HYPERPARAMETER-FREE DOA ESTIMATION UNDER POWER CONSTRAINTS

Chundi Zheng[1,2]*, Gang Li*[1]*, Pan Li*[1]*, and Xiqin Wang*[1]

1. Department of Electronic Engineering, Tsinghua University, Beijing 100084, China
2. Marine Corps College, Guangzhou 510430, China

ABSTRACT

Based on the covariance-like fitting criterion we propose a direction of arrival (DOA) estimation algorithm that embeds a weighting scheme in the objective function without selection of any hyperparameters. With an assumption of uncorrelated sources, we formulate the problem of DOA estimation as a linearly constrained quadratic optimization under power constraints. Numerical results show that the proposed method not only is robust to this assumption but also has the superior performance in comparison with some other sparsity-driven methods.

Index Terms— Direction of arrival estimation, sparse recovery, weighted ℓ_2 minimization, array signal processing

1. INTRODUCTION

Direction of arrival (DOA) estimation is an interesting issue in array processing field, which has wide applications from electromagnetics to acoustics [1]. Many algorithms have been proposed for achieving high resolution and high accuracy of DOA estimation. In general, these algorithms can be categorized in two classes according to the use of hyperparameters: hyperparameter-free and hyperparameter-dependent. Here, the hyperparameter refers to the noise power, and other user-dependent parameters. In practice, determining proper values of hyperparameters is usually difficult to be carried out [2, 3]. Therefore, hyperparameter-free methods are more attractive in practical applications.

It is well known that Fourier-based spectral analysis techniques (e.g. the Bartlett beamformer) can estimate DOA without any hyprparameters, but they suffer from the Rayleigh resolution limit [1]. To overcome this disadvantage, the classical Capon method based on subspace processing has been exploited to achieve high resolution DOA estimation without any hyperparameters [4]. Recently, the sparse representation has been introduced into DOA estimation [5, 6], and designing high-performance sparsity-driven DOA estimator without knowledge of hyperparameters is also challenging [3, 7, 8]. Sparse Iterative Covariance-based Estimation (SPICE) does not require any hyperparameter and ensures reliable DOA estimates with global convergence [3,7]. However, it is reported in [9] that the accuracy of SPICE is unsatisfactory. By employing the maximum likelihood principle, LIKelihood-based Estimation of Sparse parameters (LIKES) has been derived to improve SPICE. LIKES performs better than SPICE in term of accuracy and resolution at cost of increased computational burden [8, 9].

In this paper, we propose a covariance-like fitting algorithm for DOA estimation under power constraints without selection of hyperparameters. The covariance matrix of the measurements is firstly represented in the sparse form, and then the DOA estimation can be regarded as a fitting problem with a overcomplete basis matrix. The hyperparameter selection is avoided by using power constraints on the estimated spectrum. Furthermore, we devise a Capon-like cepstrum (in the context, the cepstrum means the reciprocal of the spectrum) weight to prompt sparsity so that the peaks of the estimated spatial spectrum are more likely to be the positions of true DOAs. That is to say, the proposed method embeds the equivalent weighted ℓ_2 norm in the objective function. From the methodological point of view in the area of sparse representation the weighted ℓ_2 norm minimization is conducive to sparsity enhancement [10–13]. Moreover, considering uncorrelated sources we formulate DOA estimation as a linearly constrained quadratic optimization problem. Numerical examples demonstrate that the proposed algorithm can obtain more accurate DOA estimates than that of ℓ_1-SVD [5], SPICE [7], and CW $\ell_{2,1}$ [14].

This paper is organized as follows. In the next section, we introduce the background and the model about the covariance-like fitting approach. In Section 3, we formulate a convex optimization function to resolve the DOA estimation problem. In Section 4, numerical experiments are provided for illustrating the performance of the proposed method. A conclusion is given in Section 5.

2. SIGNAL MODEL

Suppose that there are K far-field narrowband signals $\{s_k(t), k = 1, 2, \cdots, K\}$, with the common center frequency f_0

