A NEW SUBBAND INFORMATION FUSION METHOD FOR WIDEBAND DOA ESTIMATION USING SPARSE SIGNAL REPRESENTATION

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ABSTRACT

We present a new subband information fusion (SIF) method for wideband direction-of-arrival (DOA) estimation using single sparse signal representation of multiple frequency-based measurement vectors. The problem of wideband DOA estimation using SIF method is to jointly utilize all the frequency bin information to recover a single sparse indicative vector (SIV). The SIF method belongs to the sparse signal representation domain and therefore it will suffer from two cases of ambiguity: algebraic aliasing and spatial aliasing. We show that these two categories of ambiguity can be reduced by combining all the frequency components. The SIF algorithm is then proposed and the SIV is recovered iteratively. The numerical simulations are performed to illustrate that the SIF method has superior performances.

Index Terms— Direction-of-arrival estimation, sparse signal representation, subband information fusion, wideband source, unconstrained optimization.

1. INTRODUCTION

Many wideband direction-of-arrival (DOA) estimation methods have been proposed over last three decades due to their various applications in radar, sonar, wireless communication and radio-astronomy etc. [1]. A classical wideband array processing is to decompose the wideband signals into many narrowband signals with a filter bank or the discrete Fourier transform (DFT), and two categories, referred to as incoherent signal subspace method (ISSM) [2] and coherent signal subspace method (CSSM) [3], are utilized to realize wideband DOA estimation. The ISSM estimates the DOAs independently and average them over all the bins. The performance of ISSM may deteriorate with low signal-to-noise ratio (SNR) frequency bins and coherent sources. The CSSM align the signal subspaces by transforming the observation vectors associated with each bin into the focusing subspace and can deal with coherent sources by averaging the subspace-aligned covariance matrices. Compared with ISSM, CSSM can enhance DOA resolution and improve the accuracy of DOA estimates at low SNR. However, CSSM requires an initial DOA estimate and the precision of DOA pre-estimates greatly influences the accuracy of DOA estimation [4, 5].

Recently, a class of sparse signal representation (SSR) methods provide a new perspective for wideband DOA estimation. The DOA estimation problem can be formulated as recovering a spatial sparse signal vector or matrix by minimizing the residual norm under sparsity constraint. One of the most successive ℓ_1-norm-based SSR algorithms for DOA estimation is ℓ_1-SVD (Singular Value Decomposition) [6], which reduces the computational complexity by SVD. Hyder and Mahata [7] present a joint $\ell_{2,0}$-norm approximation (JLZA) method and extend it to wideband DOA estimation. The main limitation of the ℓ_1-SVD and JLZA algorithms is that they could not jointly use all the subband information to estimate DOA and therefore lose their performance. More recently, Liu *et al*. [8] present a wideband covariance matrix sparse representation (W-CMSR) method for DOA estimation. The W-CMSR method uses time domain measurement information and has its limitation for spatial nonambiguity because the spatial aliasing is frequency-dependent. To deal with the spatial aliasing for SSR problem, Tang *et al*. [9] shows that such ambiguity can be removed by using multiple dictionaries, each dictionary corresponding to a judiciously chosen frequency. However, it still does not consider using all the frequency bin information to reduce the spatial ambiguity.

In this paper, we present a new subband information fusion (SIF) method for wideband DOA estimation. This method can jointly integrate all the frequency bins together to estimate a sparse indicative vector (SIV) that is used to indicate the location of sources. By using the SIF method, the spatial nonambiguity condition can be extended dramatically compared with the classical beamforming technique. We then develop the SIF algorithm and compare the SIF al-

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gorithm with the W-CMSR algorithm. The simulation results show that the proposed algorithm has better performance.

