

# SPARSE MIMO RADAR WITH PHASE MISMATCH

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## ABSTRACT

In this paper we propose a sparse model to accurately estimate target locations in a distributed multiple-input multiple-output (MIMO) radar system with phase mismatches at transmitters and receivers. We formulate the localization problem based on maximum a posteriori (MAP) estimation. To reduce the effect of phase mismatches we develop a novel alternating minimization approach based on sparse signal recovery and structured matrix perturbation. Using numerical simulations, we show that our algorithms significantly improve the performance of the distributed MIMO radar system.

**Index Terms**— distributed MIMO radar, phase mismatch, alternating minimization

## 1. INTRODUCTION

Multiple-input multiple output (MIMO) radar systems have attracted much attention recently [1]-[3]. They have two configurations, namely colocated MIMO radar [4], [5], and distributed MIMO radar [6]-[8]. Colocated radars use multiple closely located transmitters and receivers. They employ waveform diversity to explore the region of interest. Unlike colocated MIMO radar, the distributed version has several scattered transmitters and receivers. It explores the spatial diversity by looking at the region of interest from different angles or distances.

Exploring sparsity is a fast growing area in the field of signal reconstruction [9]-[11]. Recent research has used compressive sensing in both colocated and distributed MIMO to explore the sparsity in the region of interest [12]-[16]. In distributed MIMO radar, phase mismatch always exists during the signal processing since perfect synchronization is impossible in a distributed configuration. The phase mismatch has been well studied from the statistical perspective, and the corresponding Cramér-Rao bound has been derived [17],[18]. Phase mismatch can be modeled as basis mismatch in the compressive sensing problem. The sensitivity of this problem has been researched in [19]. In [20], an alternating minimization method based on total least squares has been proposed to

solve the sparse signal reconstruction with matrix perturbation.

In this paper, we set up a more realistic distributed MIMO radar system. Unlike [14], our signal model does not include a matching filter in each receiver. In addition, we consider the phase mismatches at the transmitters and receivers due to the imperfect synchronization. The phase mismatches at the transmitters and receivers are assumed to be virtually static during the entire coherent processing interval (CPI) [17]. They are independently and identically distributed random variables with uniform distribution. The method underlying this paper is inspired by the concept of sparsity-cognizant total least squares [20]. But unlike sparsity-cognizant total least squares, which considers the perturbed error only as a Gaussian distribution, our method can be applied to other distributions. Compared to the traditional sparsity signal reconstruction method we show improvements in the reconstructed signal in terms of correctly detecting the number of targets and increasing the probability of detection while maintaining a desired level of false alarm.

The rest of this paper is organized as follows. In Section 2, we present the signal model for the distributed MIMO radar with phase mismatches. In Section 3, we formulate the optimization problem from the maximum a posteriori point of view, and propose the Alternating LASSO (ALASSO) algorithm. In Section 4, we give numerical results to show the performance of our algorithm. In Section 5, we conclude the paper.

## 2. PHASE MISMATCH MODEL

We consider a distributed MIMO radar with  $M_T$  transmitters,  $M_R$  receivers, and  $K$  targets. The transmitters and receivers are widely separated. We denote the location of the  $i$ -th transmitter by  $[T_x^i, T_y^i]$  and the location of the  $j$ -th receiver by  $[R_x^j, R_y^j]$ . The location of the  $k$ -th target is indicated by  $[P_x^k, P_y^k]$ . Suppose  $x_i(t)$  indicates the waveform generated by the  $i$ -th transmitter at time  $t$ , then it has the form

$$x_i(t) = \sum_{n=0}^L \beta_{in} c_{in} u(t - nT), \quad i = 1, \dots, M_T, \quad (1)$$

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where  $u(t)$  is a pulse function which can be written as

$$u(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq T, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

The parameter  $c_{in}$  follows an i.i.d. Gaussian distribution and  $\beta_{in}$  is a random on-off pattern which determines whether the waveform is transmitted at  $n$ -th time or not. Some of them can be set to zeros if we want to save transmitting energy in the application. We assume all  $\beta_{in}$  equal to 1 for now.  $T$  is the interval length of each pulse and is also the sampling interval for the receivers. With the narrow band assumption (since  $x_i$  is piecewise stationary) we can ignore the delay terms and time mismatch terms in  $x_i(t)$  [5] [16] and only consider the mismatch terms in the exponential terms.

