PERFORMANCE ANALYSIS OF ESPRIT-TYPE ALGORITHMS FOR NON-CIRCULAR SOURCES

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ABSTRACT

High-resolution parameter estimation algorithms designed to benefit from the presence of non-circular (NC) source signals allow for an increased identifiability and a lower estimation error. In this paper, we present a 1-D first-order performance analysis of the NC standard ESPRIT and NC Unitary ESPRIT estimation schemes for strictly second-order (SO) non-circular sources, where NC Unitary ESPRIT has a lower complexity and a better performance in the low signal-to-noise ratio (SNR) regime. Our derived expressions are asymptotic in the effective SNR and explicit in the noise realizations, i.e., no assumptions about the noise statistics are necessary. As a main result, we show that the asymptotic performance of both NC ESPRIT-type algorithms is identical in the effective SNR and that NC Unitary ESPRIT is even applicable to array geometries without a centro-symmetric structure as required for Unitary ESPRIT.

Index Terms— Performance analysis, Unitary ESPRIT, widelylinear processing, non-circular sources, DOA estimation.

1. INTRODUCTION AND STATE OF THE ART

Estimating the directions of arrival (DOA) of incident signals captured by a sensor array has long been of great research interest, given its importance in a variety of signal processing applications such as radar, sonar, biomedical imaging, and wireless communications. Among other subspace-based parameter estimation schemes, standard ESPRIT [1] and Unitary ESPRIT [2] are some of the most powerful estimators due to their high-resolution capabilities and their low complexity as they provide closed-form estimates.

Since the development of the first parameter estimation algorithms, the assessment of their analytical performances has constantly been demanded. The two most prominent strategies have been proposed in [3] and [4]. The derivations in [3] analyze the distribution of the eigenvectors of the sample covariance matrix, which is a complex methodology that requires strong Gaussianity assumptions on the source symbols and the noise, and is only asymptotic in the sample size. In contrast, [4] provides a first-order approximation of the estimation error caused by the perturbed subspace estimate due to a small noise contribution. Unlike [3], it is asymptotic in the effective signal-to-noise ratio (SNR), i.e., the result becomes accurate as either the number of snapshots or the SNR approaches infinity, and no assumptions about the statistics of the signals or the noise are necessary. However, for the mean squared error (MSE) expressions, [4] assumes circular symmetry of the noise distribution. In [5], the authors have extended the framework of [4] to the case of multi-dimensional (R-D) parameter estimation and have derived MSE expressions that only require the noise to be zero-mean without any assumptions about its statistics. Perturbation analyses for a tensor-based subspace estimation scheme and Tensor-ESPRIT have been presented in [6] and [7], respectively.

Recently, a number of improved subspace-based parameter estimation schemes, e.g., NC MUSIC [8], NC Root-MUSIC [9], NC standard ESPRIT [10] and NC Unitary ESPRIT [11] were developed to take advantage of the strict second-order (SO) non-circularity of a class of source signals also referred to as rectilinear signals [12]. Examples of such signals are BPSK, Offset-QPSK, PAM, and ASKmodulated signals. By applying widely-linear (WL) processing, which virtually doubles the array aperture, the estimation error of the aforementioned algorithms is significantly reduced and the number of detectable sources is doubled. The performance of NC MUSIC has been investigated in [8], [13] and [14]. However, a performance analysis of NC standard ESPRIT and NC Unitary ESPRIT has not been reported in the literature.

In this paper, we further extend [5] by incorporating WL preprocessing for strictly SO non-circular sources and present an analytical performance assessment for 1-D NC standard ESPRIT and 1-D NC Unitary ESPRIT. For both algorithms, we resort to least squares (LS) to solve the shift invariance equations. We find the explicit first-order expansion for the estimation error in terms of the noise realization and generic MSE expressions, where we only assume that the noise has a zero mean, but no assumptions about the noise statistics are needed; the noise could also be non-Gaussian or non-circular. Moreover, we prove that NC standard ESPRIT and NC Unitary ESPRIT show the same asymptotic performance in the effective SNR and that NC Unitary ESPRIT does not require a centro-symmetric array structure as is the case for Unitary ESPRIT. Extensions to multiple dimensions [5] and the integration of structured least squares as in [15] are also possible.