2. STATISTICAL MODEL

Consider a uniform linear array (ULA) of P omnidirectional sensors is exposed to a wavefield generated by Q far-field wideband sources in presence of noise. For wideband processing, a standard way is to split the time-samples at each sensor into N segments, where for each segment, K narrowband signals are obtained by a bank of narrowband filters or the discrete Fourier transform (DFT) [5]. Assume that the frequencies of all related sources overlap from $\omega_0 - \frac{B}{2}$ rad/s to $\omega_0 + \frac{B}{2}$ rad/s, where ω_0 is the center frequency and B is the bandwidth of some frequency band where the frequency bands of all sources intersect. The array output vector for fixed frequency $\omega_k \in [\omega_0 - \frac{B}{2}, \omega_0 + \frac{B}{2}]$, $k = 1, \ldots, K$, is formed and can be expressed by

$$\mathbf{x}_{k,n} = \mathbf{A}_k(\boldsymbol{\theta})\mathbf{s}_{k,n} + \mathbf{w}_{k,n} \quad (1)$$

where $\mathbf{x}_{k,n}$ is the $P \times 1$ array measurement vector, $\mathbf{s}_{k,n}$ denotes the $Q \times 1$ source signal vector at frequency ω_k and $\mathbf{w}_{k,n}$ represents the $P \times 1$ additive noise vector. $\mathbf{A}_k(\boldsymbol{\theta})$ is the manifold matrix at frequency ω_k

$$\mathbf{A}_k(\boldsymbol{\theta}) = [\mathbf{a}_k(\theta_1), \ldots, \mathbf{a}_k(\theta_Q)] \quad (2)$$

where the steering vector $\mathbf{a}_k(\theta_q)$ can be written as

$$\mathbf{a}_k(\theta_q) = \left[1, e^{-j\omega_k \frac{d}{c}\sin\theta_q}, \ldots, e^{-j\omega_k (P-1)\frac{d}{c}\sin\theta_q}\right]^T \quad (3)$$

where θ_q, $q = 1, \ldots, Q$, is the DOA of the q-th source, d is the distance between the two adjacent sensors, and c is the speed of the signal propagation.

The DOA estimation appears in (1) is a nonlinear parameter estimation problem, where the DOA parameter $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_Q]^T$ need to be estimated. The sparse representation model transforms a parameter estimation problem into sparse spectrum estimation. We denote $\Theta = \{\theta_l\}_{l=1}^{L}$ as the set of a sampling grid of all possible source locations, $L \gg Q$. We assume that the grid is fine enough that Θ can represent the true source locations exactly or closely. Then, the array output model (1) is changed into

$$\mathbf{x}_{k,n} = \sum_{l=1}^{L} \mathbf{a}_k(\theta_l) v_{k,l}(n) + \mathbf{w}_{k,n} \quad (4)$$

where $v_{k,l}(n)$ and $\mathbf{a}_k(\theta_l)$ are the visual source and steering vector corresponding to the l-th grid. We introduce an overcomplete basis matrix $\mathbf{A}_k = [\mathbf{a}_k(\theta_1), \ldots, \mathbf{a}_k(\theta_L)]$. The sparse representation model in (4) can be represented by the compact model

$$\mathbf{x}_{k,n} = \mathbf{A}_k \mathbf{v}_{k,n} + \mathbf{w}_{k,n} \quad (5)$$

where $\mathbf{v}_{k,n} = [v_{k,1}(n), \ldots, v_{k,L}(n)]^T$ is the sparse representation of source vector. The nonzero entries of $\mathbf{v}_{k,n}$ represent true sources and zero otherwise. Assume that the nonzero indexes of $\mathbf{v}_{k,n}$ are contained in the set $\mathcal{I} := \{l_1, \ldots, l_Q\}$. Obviously, $v_{k,l_q}(n) = s_{k,q}(n)$, where $s_{k,q}(n)$ denotes the k-th subband signal radiated from the q-th target at the n-th snapshot.

3. WIDEBAND DOA ESTIMATION USING A SUBBAND INFORMATION FUSION METHOD

3.1. A New SIF Method

From the sparse representation model in (5), it is clear that the matrices $\mathbf{V}_1, \ldots, \mathbf{V}_N$ share the identical sparse structure, where $\mathbf{V}_n = [\mathbf{v}_{1,n}, \ldots, \mathbf{v}_{K,n}]$, $n = 1, \ldots, N$. Let $\mathbf{V} = [\mathbf{V}_1, \ldots, \mathbf{V}_N]$. The nonzero rows of matrix $\mathbf{V}$ indicate the source locations. The sparse indicative vector employed in [10, 11] for narrowband DOA estimation can also be used to represent the sparsity structure of $\mathbf{V}$ in wideband processing. Thus, the estimating of entire matrix $\mathbf{V}$ can be avoided. To develop the SIF method, we introduce an over-complete dictionary [10]