In a compressive sensing model we normally discretize the space into several grids and use the vector  $\mathbf{s} = [s_1, \dots, s_P]^T$  to present the reflection factors from each of the grids. In our model  $\mathbf{s}$  is defined as

$$s_p = \begin{cases} \alpha_k & \text{k-th target is at this point,} \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Now we rewrite the signal model in a sampled format and we neglect the sample interval  $T$  in the equations for simplicity. The mixed signal at the  $p$ -th grid equals

$$y_p(n) = \sum_{i=1}^{M_T} x_i(n) e^{-j \frac{2\pi f_c}{c} d_{ip}^t + j \theta_i^t}, \quad (4)$$

where  $f_c$  indicates the transmitting frequency of the radar system,  $c$  indicates the speed of radar signal,  $d_{ip}^t$  indicates the distance between  $i$ -th transmitter and  $p$ -th grid,  $\theta_i^t$  is the phase mismatch of  $i$ -th transmitter, and  $n$  is the time index for the  $n$ -th sample. After rewriting equation (4) in its vector form, we get

$$y_p(n) = \mathbf{x}^T(n) \mathbf{u}_p, \quad (5)$$

where

$$\mathbf{x} = [x_1(n), \dots, x_{M_T}(n)]^T, \quad (6)$$

$$\mathbf{u}_p = [e^{j\theta_1^t} e^{-j \frac{2\pi f_c}{c} d_{1p}^t}, \dots, e^{j\theta_{M_T}^t} e^{-j \frac{2\pi f_c}{c} d_{M_T p}^t}]^T. \quad (7)$$

Then the signal received by the  $j$ -th receiver can be written as

$$z_j(n) = \sum_{p=1}^P s_p e^{-j \frac{2\pi f_c}{c} d_{jp}^r + j \theta_j^r} y_p(n), \quad j = 1, \dots, M_R, \quad (8)$$

in which  $\theta_j^r$  presents the phase mismatch between the  $j$ -th receiver and the information of the fusion center, where all the received signals are collected.  $d_{jp}^r$  indicates the distance between  $j$ -th receiver and  $p$ -th grid.

Suppose we take  $L$  snapshots, and then stack all the measurements from the  $j$ -th receiver in one vector. We will have

$$\mathbf{z}_j = \begin{pmatrix} z_j(0) \\ \vdots \\ z_j(L-1) \end{pmatrix} = \sum_{p=1}^P s_p e^{-j \frac{2\pi f_c}{c} d_{jp}^r + j \theta_j^r} \mathbf{X} \mathbf{u}_p + \mathbf{e}_j = \Psi_j \mathbf{s} + \mathbf{e}_j, \quad (9)$$

where  $\mathbf{X} = [\mathbf{x}(0), \dots, \mathbf{x}(L-1)]^T$ .  $\mathbf{e}_j$  denotes the noise received by the  $j$ -th receiver during sampling. In our work we assume the noise is i.i.d. Gaussian.  $\mathbf{s}$  indicates the locational signal with  $\mathbf{s} = [s_1, \dots, s_P]^T$ , and  $\Psi_j$  represents the basis for the  $j$ -th receiver:

$$\Psi_j = [e^{-j \frac{2\pi f_c}{c} d_{j1}^r + j \theta_j^r} \mathbf{X} \mathbf{u}_1, \dots, e^{-j \frac{2\pi f_c}{c} d_{jP}^r + j \theta_j^r} \mathbf{X} \mathbf{u}_P]. \quad (10)$$

In order to make the model more concise, we stack all the information received by the fusion center into one vector as:

$$\mathbf{z} = \begin{pmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_{M_R} \end{pmatrix} = \mathbf{H}(\boldsymbol{\theta}_t, \boldsymbol{\theta}_r) \mathbf{s} + \mathbf{e} \quad (11)$$

where

$$\mathbf{H}(\boldsymbol{\theta}_t, \boldsymbol{\theta}_r) = \begin{pmatrix} \Psi_1 \\ \vdots \\ \Psi_{M_R} \end{pmatrix} \quad (12)$$

which is a matrix function determined by the unknown phase mismatches  $\boldsymbol{\theta}_t = [\theta_1^t, \dots, \theta_{M_T}^t]^T$  and  $\boldsymbol{\theta}_r = [\theta_1^r, \dots, \theta_{M_R}^r]^T$ .