2. DATA MODEL

Let a shift-invariance-structured sensor array composed of M elements receive narrowband signals from d far-field sources. Considering N subsequent snapshots, the noisy measurement data can be modeled as

$$\boldsymbol{X} = \boldsymbol{A}\boldsymbol{S} + \boldsymbol{N} \in \mathbb{C}^{M \times N}, \tag{1}$$

where $A = [a(\mu_1), \ldots, a(\mu_d)] \in \mathbb{C}^{M \times d}$ is the array steering matrix, which consists of the array steering vectors $a(\mu_i)$ corresponding to the *i*-th spatial frequency with $i = 1, \ldots, d$, $S \in \mathbb{C}^{d \times N}$ represents the zero-mean source symbol matrix and $N \in \mathbb{C}^{M \times N}$ contains the samples of the additive sensor noise. Due to the assumption of strictly SO non-circular sources, where the complex symbol

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amplitudes lie on a line in the I/Q diagram, we can decompose the source symbol matrix as [11]

$$\boldsymbol{S} = \boldsymbol{\Psi} \boldsymbol{S}_0, \tag{2}$$

where $S_0 \in \mathbb{R}^{d \times N}$ is a real-valued symbol matrix and $\Psi = \text{diag}\{e^{j\varphi_i}\}_{i=1}^d$ contains arbitrary complex phase shifts on its diagonal that are usually different for each received signal.

3. REVIEW OF NC STANDARD ESPRIT

In this section, we briefly review the NC standard ESPRIT algorithm [10] but extend it from the uniform linear array (ULA) case to arbitrarily formed shift-invariant array geometries. Then, these results serve as a basis for the presented perturbation analysis.

In order to take advantage of the benefits associated with noncircular source signals, we apply widely-linear processing and define the augmented measurement matrix $X^{(nc)}$ according to [11] as

$$\boldsymbol{X}^{(\mathrm{nc})} = \begin{bmatrix} \boldsymbol{X} \\ \boldsymbol{\Pi}_{M} \boldsymbol{X}^{*} \end{bmatrix} \in \mathbb{C}^{2M \times N}, \qquad (3)$$

where Π_M is the $M \times M$ exchange matrix with ones on its antidiagonal and zeros elsewhere, which is used to facilitate the real-valued implementation of NC Unitary ESPRIT. The expansion of (3) using (1) and (2) gives

$$\begin{aligned} \boldsymbol{X}^{(\mathrm{nc})} &= \begin{bmatrix} \boldsymbol{AS} \\ \boldsymbol{\Pi}_{M} \boldsymbol{A}^{*} \boldsymbol{S}^{*} \end{bmatrix} + \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{\Pi}_{M} \boldsymbol{N}^{*} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{\Pi}_{M} \boldsymbol{A}^{*} \boldsymbol{\Psi}^{*} \boldsymbol{\Psi}^{*} \end{bmatrix} \boldsymbol{S} + \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{\Pi}_{M} \boldsymbol{N}^{*} \end{bmatrix} \tag{4} \end{aligned}$$

$$= \boldsymbol{A}^{(\mathrm{nc})}\boldsymbol{S} + \boldsymbol{N}^{(\mathrm{nc})} = \boldsymbol{X}_{0}^{(\mathrm{nc})} + \boldsymbol{N}^{(\mathrm{nc})}, \qquad (5)$$

where $\boldsymbol{X}_{0}^{(\mathrm{nc})} \in \mathbb{C}^{2M \times N}$ is the unperturbed augmented measurement matrix and we have used the fact that $\boldsymbol{S}_{0} = \boldsymbol{\Psi}^{*}\boldsymbol{S}$ in (4). Equation (5) shows that (3) can be written similarly to (1), where the extended dimensions of $\boldsymbol{A}^{(\mathrm{nc})} \in \mathbb{C}^{2M \times N}$ can be interpreted as a virtual doubling of the sensor elements, which improves the estimation performance and doubles the number of detectable sources.