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impinging on the array from distinct directions $\{\theta_k, k = 1, 2, \cdots, K\}$. The measurements can be described in vector form as:

$$\mathbf{y}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), t = 1, 2, \cdots, T., \quad (1)$$

where the measurements $\mathbf{y}(t) = [y_1(t), \cdots, y_M(t)]^{\mathrm{T}} \in \mathbb{C}^{M\times 1}$, the source signal vector $\mathbf{s}(t) = [s_1(t), \cdots, s_K(t)]^{\mathrm{T}} \in \mathbb{C}^{K\times 1}$, and the noise vector $\mathbf{n}(t) = [n_1(t), \cdots, n_M(t)]^{\mathrm{T}} \in \mathbb{C}^{M\times 1}$, M denotes the number of sensors, the notation $[\cdot]^{\mathrm{T}}$ denotes the transposition. The matrix $\mathbf{A} \in \mathbb{C}^{M\times K}$ is the array response matrix given by $\mathbf{A} = [\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_K)]$ with $\mathbf{a}(\theta_k) = [1, e^{j2\pi f_0 d sin(\theta_k)/c}, \cdots, e^{j(M-1)2\pi f_0 d sin(\theta_k)/c}]^{\mathrm{T}}$, where c and d denote the propagation speed and the sensor spacing, respectively. The vector $\mathbf{n}(t)$ is an additive noise vector with zero-mean. Without loss of generality, $\mathbf{n}(t)$ is assumed to be uncorrelated with $\mathbf{s}(t)$.

Assuming the source signals are uncorrelated and the observed space is uniformly sampled with N angular grids, thus the covariance matrix of the measurements $\mathbf{y}(t)$ can be expressed as [7]:

$$\begin{aligned} \mathbf{R} &= E[\mathbf{y}(t)\mathbf{y}^{\mathrm{H}}(t)] \\ &= \sum_{n=1}^{N} p_n \mathbf{a}_n \mathbf{a}_n^{\mathrm{H}} + \mathbf{R}_N \\ &\triangleq \mathbf{\Phi}\mathbf{P}\mathbf{\Phi}^{\mathrm{H}}, \end{aligned} \quad (2)$$

where

$$\mathbf{\Phi} \triangleq [\mathbf{a}_1, \cdots, \mathbf{a}_n, \cdots, \mathbf{a}_N, \mathbf{I}_M] = [\bar{\mathbf{A}}, \mathbf{I}_M], \quad (3)$$

$$\bar{\mathbf{A}} = [\mathbf{a}_1, \cdots, \mathbf{a}_n, \cdots, \mathbf{a}_N], \quad (4)$$

$$\mathbf{R}_N \triangleq \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \sigma_M^2 \end{bmatrix}, \quad (5)$$

$$\mathbf{P} \triangleq \begin{bmatrix} p_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & p_2 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & p_N & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & p_{N+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & p_{N+M} \end{bmatrix}, \quad (6)$$

$\mathbf{I}_M$ is a $M \times M$ identity matrix, and the sign $(\cdot)^{\mathrm{H}}$ denotes the conjugate transpose, $p_{N+1} = \sigma_1^2, \cdots, p_{N+M} = \sigma_M^2$.

Ideally, if and only if $\theta_n = \theta_k$, $p_n > 0$ for $n \leq N$. In other words, the DOA estimation can be achieved by determining the peaks of the spatial spectrum $\mathbf{p}$ for $n \leq N$, where $\mathbf{p} = [p_1, \cdots, p_{N+M}]^{\mathrm{T}} \in \mathbb{R}^{(N+M)\times 1}$ corresponds to the diagonal elements of $\mathbf{P}$.

3. THE PROPOSED ALGORITHM

Utilizing the eigenvalue decomposition of $\mathbf{R}$, we have

$$\mathbf{R} = \mathbf{U}\mathbf{\Sigma}\mathbf{U}^{\mathrm{H}}, \quad (7)$$

where the columns of $\mathbf{U}$ denote the eigenvectors of $\mathbf{R}$ and $\mathbf{U}\mathbf{U}^{\mathrm{H}} = \mathbf{U}^{\mathrm{H}}\mathbf{U} = \mathbf{I}_M$, the matrix $\mathbf{\Sigma}$ is a diagonal matrix and its diagonal elements are the eigenvalues of $\mathbf{R}$.

Substituting (7) into (2) yields the following results:

$$\mathbf{U}\mathbf{\Sigma}\mathbf{U}^{\mathrm{H}} = \mathbf{\Phi}\mathbf{P}\mathbf{\Phi}^{\mathrm{H}}, \quad (8)$$

$$\mathbf{I}_M = \mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{U}^{\mathrm{H}}\mathbf{\Phi}\mathbf{P}\mathbf{\Phi}^{\mathrm{H}}\mathbf{U}\mathbf{\Sigma}^{-\frac{1}{2}}. \quad (9)$$

Now, we devise a new DOA estimation algorithm by employing the covariance-like fitting criterion:

$$\begin{aligned} \min_{p_n \geq 0} \quad & \|\mathbf{I}_M - \mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{U}^{\mathrm{H}}\mathbf{\Phi}\mathbf{P}\mathbf{\Phi}^{\mathrm{H}}\mathbf{U}\mathbf{\Sigma}^{-\frac{1}{2}}\|_{\mathrm{F}}^2 \\ s.t. \quad & \sum_{n=1}^{N+M} p_n = \mathrm{trace}(\mathbf{R})/M, \end{aligned} \quad (10)$$

where the sign $\mathrm{trace}(\cdot)$ and $\|\cdot\|_{\mathrm{F}}$ denote the trace of matrix and the Frobenius norm, respectively.