$$\boldsymbol{\Psi}_{k,n} = \left[\mathbf{a}_k(\theta_1)\hat{v}_{k,1}(n) \quad \cdots \quad \mathbf{a}_k(\theta_L)\hat{v}_{k,L}(n)\right], \quad (6)$$

where

$$\hat{v}_{k,l}(n) = \left(\mathbf{a}_k^H(\theta_l)\mathbf{a}_k^H(\theta_l)\right)^{-1} \mathbf{a}_k^H(\theta_l)\mathbf{x}_{k,n}. \quad (7)$$

Let us use $\mathbf{D} \in \mathbb{C}^{L \times L}$ denote a new dictionary that $\mathbf{D}^H\mathbf{D} = \boldsymbol{\Phi}$, where

$$\boldsymbol{\Phi} = \sum_{k=1}^{K}\sum_{n=1}^{N} \boldsymbol{\Psi}_{k,n}^H \boldsymbol{\Psi}_{k,n}. \quad (8)$$

One realization of $\mathbf{D}$ is given by the eigenvalue decomposition (EVD) of the Hermitian matrix $\boldsymbol{\Phi}$. Assume that $\boldsymbol{\Phi}$ can be written as $\boldsymbol{\Phi} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^H$, where $\mathbf{U}$ is a unitary matrix whose columns are composed by L orthonormal eigenvectors and $\boldsymbol{\Lambda}$ is a diagonal matrix of eigenvalues. Thus, the dictionary $\mathbf{D}$ has the expression $\mathbf{D} = \mathbf{U}\boldsymbol{\Lambda}^{\frac{1}{2}}$. Let M denote the rank of $\boldsymbol{\Phi}$. We have $\text{rank}(\mathbf{D}) = \text{rank}(\boldsymbol{\Phi}) = M$.

We define a new SIV $\mathbf{g}$ which has the same sparsity structure with $\mathbf{V}$. $\mathbf{g}$ can be used to indicate the locations of sources. Recovering $\mathbf{g}$ needs to jointly minimize a residual term $\|\mathbf{y} - \mathbf{D}\mathbf{g}\|^2$ and a ℓ_p-norm ($0 \le p \le 1$) penalty, which leads to the SIF method:

$$\min_{\mathbf{g}} f(\mathbf{g}): \quad f(\mathbf{g}) = \|\mathbf{y} - \mathbf{D}\mathbf{g}\|^2 + \lambda\|\mathbf{g}\|_p, \quad (9)$$

where λ is a regularization parameter, $\mathbf{y}$ is a new vector defined by $\mathbf{D}^H\mathbf{y} = \mathbf{h}$,

$$\mathbf{h} = \sum_{k=1}^{K}\sum_{n=1}^{N} \boldsymbol{\Psi}_{k,n}^H \mathbf{x}_{k,n}. \quad (10)$$

If the number of sources is fixed during N snapshots, $\mathbf{g}$ is Q-sparse and $\|\mathbf{g}\|_0 = Q$. The indexes of nonzero elements in $\mathbf{g}$ are contained in the set $\mathcal{I}$ which indicates the source locations. Intuitively, $\mathbf{g}$ captures the sparsity structure of $\mathbf{y}$ in the basis of $\mathbf{D}$. Therefore, $\mathbf{g}$ can be recovered as sparse as possible provided that the residual term $\|\mathbf{y} - \mathbf{D}\mathbf{g}\|^2$ is minimized. The problem of DOA estimation can then be solved by recovering $\mathbf{g}$ instead of estimating $\mathbf{V}$.

Note that the derivation of (9) has an equivalent expression which is given by

$$\begin{aligned}\arg\min_{\mathbf{g}} & f(\mathbf{g}) \\ & \equiv \arg\min_{\mathbf{g}} \sum_{k=1}^{K}\sum_{n=1}^{N} \|\mathbf{x}_{k,n} - \mathbf{\Psi}_{k,n}\mathbf{g}\|^2 + \lambda\|\mathbf{g}\|_p. \quad (11)\end{aligned}$$

The above expression (11) shows that all the frequency bins and snapshots are integrated together to estimate $\mathbf{g}$. We then find that the over-complete dictionary $\mathbf{\Psi}_{k,n}$ can only be used for specific frequency and snapshot in which $\mathbf{x}_{k,n}$ can be expressed as a sparse signal. Yet $\mathbf{D}$ includes all the frequency and snapshot measurements information and therefore (9) is actually an information fusion formulation for the wideband DOA estimation problem.