It can then be shown that if the array is shift-invariant, i.e.,

$$\boldsymbol{J}_1 \boldsymbol{A} \boldsymbol{\Phi} = \boldsymbol{J}_2 \boldsymbol{A}, \tag{6}$$

where J_1 and $J_2 \in \mathbb{R}^{M^{(sel)} \times M}$ are the selection matrices for the first and the second subarray, and $\Phi = \text{diag}\{[e^{j\mu_1}, \ldots, e^{j\mu_d}]\} \in \mathbb{C}^{d \times d}$ contains the spatial frequencies to be estimated, then $A^{(nc)}$ also possesses the shift invariance property

$$\boldsymbol{J}_{1}^{(\mathrm{nc})}\boldsymbol{A}^{(\mathrm{nc})}\boldsymbol{\Phi} = \boldsymbol{J}_{2}^{(\mathrm{nc})}\boldsymbol{A}^{(\mathrm{nc})},\tag{7}$$

where

$$\boldsymbol{J}_{1}^{(\mathrm{nc})} = \begin{bmatrix} \boldsymbol{J}_{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Pi}_{M^{(\mathrm{sel})}} \boldsymbol{J}_{2} \boldsymbol{\Pi}_{M} \end{bmatrix} \in \mathbb{R}^{2M^{(\mathrm{sel})} \times 2M}, \quad (8)$$

$$\boldsymbol{J}_{2}^{(\mathrm{nc})} = \begin{bmatrix} \boldsymbol{J}_{2} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Pi}_{M^{(\mathrm{sel})}} \boldsymbol{J}_{1} \boldsymbol{\Pi}_{M} \end{bmatrix} \in \mathbb{R}^{2M^{(\mathrm{sel})} \times 2M}.$$
(9)

Based on the noisy augmented data model (5), we estimate the signal subspace $\hat{U}_{s}^{(nc)} \in \mathbb{C}^{2M \times d}$ by computing the *d* dominant left singular vectors of $X^{(nc)}$. As $A^{(nc)}$ and $\hat{U}_{s}^{(nc)}$ span approximately the same column space, we can find a non-singular matrix $T \in \mathbb{C}^{d \times d}$ such that $A^{(nc)} \approx \hat{U}_{s}^{(nc)}T$. Using this relation, the shift invariance equation (7) can be expressed in terms of the estimated

signal subspace yielding

$$\boldsymbol{J}_{1}^{(\mathrm{nc})}\hat{\boldsymbol{U}}_{\mathrm{s}}^{(\mathrm{nc})}\boldsymbol{\Upsilon}\approx\boldsymbol{J}_{2}^{(\mathrm{nc})}\hat{\boldsymbol{U}}_{\mathrm{s}}^{(\mathrm{nc})}$$
(10)

with $\Upsilon = T \Phi T^{-1}$. Often, the unknown matrix Υ is estimated using least squares (LS), i.e.,

$$\hat{\mathbf{\Upsilon}} = \left(\boldsymbol{J}_1^{(\mathrm{nc})} \hat{\boldsymbol{U}}_{\mathrm{s}}^{(\mathrm{nc})} \right)^+ \boldsymbol{J}_2^{(\mathrm{nc})} \hat{\boldsymbol{U}}_{\mathrm{s}}^{(\mathrm{nc})} \in \mathbb{C}^{d \times d}, \tag{11}$$

where ⁺ stands for the Moore-Penrose pseudo inverse. Finally, the spatial frequency estimates are obtained by $\hat{\mu}_i = \arg\{\text{EV}_i\{\hat{\Upsilon}\}\}, i = 1, \ldots, d$, where $\text{EV}_i\{\hat{\Upsilon}\}$ is the *i*-th eigenvalue of $\hat{\Upsilon}$.