The objective function in (10) can be written with the trace of matrix as:

$$\begin{aligned} & \|\mathbf{I}_M - \mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{U}^{\mathrm{H}}\mathbf{\Phi}\mathbf{P}\mathbf{\Phi}^{\mathrm{H}}\mathbf{U}\mathbf{\Sigma}^{-\frac{1}{2}}\|_{\mathrm{F}}^2 \\ = \; & M - 2\mathrm{trace}(\mathbf{P}\mathbf{B}) + \mathrm{trace}(\mathbf{P}\mathbf{B}\mathbf{P}\mathbf{B}), \end{aligned} \quad (11)$$

where $\mathbf{B} = \mathbf{\Phi}^{\mathrm{H}}\mathbf{U}\mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{U}^{\mathrm{H}}\mathbf{\Phi} = \mathbf{\Phi}^{\mathrm{H}}\mathbf{R}^{-1}\mathbf{\Phi}$. Thus, (10) can be written as:

$$\begin{aligned} \min_{p_n \geq 0} \quad & \mathrm{trace}(\mathbf{P}\mathbf{B}\mathbf{P}\mathbf{B}) - 2\mathrm{trace}(\mathbf{P}\mathbf{B}) \\ s.t. \quad & \sum_{n=1}^{N+M} p_n = \mathrm{trace}(\mathbf{R})/M. \end{aligned} \quad (12)$$

Note that the minimization problem (12) can be simplified as a linearly constrained quadratic problem by utilizing the diagonal matrix property of $\mathbf{P}$:

$$\begin{aligned} \min_{p_n \geq 0} \quad & \mathbf{p}^{\mathrm{T}}\mathbf{D}\mathbf{p} - 2\mathbf{d}^{\mathrm{T}}\mathbf{p} \\ s.t. \quad & \mathbf{1}^{\mathrm{T}}\mathbf{p} = \mathrm{trace}(\mathbf{R})/M, \end{aligned} \quad (13)$$

where $\mathbf{D} = \mathbf{B} \odot \mathbf{B}^*$, $\mathbf{d}$ is a real value vector that corresponds to the diagonal elements of $\mathbf{B}$, $\mathbf{1}$ is the vector of all ones, and the sign $(\cdot)^*$ and $\odot$ denote the conjugate and the Hadamart product, respectively. In practice, we have to replace the covariance $\mathbf{R}$ with the sampling covariance $\hat{\mathbf{R}}$, where $\hat{\mathbf{R}} = (1/T)\sum_{t=1}^{T}\mathbf{y}(t)\mathbf{y}^{\mathrm{H}}(t)$. Here, we employ the CVX package [15] to implement the quadratic optimization problem. We leave an effective computational procedure that is tailored for this problem to further work.

In order to compare with the Capon cepstrum, we define the Capon-like cepstrum $w_{n,m}$ as below:

$$w_{n,m} = |\mathbf{a}_n^{\mathrm{H}}\mathbf{U}\mathbf{\Sigma}^{-1}\mathbf{U}^{\mathrm{H}}\mathbf{a}_m| = |\mathbf{a}_n^{\mathrm{H}}\mathbf{R}^{-1}\mathbf{a}_m|. \quad (14)$$

If and only if $n = m$, $w_{n,m}$ becomes the Capon cepstrum $w_{n,n}$.

