it was shown that if the locations $\zeta_{1:N}$ are drawn i.i.d. from an even distribution $p(\zeta)$, and if the uniform grid spacing $\phi_{1:G}$ satisfies $\mathbb{E}\left[\beta_\zeta(u_{1,i})\right] = \mathbb{E}\left[\beta_\zeta(2u_{1,i})\right] = 0$, then the random variable $\beta_\zeta(u_{1,i}) = \frac{1}{N}\mathbf{b}_1^H\mathbf{b}_i$ follows a $\mathcal{CN}(0, 1/N)$ distribution. Under similar conditions, $\beta_\xi(u_{1,i}) = \frac{1}{M}\mathbf{c}_1^H\mathbf{c}_i \sim \mathcal{CN}(0, 1/M)$. Also $\beta_\zeta(u_{1,i})$ and $\beta_\xi(u_{1,i})$ are independent. Thus $\frac{1}{MN}\left|\mathbf{a}_1^H\mathbf{a}_i\right|$ is distributed as the product of two independent Rayleigh random variables with parameters $\sigma = 1/\sqrt{2N}$ and $\sigma = 1/\sqrt{2M}$, respectively. The closed form cumulative distribution function (cdf) of such random variable is given in [15], $\Pr\left(\frac{1}{MN}\left|\mathbf{a}_1^H\mathbf{a}_i\right| \leq q\right) = 1 - 2\sqrt{MN}q\, K_1\left(2\sqrt{MN}q\right)$, where $K_1(\cdot)$ is the modified Bessel function of the second kind. From $\mu = \max_{i>1} \frac{1}{MN}\left|\mathbf{a}_1^H\mathbf{a}_i\right|$, and making the conservative assumption of independence between the $G-1$ variables $\frac{1}{MN}\left|\mathbf{a}_1^H\mathbf{a}_i\right|$, for $i = 2$ to G, the complementary cdf of the maximum among these $G-1$ variables is upper bounded by $\Pr(\mu > q) < 1 - \left[\Pr\left(\frac{1}{MN}\left|\mathbf{a}_1^H\mathbf{a}_i\right| \leq q\right)\right]^{G-1}$, obtaining (9).

6.2. Theorem 2: Outline of proof

Here, we outline the proof of Theorem 2. Using (1), for the element index t in the vectors $\mathbf{a}(\phi_i)$ and $\mathbf{a}(\phi_l)$, we have $\mathbf{a}_t^*(\phi_i)\,\mathbf{a}_t(\phi_l) = \exp\left[ju_{i,l}(\zeta_n + \xi_m)\right]$, where $u_{i,l} \triangleq \pi Z(\phi_i - \phi_l)/\lambda$ and $t = N(m-1)+n$. Also, the average of $\exp\left[ju_{i,l}(\zeta_n + \xi_m)\right]$ does not depend on the index n and m, since $\zeta_{1:N}$ are identically distributed, and so are $\xi_{1:M}$. By dropping the indexes of ζ_n and ξ_m and using $z = \xi + \zeta$ we have $\mathbb{E}\left[\mathbf{a}_t^*(\phi_i)\,\mathbf{a}_t(\phi_l)\right] = \mathbb{E}\left[\exp(ju_{i,l}z)\right]\ \forall t$. Combining this with (11), and noticing that $\exp(jzu_{i,i}) = 1$ for every i, we obtain the "if" direction of the claim. The "only if" direction follows since when (11) is not satisfied, there exist one pair of indexes i and l such that $\mathbb{E}\left[\mathbf{a}_t^*(\phi_i)\,\mathbf{a}_t(\phi_l)\right] = \mathbb{E}\left[\exp(jzu_{i,l})\right] \neq 0$.

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