4. PERFORMANCE OF NC STANDARD ESPRIT

To obtain a first-order perturbation analysis of the subspace estimate, we adopt the analytical framework proposed in [4], which derives an explicit first-order error expansion assuming that a desired signal is distorted by an additive perturbation. This perturbation is deterministic and we only assume that it is small compared to the desired signal. Thus, no assumptions about its statistics such as Gaussianity or non-/circular symmetry are made. It is evident that the augmented measurement matrix in (5) does not affect the assumption that only a small noise perturbation is observed. Hence, we can apply the concept of [4] directly to the augmented measurement matrix in (5). The results are asymptotic in the effective SNR and explicit in the noise term $N^{(nc)}$. As in [4], we develop a first-order subspace error expansion to find a corresponding first-order expression for the parameter estimation error.

Based on (5), we can express the SVD of the noise-free observations $\boldsymbol{X}_{0}^{(\mathrm{nc})}$ as

$$\boldsymbol{X}_{0}^{(\mathrm{nc})} = \begin{bmatrix} \boldsymbol{U}_{\mathrm{s}}^{(\mathrm{nc})} & \boldsymbol{U}_{\mathrm{n}}^{(\mathrm{nc})} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Sigma}_{\mathrm{s}}^{(\mathrm{nc})} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{V}_{\mathrm{s}}^{(\mathrm{nc})} & \boldsymbol{V}_{\mathrm{n}}^{(\mathrm{nc})} \end{bmatrix}^{H},$$
(12)

where $U_{s}^{(nc)} \in \mathbb{C}^{M \times d}$, $U_{n}^{(nc)} \in \mathbb{C}^{M \times (M-d)}$, and $V_{s}^{(nc)} \in \mathbb{C}^{N \times d}$ span the signal subspace, the noise subspace, and the row space respectively, and $\Sigma_{s}^{(nc)} \in \mathbb{C}^{d \times d}$ contains the non-zero singular values on its diagonal. Next, we write the perturbed signal subspace estimate of $\hat{U}_{s}^{(nc)}$ from the previous section as $\hat{U}_{s}^{(nc)} = U_{s}^{(nc)} + \Delta U_{s}^{(nc)}$, where $\Delta U_{s}^{(nc)}$ denotes the estimation error.

From [4] and its application to (5), we get the first-order approximation

$$\Delta \boldsymbol{U}_{\mathrm{s}}^{(\mathrm{nc})} = \boldsymbol{U}_{\mathrm{n}}^{(\mathrm{nc})} \boldsymbol{U}_{\mathrm{n}}^{(\mathrm{nc})H} \boldsymbol{N}^{(\mathrm{nc})} \boldsymbol{V}_{\mathrm{s}}^{(\mathrm{nc})} \boldsymbol{\Sigma}_{\mathrm{s}}^{(\mathrm{nc})^{-1}} + \mathcal{O}\{\Delta^{2}\}, \quad (13)$$

where $\Delta = \|\boldsymbol{N}^{(nc)}\|$, and $\|\cdot\|$ represents an arbitrary norm. Equation (13) models the leakage of the noise subspace spanned by the columns of $\boldsymbol{U}_n^{(nc)}$ into the signal subspace due to the effect of the noise. The perturbation within the signal subspace $\boldsymbol{U}_s^{(nc)}$ provided in [16], [17] is not taken into account as it does not have an impact on the performance of NC standard ESPRIT.