Substituting (14) into the objective function of (13) yields the following result:

$$\begin{aligned} f(\mathbf{p}) &= \mathbf{p}^{\mathrm{T}}\mathbf{D}\mathbf{p} - 2\mathbf{d}^{\mathrm{T}}\mathbf{p} \\ &= \sum_{n=1}^{N+M}\sum_{m=1}^{N+M} w_{n,m}^2 p_n p_m - 2\sum_{n=1}^{N+M} w_{n,n} p_n \\ &= \sum_{n=1}^{N+M}\sum_{m\neq n}^{N+M} w_{n,m}^2 p_n p_m \\ &\quad + \sum_{n=1}^{N+M} w_{n,n} p_n (w_{n,n} p_n - 2). \end{aligned} \quad (15)$$

From the second term of (15) , it is noted that the Capon cepstrum $w_{n,n}$ and the corresponding spectrum element p_n are linked together. In addition, for the first term of (15) the Capon-like cepstrum $w_{n,m}$ can be regraded as the corresponding weight on p_n and p_m, where the Capon-like cepstrum has similar property with the Capon cepstrum (see Appendix for details). This means that smaller weights are appointed to these positions that more likely correspond to the DOAs, which is helpful to improve the sparseness of the solution as reported in [10–13].

Although the source signals are assumed to be spatially uncorrelated in (2), the proposed method still can deal with the correlated source case. It is easy to deduce this conclusion as done in [7, 16]. In next section, we demonstrate this fact with numerical examples.

4. NUMERICAL EXAMPLES

In this section, we demonstrate the performance of the proposed method with some numerical examples. The results of several sparse signal recovery methods, i.e.,ℓ_1-SVD, SPICE, and CW $\ell_{2,1}$, are also provided for comparison. In all experiments we consider a uniformly-spaced linear array (ULA) composed of $M = 10$ sensors with a spacing of $d = c/(2f_0)$. The angle interval $[-90°, 90°]$ is uniformly sampled with 1801 grids.

In the first experiment, four uncorrelated sources with the same amplitude are impinging on the array from $\{-12°, 18°, 26°, 36°\}$. As shown in Fig.1, the proposed method can obtain more accurate estimates than that of ℓ_1-SVD, CW $\ell_{2,1}$, and SPICE, especially for relatively high Signal Noise Ratio (SNR). Although CW $\ell_{2,1}$ and the proposed method has a similar performance when SNR is lower, it is noted that the former requires some hyperparameters [1], and the latter does not. In addition, as can be seen from Fig.2, when the number of snapshots is very few, ℓ_1-SVD and CW $\ell_{2,1}$ can obtain higher accuracy with proper hyperparameters. With increasing the number of snapshots, the performance of the proposed method exceeds that of ℓ_1-SVD and CW $\ell_{2,1}$. Furthermore, in contrast to SPICE that also does not require the hyperparameters, the proposed method performs better when the number of snapshots becomes larger.

[1]For adjusting the sparseness of the solution ℓ_1-SVD and CW $\ell_{2,1}$ have to use the noise power and a user-dependent parameter (i.e., the confidence level of the estimated noise energy), which results in the performance degradation providing these hyperparameters are improper.

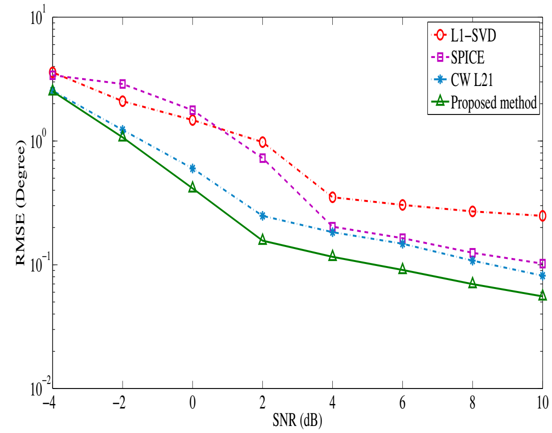

Fig. 1. RMSE versus SNR. The number of snapshots is 200, the number of sensors is 10, the sources is uncorrelated, and DOAs:$\{-12°, 18°, 26°, 36°\}$, 500 Monte Carlo trials.

In the second experiment, we plot the estimated spatial spectrum for two correlated sources that have the same amplitude with correlation coefficient of 0.8. It can be seen that the peaks over 200 Monte carlo trials concentrate on the true DOA positions. This shows that the proposed method is robust to the assumption of uncorrelated sources and works well for correlated source scenario.

5. CONCLUSION

We present a DOA estimation method by using the Capon-like weighted ℓ_2 minimization scheme. The hyperparameter selection is avoided by covariance fitting under power constraints. Numerical experiments demonstrate that the proposed method outperforms ℓ_1-SVD, SPICE, and CW $\ell_{2,1}$ for uncorrelated sources and works well for correlated sources. Future work includes the theoretical performance analysis and the way of computational complexity reduction.