3.2. Nonambiguity Guarantee for the SIF Method

As discussed in [9], the sparse signal representation (SSR) approaches are generally subject to two kinds of ambiguity. One is referred as algebraic aliasing, which arises from the over-completeness of the dictionary. The other is called spatial aliasing, just like the classical beamforming. If the array spacing d is larger than half the apparent wavelength, it will be possible to find at least two angles θ_l and $\theta_{l'}$. The corresponding columns in $\mathbf{A}_k$ will be identical.

For the algebraic aliasing-free problem, we investigate the sufficient condition for the existence of a unique solution to the noise-free equations $\mathbf{y} = \mathbf{D}\mathbf{g}$. The following theorem shows that the algebraic nonambiguity is guaranteed if the number of sources satisfies certain condition.

Theorem 1. *Consider the equations $\boldsymbol{y} = \boldsymbol{D}\boldsymbol{g}$, where $\boldsymbol{D}$ is a $L \times L$ dictionary, $\boldsymbol{y}$ is a $L \times 1$ vector and $\boldsymbol{g}$ is a sparse indicative vector whose nonzero elements indicate the locations of sources. $\boldsymbol{D}$ and $\boldsymbol{y}$ are defined by $\boldsymbol{D}^H\boldsymbol{D} = \mathbf{\Phi}$, $\boldsymbol{D}^H\boldsymbol{y} = \boldsymbol{h}$ respectively, where $\mathbf{\Phi}$ and $\boldsymbol{h}$ is given in* (8) *and* (10)*. Assume that no spatial nonambiguity exists. A unique solution of sparse indicative vector $\boldsymbol{g}$ is guaranteed if $Q \le \min(\lceil (M+1)/2\rceil - 1, \lceil (P+NK)/2\rceil - 1)$ for $NK < P$ and $Q \le \min(\lceil (M+1)/2\rceil - 1, P-1)$ for $NK \ge P$, where $\lceil\cdot\rceil$ denotes the ceiling operation.*

Proof: See [12]. In Subsection 3.1, we convert the multiple measurement vectors (MMV) problem to a single measurement vector (SMV) problem. The proof of theorem 1 can use the unique solution condition (see [13], theorem 2.4) for the SMV problem. At the same time, the unique solution for $\mathbf{y} = \mathbf{D}\mathbf{g}$ is under the MMV problem, since it is derived from the MMV problem.

We then investigate the spatial aliasing problem. Assume that two angles θ_{l_1} and θ_{l_2} satisfy

$$\omega_k d \sin\theta_{l_1}/c = \omega_k d \sin\theta_{l_2}/c + 2\pi I, \quad (12)$$

where I is an arbitrary integer. Let us use θ_{l_1} and θ_{l_2} to generate two corresponding columns in dictionary $\mathbf{D}$. If these two columns are identical, the spatial aliasing occurs. Unlike the narrowband DOA estimation problem, the spatial nonambiguity condition for wideband DOA estimation can be more relaxed. It is clear that the spatial aliasing is dependent on the frequency. For the SIF method, all the frequency bin information can be combined to reduce the spatial ambiguity, which leads to the following theorem.

Theorem 2. *Assume that the number of sources satisfies the unique solution condition proposed in theorem 1. Let $\Delta\omega = \omega_k - \omega_{k-1}$. With a ULA interspaced by a unit distance d, the spatial nonambiguity is guaranteed if $d < \frac{\pi c}{\Delta\omega}$, where c is the propagation speed of the source signal.*

Proof: See [12]. The proof of theorem 2 is similar to the proof of theorem 1 in [9]. We first assume that θ_{l_1} is one of the target angles and θ_{l_2} is the spatial aliasing angle, which lead to the identical column as θ_{l_1} in Hermitian matrix $\mathbf{\Phi}$. We then prove that under the condition $d < \frac{\pi c}{\Delta\omega}$, $\theta_{l_1} = \theta_{l_2}$ holds and the spatial nonambiguity is guaranteed.