For the estimation error of the i-th spatial frequency obtained by the LS solution in (11), we follow the lines of [4] to obtain

$$\Delta \mu_{i} = \operatorname{Im} \left\{ \boldsymbol{p}_{i}^{T} \left(\boldsymbol{J}_{1}^{(\mathrm{nc})} \boldsymbol{U}_{\mathrm{s}}^{(\mathrm{nc})} \right)^{+} \left[\boldsymbol{J}_{2}^{(\mathrm{nc})} / \lambda_{i} - \boldsymbol{J}_{1}^{(\mathrm{nc})} \right] \Delta \boldsymbol{U}_{\mathrm{s}}^{(\mathrm{nc})} \boldsymbol{q}_{i} \right\} + \mathcal{O}\{\Delta^{2}\},$$
(14)

where $\lambda_i = e^{j\mu_i}$ is the *i*-th eigenvalue of Υ , q_i represents the *i*-th eigenvector of Υ and the *i*-th column vector of the eigenvector

matrix Q, and p_i^T is the *i*-th row vector of $P = Q^{-1}$. Hence, the eigendecomposition of Υ is given by

$$\Upsilon = Q\Lambda Q^{-1},\tag{15}$$

where Λ contains the eigenvalues λ_i on its diagonal. Then, by inserting (13) into (14), we can write the first-order approximation for the estimation errors $\Delta \mu_i$ explicitly in terms of the noise perturbation $N^{(nc)}$.

In order to derive an analytical expression for the MSE of NC standard ESPRIT, we resort to [5]-[7], in which the authors prove that the MSE only depends on the SO statistics of the noise, i.e., the covariance matrix and the pseudo-covariance matrix, assuming the noise only to be zero-mean. Note that the physical noise N can be potentially non-circular. As the widely-linear transformation in (3) does not violate the zero-mean assumption, [5] is directly applicable once the corresponding SO statistics are found. Thus, if $n^{(nc)} = \text{vec}\{N^{(nc)}\} \in \mathbb{C}^{2MN \times 1}$, we state that for the covariance matrix $R_{nn}^{(nc)} = \mathbb{E}\{n^{(nc)}n^{(nc)^{H}}\} \in \mathbb{C}^{2MN \times 2MN}$ and the pseudo-covariance matrix $C_{nn}^{(nc)} = \mathbb{E}\{n^{(nc)}n^{(nc)^{T}}\} \in \mathbb{C}^{2MN \times 2MN}$, the MSE for the *i*-th spatial frequency is given by

$$\mathbb{E}\left\{\left(\Delta\mu_{i}\right)^{2}\right\} = \frac{1}{2}\left(\boldsymbol{r}_{i}^{(\mathrm{nc})^{H}}\boldsymbol{W}^{(\mathrm{nc})^{*}}\boldsymbol{R}_{\mathrm{nn}}^{(\mathrm{nc})^{T}}\boldsymbol{W}^{(\mathrm{nc})^{T}}\boldsymbol{r}_{i}^{(\mathrm{nc})}\right) - \operatorname{Re}\left\{\boldsymbol{r}_{i}^{(\mathrm{nc})^{T}}\boldsymbol{W}^{(\mathrm{nc})^{T}}\boldsymbol{W}^{(\mathrm{nc})^{T}}\boldsymbol{r}_{i}^{(\mathrm{nc})}\right\} + \mathcal{O}\{\Delta^{2}\},$$
(16)

where

$$oldsymbol{r}_i^{(
m nc)} = oldsymbol{q}_i \otimes \left(\left[\left(oldsymbol{J}_1^{(
m nc)} oldsymbol{U}_{
m s}^{(
m nc)}
ight)^+ \left(oldsymbol{J}_2^{(
m nc)} / \lambda_i - oldsymbol{J}_1^{(
m nc)}
ight)
ight]^T oldsymbol{p}_i
ight)$$

and

$$oldsymbol{W}^{(\mathrm{nc})} = \left(oldsymbol{\Sigma}_{\mathrm{s}}^{(\mathrm{nc})^{-1}} oldsymbol{V}_{\mathrm{s}}^{(\mathrm{nc})^{T}}
ight) \otimes \left(oldsymbol{U}_{\mathrm{n}}^{(\mathrm{nc})} oldsymbol{U}_{\mathrm{n}}^{(\mathrm{nc})^{H}}
ight)$$