### 3. ALTERNATING MINIMIZATION METHOD

According to Bayesian compressive sensing [21], we can assume that the spatial signal  $\mathbf{s}$  follows the widely used prior Laplacian distribution. In the meantime we assume the phase mismatches from the transmitters and receivers follow uniform distribution  $\mathcal{U}(\theta_1, \theta_2)$  and  $\mathcal{U}(\theta_3, \theta_4)$  respectively [17]. Therefore the maximization of the posterior log-likelihood function can be formulated into the optimization as follows:

$$\min_{\boldsymbol{\theta}_t, \boldsymbol{\theta}_r, \mathbf{s}} \|\mathbf{z} - \mathbf{H}(\boldsymbol{\theta}_t, \boldsymbol{\theta}_r) \mathbf{s}\|_2^2 + \rho \|\mathbf{s}\|_1 \quad (13)$$

$$\text{subject to } \boldsymbol{\theta}_1 \leq \boldsymbol{\theta}_t \leq \boldsymbol{\theta}_2, \boldsymbol{\theta}_3 \leq \boldsymbol{\theta}_r \leq \boldsymbol{\theta}_4. \quad (14)$$

Then we use the equation  $e^{j\theta} = \cos\theta + j\sin\theta = 1 + j\theta$  when  $\theta$  is small enough, i.e.,  $\theta \leq 30^\circ$ . With this, we can rewrite our objective function and use coordinate descent to find a solution.

After approximation of phase mismatch terms, we have our system matrix  $\mathbf{H}$  as

$$\mathbf{H} = [\mathbf{a}_1 \otimes \mathbf{X} \tilde{\mathbf{u}}_1, \mathbf{a}_2 \otimes \mathbf{X} \tilde{\mathbf{u}}_2, \dots, \mathbf{a}_P \otimes \mathbf{X} \tilde{\mathbf{u}}_P], \quad (15)$$

in which

$$\mathbf{a}_p = \begin{pmatrix} (1 + j\theta_1^r)e^{-j\frac{2\pi f_c}{c}d_{1p}^r} \\ \vdots \\ (1 + j\theta_{M_R}^r)e^{-j\frac{2\pi f_c}{c}d_{M_R p}^r} \end{pmatrix}, \quad (16)$$

$$\tilde{\mathbf{u}}_p = \begin{pmatrix} (1 + j\theta_1^t)e^{-j\frac{2\pi f_c}{c}d_{1p}^t} \\ \vdots \\ (1 + j\theta_{M_T}^t)e^{-j\frac{2\pi f_c}{c}d_{M_T p}^t} \end{pmatrix}. \quad (17)$$

The symbol  $\otimes$  is used to indicate the Kronecker product of two matrices. With this  $\mathbf{H}$  we can apply alternating minimization among the three variables  $\theta_t$ ,  $\theta_r$ , and  $\mathbf{s}$ . When we fix  $\theta_t$  and  $\theta_r$ , the minimization is just the standard LASSO problem as follows:

$$\min_{\mathbf{s}} \|\mathbf{z} - \mathbf{H}\mathbf{s}\|_2^2 + \rho\|\mathbf{s}\|_1. \quad (18)$$

Normally we can choose  $\rho = \sigma\sqrt{2\log(NLM_R)}$  when we solve the LASSO problem according to [22]. In [20], instead of finding the optimal solution by directly solving equation (18), in every iteration  $\mathbf{s}$  is solved only by solving several one dimension LASSO problems sequentially. When we fix  $\theta_r$  and  $\mathbf{s}$  or  $\theta_t$  and  $\mathbf{s}$ , we will have a linear structured matrix perturbation problem, which can be solved when we write  $\mathbf{H}$  in the form of the original matrix and basis matrices for the perturbation.

First we write the measurement model as a linear combination of elements of  $\theta_t$  when we fix  $\theta_r$  and  $\mathbf{s}$ . We have

$$\mathbf{z} = \mathbf{H}_0^t \mathbf{s} + j\theta_1^t \mathbf{H}_1^t \mathbf{s} + \cdots + j\theta_{M_T}^t \mathbf{H}_{M_T}^t \mathbf{s} + \mathbf{e}, \quad (19)$$

where  $\mathbf{H}_0^t$  and  $\mathbf{H}_i^t$  ( $i = 1, \dots, M_T$ ) can be obtained by rewriting equation (15). Therefore we have the optimization problem as follows:

$$\min_{\theta_t} \|\mathbf{z} - \mathbf{H}_0^t \mathbf{s} - \sum_{i=1}^{M_T} j\theta_i^t \mathbf{H}_i^t \mathbf{s}\|_2^2 \quad (20)$$

$$\text{subject to} \quad \theta_1 \leq \theta_t \leq \theta_2. \quad (21)$$

This is a constrained least squares problem which is convex and has a feasible solution, so the optimal solution can be found easily through the application of KKT conditions. Now if we fix  $\theta_t$  and  $\mathbf{s}$ , we will have a similar optimization problem as the above one. The measurement model can be rewritten as

$$\mathbf{z} = \mathbf{H}_0^r \mathbf{s} + j\theta_1^r \mathbf{H}_1^r \mathbf{s} + \cdots + j\theta_{M_R}^r \mathbf{H}_{M_R}^r \mathbf{s} + \mathbf{e}. \quad (22)$$