3.3. SIF Algorithm

Consider the following unconstrained problem:

$$\min_{\mathbf{g}} \sum_{k=1}^{K}\sum_{n=1}^{N} \|\mathbf{x}_{k,n} - \mathbf{\Psi}_{k,n}\mathbf{g}\|^2 + \lambda\|\mathbf{g}\|_p. \quad (13)$$

The problem (13) can be solved by generating a sequence of iterates $\{\mathbf{g}_0, \mathbf{g}_1, \ldots\}$. For each iteration, the optimization problem of (13) is given by

$$\mathbf{g}_{t+1} = 2\mathbf{\Pi}(\mathbf{g}_t)\mathbf{H}^{-1}(\mathbf{g}_t)\mathbf{h}, \quad (14)$$

where

$$\mathbf{H}(\mathbf{g}) = 2\mathbf{\Phi}\mathbf{\Pi}(\mathbf{g}) + p\lambda\mathbf{I} \quad (15)$$

$$\mathbf{\Pi}(\mathbf{g}) = \begin{bmatrix} (g_1^2+\epsilon)^{1-p/2} & & 0 \\ & \ddots & \\ 0 & & (g_L^2+\epsilon)^{1-p/2} \end{bmatrix}, \quad (16)$$

$\mathbf{I}$ is a $L \times L$ identity matrix, ϵ is a very small smooth parameter and the expressions of $\mathbf{\Phi}$ and $\mathbf{h}$ are given in (8) and (10). The computational complexity of computing (14) can be reduced by using the conjugate gradient (CG) technique [14, 15]. Let $\mathbf{H}^{-1}(\mathbf{g})\mathbf{h}$ be replaced by $\mathrm{cg}(\mathbf{H}(\mathbf{g}), \mathbf{h})$, where $\mathrm{cg}(\mathbf{H}(\mathbf{g}), \mathbf{h})$ is a CG solution of linear equation $\mathbf{H}(\mathbf{g})\mathbf{b} = \mathbf{h}$ for $\mathbf{b} = \mathbf{H}^{-1}(\mathbf{g})\mathbf{h}$.

4. EXPERIMENTAL RESULTS

In this section, we illustrate the performance of SIF algorithm via various numerical simulations. We consider the same numerical examples given in [8] for ease of comparison. Assume that two BPSK signals with central frequency of 70 MHz and bandwidth of 40% impinge on a ULA with 7 sensors. Fig. 1 and Fig. 2 show the DOA estimation results of two sources from the directions of $-10°$ and $10°$ obtained by the W-CMSR and SIF respectively. We take $K = 256$, $N = 1$ and SNR $= 0$ dB. The parameters p, λ and ϵ are set to 0.1, $\lambda = 0.2 \times \|2\boldsymbol{h}\|_\infty$ and $\epsilon = 10^{-18}$ for the SIF algorithm. ϵ can be arbitrary small. In Fig. 1, the interspace d of the ULA satisfies $d = \pi c/(\omega_0 + \frac{B}{2})$. We then extend d to 100 times in Fig. 2. It can be seen from Fig. 1 that both the W-CMSR and SIF algorithms perform well. However, the W-CMSR algorithm fails in Fig. 2 while the SIF algorithm still has good results. Fig. 2 shows that the spatial ambiguity is reduced by the SIF algorithm.

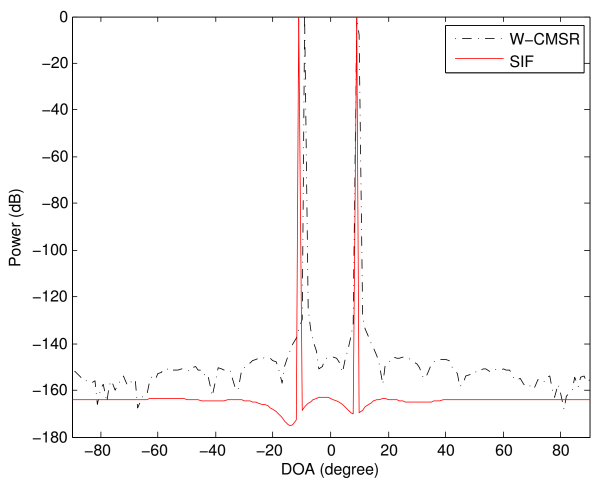

Fig. 1. Spatial spectrum of two sources from $-10°$ and $10°$ obtained by the W-CMSR and SIF algorithms, P: 7, SNR: 0 dB, grid resolution: $1°$, d: 1.948.