In the next step, we derive the covariance matrix and the pseudo-covariance matrix of the augmented noise contribution $n^{(nc)} = \text{vec}\{N^{(nc)}\}$ needed in (16). To this end, we use the commutation matrix $K_{M,N}$ of size $MN \times MN$, which satisfies [18]

$$\mathbf{K}_{M,N} \cdot \operatorname{vec}\{\mathbf{A}\} = \operatorname{vec}\{\mathbf{A}^T\}$$
(17)

for arbitrary matrices $\boldsymbol{A} \in \mathbb{C}^{M \times N}$. We first expand $\boldsymbol{n}^{(\mathrm{nc})}$ as

$$\boldsymbol{n}^{(\mathrm{nc})} = \mathrm{vec}\{\boldsymbol{N}^{(\mathrm{nc})}\} = \mathrm{vec}\left\{\begin{bmatrix}\boldsymbol{N}\\\boldsymbol{\Pi}_{M}\boldsymbol{N}^{*}\end{bmatrix}\right\}$$
(18)

$$= \boldsymbol{K}_{2M,N}^{T} \begin{bmatrix} \operatorname{vec}\{\boldsymbol{N}\} \\ \operatorname{vec}\{(\boldsymbol{\Pi}_{M}\boldsymbol{N}^{*})^{T}\} \end{bmatrix}$$
(19)

$$= \mathbf{K}_{2M,N}^{T} \begin{bmatrix} \mathbf{K}_{M,N} \cdot \operatorname{vec}\{\mathbf{N}\} \\ \mathbf{K}_{M,N} \cdot \operatorname{vec}\{\mathbf{\Pi}_{M}\mathbf{N}^{*}\} \end{bmatrix}$$
$$= \underbrace{\mathbf{K}_{2M,N}^{T} (\mathbf{I}_{2} \otimes \mathbf{K}_{M,N})}_{\tilde{\mathbf{K}} \in \{\mathbf{\Pi}_{M}\mathbf{N}^{*}\}} \begin{bmatrix} \operatorname{vec}\{\mathbf{N}\} \\ \operatorname{vec}\{\mathbf{\Pi}_{M}\mathbf{N}^{*}\} \end{bmatrix}$$
$$= \widetilde{\mathbf{K}} \cdot \begin{bmatrix} \operatorname{vec}\{\mathbf{N}\} \\ \operatorname{vec}\{\mathbf{\Pi}_{M}\mathbf{N}^{*}\} \end{bmatrix}, \quad (20)$$

where \hat{K} is of size $2MN \times 2MN$ and we have applied property (17) to equation (18) and (19). By defining $n = \text{vec}\{N\} \in \mathbb{C}^{MN \times 1}$ and using the property $\text{vec}\{AXB\} = (B^T \otimes A) \cdot \text{vec}\{X\}$ for arbitrary matrices A, B, and X of appropriate sizes, we can formulate (20) as

$$\boldsymbol{n}^{(\mathrm{nc})} = \tilde{\boldsymbol{K}} \begin{bmatrix} \boldsymbol{n} \\ (\boldsymbol{I}_N \otimes \boldsymbol{\Pi}_M) \boldsymbol{n}^* \end{bmatrix}.$$
 (21)

This enables us to express the SO statistics of $n^{(nc)}$ by means of the covariance matrix $R_{nn} = \mathbb{E}\{nn^H\}$ and the pseudo-covariance matrix $C_{nn} = \mathbb{E}\{nn^T\}$ of the physically present noise component n. Thus, we get

$$\mathbf{R}_{nn}^{(nc)} = \mathbb{E}\{\mathbf{n}^{(nc)}\mathbf{n}^{(nc)^{H}}\}$$