6. APPENDIX

In this Appendix, we try to confirm the Capon-like cepstrum $w_{n,m}$ has the similar property with the Capon cepstrum. Here, The "similar property" means that $w_{n,m} < w_{m,m}$ if θ_n is the

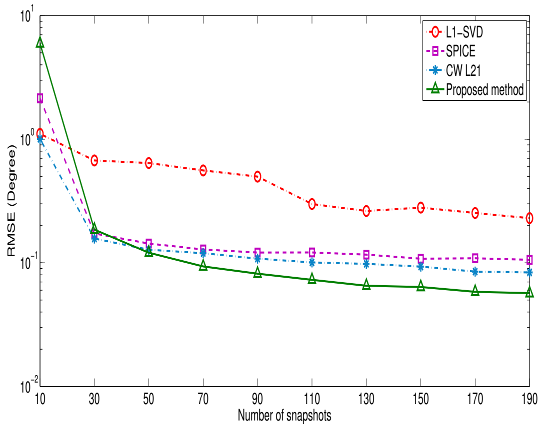

Fig. 2. RMSE versus number of snapshots. SNR is 10dB, the number of sensors is 10, the sources is uncorrelated, and DOAs:$\{-12°, 18°, 26°, 36°\}$, 500 Monte Carlo trials.

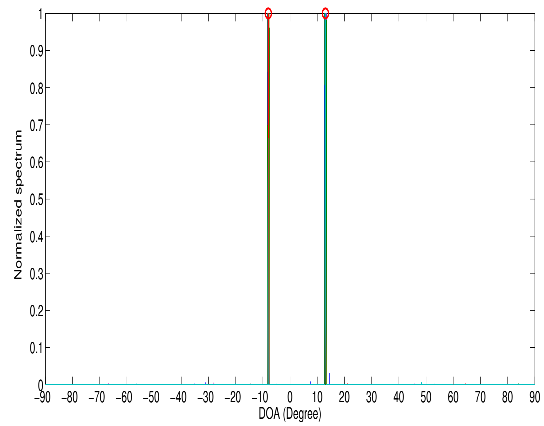

Fig. 3. Normalized spatial spectrum of the proposed method. SNR is 10dB, the number of snapshots is 200, the correlation coefficient of two sources is 0.8, DOAs: $\{-8°, 13°\}$, two red circles denote the true DOAs, 200 Monte Carlo trials.

true DOA and θ_m is not. Let $\mathbf{v}_n \triangleq \mathbf{U}^{\mathrm{H}}\mathbf{a}_n$ and $\mathbf{v}_m \triangleq \mathbf{U}^{\mathrm{H}}\mathbf{a}_m$, we have

$$\begin{aligned} w_{n,m} &= |\mathbf{a}_n^{\mathrm{H}}\mathbf{U}\mathbf{\Sigma}^{-1}\mathbf{U}^{\mathrm{H}}\mathbf{a}_m| \\ &= |\sum_{i=1}^{M}\frac{v_n^i v_m^i}{\lambda_i}| \\ &\leq \sum_{i=1}^{M}\frac{|v_n^i|}{\lambda_i^{1/2}}\frac{|v_m^i|}{\lambda_i^{1/2}} \\ &\leq \sqrt{\sum_{i=1}^{M}\frac{|v_n^i|^2}{\lambda_i}}\sqrt{\sum_{i=1}^{M}\frac{|v_m^i|^2}{\lambda_i}} \\ &= \sqrt{|\mathbf{a}_n^{\mathrm{H}}\mathbf{U}\mathbf{\Sigma}^{-1}\mathbf{U}^{\mathrm{H}}\mathbf{a}_n|}\sqrt{|\mathbf{a}_m^{\mathrm{H}}\mathbf{U}\mathbf{\Sigma}^{-1}\mathbf{U}^{\mathrm{H}}\mathbf{a}_m|} \\ &= \sqrt{w_{n,n}}\sqrt{w_{m,m}} \end{aligned} \tag{16}$$

where λ_i is the ith diagonal elements of $\mathbf{\Sigma}$, v_n^i and v_m^i denote the ith elements of $\mathbf{v}_n$ and $\mathbf{v}_m$, respectively.

Note that $0 < w_{n,n} < w_{m,m}$ since θ_n is the true DOA and θ_m is not, so from (16) we have

$$\begin{aligned} w_{n,m} &\leq \sqrt{w_{n,n}}\sqrt{w_{m,m}} \\ &< \sqrt{w_{m,m}}\sqrt{w_{m,m}} \\ &= w_{m,m}. \end{aligned} \tag{17}$$

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