Likewise, the optimization problem can be characterized as

$$\min_{\theta_r} \|\mathbf{z} - \mathbf{H}_0^r \mathbf{s} - \sum_{i=1}^{M_R} \theta_i^r \mathbf{H}_i^r \mathbf{s}\|_2^2 \quad (23)$$

$$\text{subject to} \quad \theta_3 \leq \theta_r \leq \theta_4 \quad (24)$$

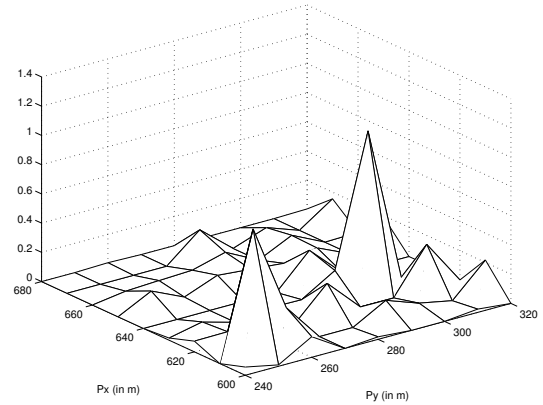
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#### Algorithm 1 (Alternating LASSO)

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**Initialize:**  $\theta_r = \mathbf{0}$ ,  $\theta_t = \mathbf{0}$ ,  $\mathbf{s}_0 = \mathbf{0}$ ,  $m = 0$  (iteration indicator)  
 $n$  = the dimension of reconstructed signal  $\mathbf{s}$   
**while** not converge **do**    **set**  $k = 1$ ,  
     **while**  $k \leq n$   
          $\mathbf{r} = \mathbf{z} - \sum_{j=1}^{k-1} \mathbf{h}_j s_j(m) - \sum_{j=k+1}^n \mathbf{h}_j s_j(m-1)$   
     **update**  
          $s_k(m) = \arg \min_{s_k(t)} \|\mathbf{r} - \mathbf{h}_k s_k\|_2^2 + \lambda|s_k|$   
          $k=k+1$   
     **end while**  
     **rewrite** the system matrix using eqn. (19)  
     **solve** the optimization problem (20), and get  $\theta_t$   
     **update**  $\mathbf{H}$ , Write the system matrix using eqn. (22)  
     **solve** the optimization problem (23), and get  $\theta_r$   
     **update**  $\mathbf{H}$  and  $m = m + 1$ .  
**end while**

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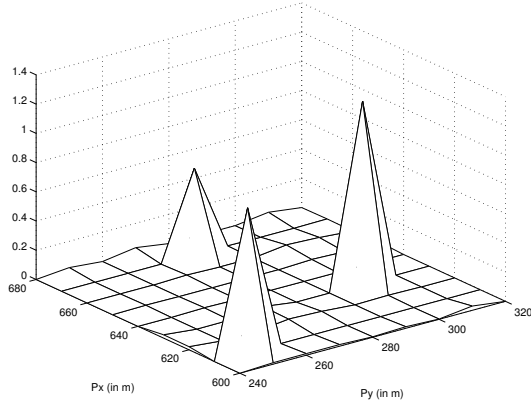


**Fig. 1:** The recovered signal using tradition LASSO.

According to [23], given arbitrary initialization, the above alternating minimization will converge monotonically to at least a stationary point of the original optimization problem. A sketch of the algorithm is given in the table named Algorithm 1, in which  $\mathbf{h}_j$  is the  $j$ -th column of the system matrix  $\mathbf{H}$ , and  $s_j(t)$  denotes the  $j$ -th element of  $\mathbf{s}(t)$ .

## 4. NUMERICAL RESULTS

In this section we recover the signal using two methods, namely ALASSO and tradition LASSO. The probability of detection for all the targets, as well as the Receiver Operating Characteristic (ROC) curve are plotted. We also show the case when some proportions of the  $\beta_{in}$  are zeros in equation 1. We demonstrate that in this case transmission energy can be saved without loss of detection performance.