The interval of two adjacent frequencies is $\Delta\omega = 2\pi$ in the simulations. According to the theorem 2, the spatial non-ambiguity is guaranteed if $d < \frac{c}{2}$, $c = 3 \times 10^8$ is the propagation speed of the signal. As d increases, the SIF algorithm captures the unique solution of (13). Moreover, the increasing of d makes the columns of dictionary $\mathbf{D}$ more and more uncorrelated. In Fig. 3, the dash line plots the"mismatch" of the DOA estimate when two sources are close at $4°$ and $10°$. The DOA estimation result for the dash line in Fig. 3 is $2°$ and $12°$ when $d = 1.948$. If the interspace d is increased to 100 times, the DOA estimation result exactly matches the source directions, see Fig. 3 the solid line.

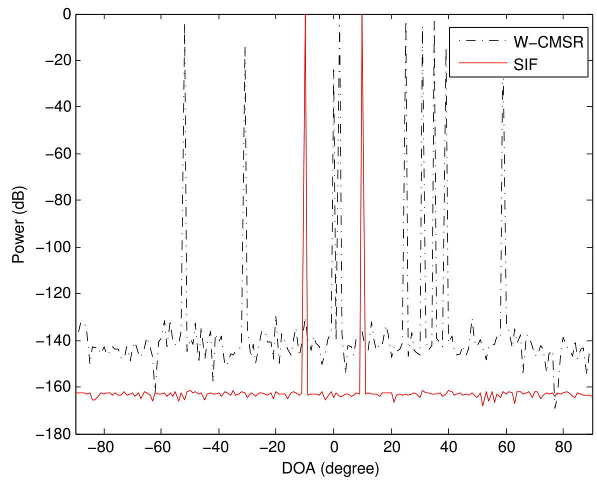

Fig. 2. Spatial spectrum of two sources from $-10°$ and $10°$ obtained by the W-CMSR and SIF algorithms, P: 7, SNR: 0 dB, grid resolution: $1°$, d: 194.8.

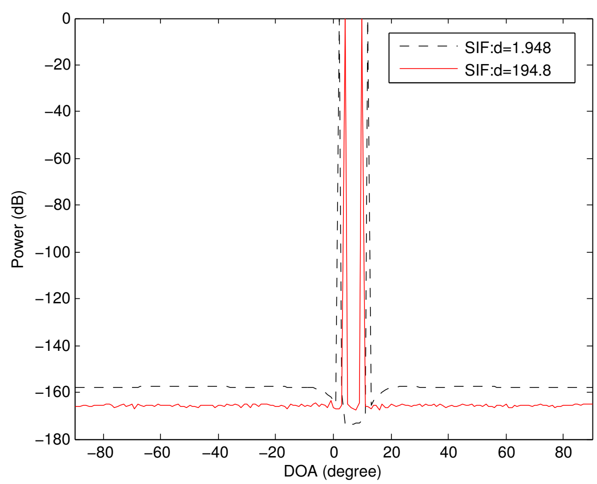

Fig. 3. Spatial spectrum of two sources from $6°$ and $10°$ obtained by the SIF algorithm, P: 7, SNR: 0 dB, resolution $1°$.

5. CONCLUSION

In this paper, we present a new subband information fusion (SIF) algorithm to solve the wideband DOA estimation problem. The SIF algorithm utilizes all the frequency bin information to recover a sparse indicative vector (SIV) iteratively. We show that the algebraic ambiguity resulting from the over-complete dictionary can be alleviated by multiple measurement vectors. Since the spatial aliasing is frequency-dependent, the spatial ambiguity due to spatial aliasing can be reduced by using all the frequency information. By using the CG method, the computational complexity of the SIF algorithm has order $O(L^2)$. The performance of the proposed algorithm is demonstrated by numerical simulations.

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