$$= \tilde{K} \begin{bmatrix} \mathbf{R}_{nn} & \mathbf{C}_{nn}(\mathbf{I}_{N} \otimes \mathbf{\Pi}_{M}) \\ (\mathbf{I}_{N} \otimes \mathbf{\Pi}_{M})\mathbf{C}_{nn}^{*} & (\mathbf{I}_{N} \otimes \mathbf{\Pi}_{M})\mathbf{R}_{nn}^{*}(\mathbf{I}_{N} \otimes \mathbf{\Pi}_{M}) \end{bmatrix} \tilde{K}^{H}$$
(22)

and

$$C_{nn}^{(nc)} = \mathbb{E}\{\boldsymbol{n}^{(nc)}\boldsymbol{n}^{(nc)^{T}}\}$$

$$= \tilde{K} \begin{bmatrix} \boldsymbol{C}_{nn} & \boldsymbol{R}_{nn}(\boldsymbol{I}_{N} \otimes \boldsymbol{\Pi}_{M}) \\ (\boldsymbol{I}_{N} \otimes \boldsymbol{\Pi}_{M})\boldsymbol{R}_{nn}^{*} & (\boldsymbol{I}_{N} \otimes \boldsymbol{\Pi}_{M})\boldsymbol{C}_{nn}^{*}(\boldsymbol{I}_{N} \otimes \boldsymbol{\Pi}_{M}) \end{bmatrix} \tilde{K}^{T}.$$
(23)

In the special case of white Gaussian circularly symmetric noise with $R_{nn} = \sigma_n^2 I_{MN}$ and $C_{nn} = \mathbf{0}_{MN}$, (22) and (23) simplify to

$$\boldsymbol{R}_{nn}^{(nc)} = \sigma_n^2 \boldsymbol{I}_{2MN} \quad \text{and} \quad \boldsymbol{C}_{nn}^{(nc)} = \sigma_n^2 (\boldsymbol{I}_N \otimes \boldsymbol{\Pi}_{2M}).$$
 (24)

5. PERFORMANCE OF NC UNITARY ESPRIT

So far, we have only derived the explicit first-order estimation error approximation and the MSE expression for NC standard ESPRIT. In this section, we investigate the analytical performance of NC Unitary ESPRIT. The Unitary ESPRIT algorithm requires the sensor array to be centro-symmetric and shift-invariant, and it includes forward-backward-averaging (FBA) and the transformation into the real-valued domain as preprocessing steps [2], [19]. However, as the analytical expressions for NC Unitary ESPRIT are asymptotic in the SNR, it can be shown that the latter transformation has no impact on the performance at high SNRs. We omit this proof due to the space limitations. Hence, the asymptotic performance of NC Unitary ESPRIT is found once FBA has been taken into account.

FBA is performed by replacing the noise-corrupted NC measurement matrix $\mathbf{X}^{(nc)} \in \mathbb{C}^{2M \times N}$ by the augmented measurement matrix $\mathbf{X}^{(nc-fba)} \in \mathbb{C}^{2M \times 2N}$ defined by

$$\boldsymbol{X}^{(\mathrm{nc-fba})} = \begin{bmatrix} \boldsymbol{X}^{(\mathrm{nc})} & \boldsymbol{\Pi}_{2M} \boldsymbol{X}^{(\mathrm{nc})^*} \boldsymbol{\Pi}_N \end{bmatrix}.$$
(25)

The next result is stated in the following theorem:

Theorem 1. Applying FBA to the noisy augmented measurement matrix $X^{(nc)}$ does not improve the signal subspace estimate.