**Fig. 2:** The recovered signal using Alternating LASSO algorithm.

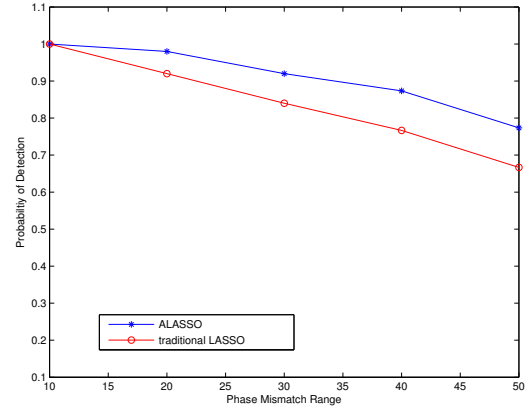
To begin with, we consider a MIMO system with four distributed transmitters and six distributed receivers. The transmitters are located at  $\vec{t}_1 = [100, 0]$  m,  $\vec{t}_2 = [150, 200]$  m,  $\vec{t}_3 = [210, 100]$  m and  $\vec{t}_4 = [800, 900]$  m. The receivers are located at  $\vec{r}_1 = [0, 100]$  m,  $\vec{r}_2 = [0, 200]$  m,  $\vec{r}_3 = [0, 500]$  m,  $\vec{r}_4 = [350, 0]$  m,  $\vec{r}_5 = [600, 1000]$  m and  $\vec{r}_6 = [800, 0]$  m. The carrier frequency is 1 GHz. The targets are located at  $\vec{p}_1 = [630, 300]$  m,  $\vec{p}_2 = [610, 250]$  m, and  $\vec{p}_3 = [670, 280]$  m with reflection factors as  $\alpha_1 = 1(1 + j)$ ,  $\alpha_2 = 0.8(1 + j)$  and  $\alpha_3 = 0.5(1 + j)$ .

The phase mismatches in the transmitters are generated with uniform distributions ranging from  $-30^\circ$  to  $30^\circ$  while the phase mismatches in the receivers are generated with the uniform distributions ranging from  $-20^\circ$  to  $20^\circ$ . The number of snapshots is set as 10. All the  $\beta_{in}$  are set to ones at this time. The SNR are fixed to be 15dB in our simulations.

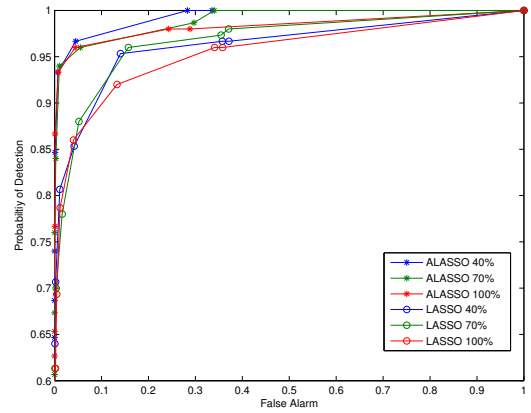
From Fig.1 we can see that the reconstructed signal is very noisy and that setting the threshold to distinguish between target and noise could be very difficult. From Fig. 2, which considers the distribution of phase mismatch as a prior information, we can see the ALASSO method recovers the signal much better than the first method, and the targets are clearly separated from the noise.

We also use Monte Carlo simulations to plot the probability of detection. The number of Monte Carlo runs is 50 for each test point. We change the range of receivers' phase mismatch from  $[-10^\circ, 10^\circ]$  to  $[-50^\circ, 50^\circ]$  while setting the phase mismatches in the transmitters as zero. From this comparison it is easy to see that with phase mismatch in the model the original method without considering the mismatch will deteriorate very quickly, while the one using the information about distribution of phase mismatches maintains a good performance.

A ROC curve is also plotted for 50 Monte Carlo runs. We also consider the case when some of  $\beta_{in}$  are zeros. We can see



**Fig. 3:** Combined probability of detection by the two methods.



**Fig. 4:** ROC curves for the LASSO and Alternating LASSO algorithms.

from Fig. 4 that by considering the prior information about phase mismatches ALASSO can increase the system performance dramatically. It also shows that when we only use 40% or 70% of the transmission waveform, the performance is the same as the case when we use all the transmission waveforms.

## 5. CONCLUSION

In this paper, we considered the case of distributed MIMO radar with imperfect synchronization. We first built the sparsity model with phase mismatches. Then by exploring the prior information about the phase mismatch, we formulated the alternating LASSO optimization problem and used the alternating minimization algorithm to get the recovered signal. In a numerical example, we demonstrated that by exploiting the information about phase mismatch we could highly improve the system's performance in locating the targets.

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