Proof. To show this result, we first expand (25) by using (3), which yields

$$\boldsymbol{X}^{(\text{nc-fba})} = \begin{bmatrix} \boldsymbol{X} & \boldsymbol{X} \boldsymbol{\Pi}_{N} \\ \boldsymbol{\Pi}_{M} \boldsymbol{X}^{*} & \boldsymbol{\Pi}_{M} \boldsymbol{X}^{*} \boldsymbol{\Pi}_{N} \end{bmatrix}.$$
 (26)

The Gram matrix $Z = X^{(nc-fba)} X^{(nc-fba)^{H}}$ is then given by

$$Z = \begin{bmatrix} X & X\Pi_N \\ \Pi_M X^* & \Pi_M X^* \Pi_N \end{bmatrix} \begin{bmatrix} X^H & X^T \Pi_M \\ \Pi_N X^H & \Pi_N X^T \Pi_M \end{bmatrix}$$
$$= 2 \cdot X^{(nc)} X^{(nc)^H}.$$
(27)

Thus, the matrices $X^{(nc-fba)}$ and $X^{(nc)}$ contain the same column spaces, which completes the proof.

As the real-valued transformation in NC Unitary ESPRIT only affects the performance at low SNRs, we can therefore conclude that the asymptotic performance of NC standard ESPRIT and NC Unitary ESPRIT for strictly SO non-circular sources is the same in the effective SNR. Next, we investigate the required properties of the array geometry for NC Unitary ESPRIT. Unitary ESPRIT is only applicable to shift-invariant antenna arrays with a centro-symmetric structure, i.e., the array is symmetric with respect to its centroid. However, for NC Unitary ESPRIT, we can formulate the theorem:

Theorem 2. Unlike Unitary ESPRIT, its extension NC Unitary ES-PRIT does not require a centro-symmetric array structure but only the shift invariance property (6).

Proof. An array is called centro-symmetric if its array steering matrix A_c satisfies [2]

$$\Pi_M A_c^* = A_c \Delta_c, \qquad (28)$$

where $\Delta_{c} \in \mathbb{C}^{d \times d}$ is a unitary diagonal matrix. Assuming that A satisfies (6) but not necessarily (28), we have

$$\Pi_{2M} \boldsymbol{A}^{(\mathrm{nc})^{*}} = \begin{bmatrix} \boldsymbol{0} & \Pi_{M} \\ \Pi_{M} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}^{*} \\ \Pi_{M} \boldsymbol{A} \boldsymbol{\Psi} \boldsymbol{\Psi} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \boldsymbol{\Psi} \\ \Pi_{M} \boldsymbol{A}^{*} \end{bmatrix}$$
$$= \begin{bmatrix} \boldsymbol{A} \\ \Pi_{M} \boldsymbol{A}^{*} \boldsymbol{\Psi}^{*} \boldsymbol{\Psi}^{*} \end{bmatrix} \boldsymbol{\Psi} \boldsymbol{\Psi} = \boldsymbol{A}^{(\mathrm{nc})} \boldsymbol{\Delta}_{\mathrm{c}}, \qquad (29)$$

which completes the proof as the centro-symmetry of A is not required.

This result shows that the virtually augmented array steering matrix always exhibits centro-symmetry even if the physical sensor array is not centro-symmetric. Hence, NC Unitary ESPRIT can be applied to a broader variety of array geometries.

6. SIMULATION RESULTS

In this section, we show simulation results to validate the asymptotic behavior of the presented performance analysis of NC standard ESPRIT and NC Unitary ESPRIT. We compare the results found analytically with the empirical estimation errors obtained by averaging over Monte Carlo trials. We employ a ULA consisting of M = 10isotropic sensor elements with interelement spacing $\delta = \lambda/2$ and assume that d = 3 sources with unit power and real-valued symbols drawn from a Gaussian distribution impinge on the array from the spatial frequencies $\mu_1 = 0.25, \mu_2 = 0.5$, and $\mu_3 = 0.75$. Moreover, we assume white Gaussian circularly symmetric sensor noise according to (24). The curves showing the root mean squared error (RMSE) of the empirical simulations ("emp") for NC standard ES-PRIT (NC SE) and NC Unitary ESPRIT (NC UE) were found by averaging over 10000 Monte Carlo trials. The square root of the analytical MSE expression (16) is denoted as ("ana"). Furthermore, we compare our results to standard ESPRIT (SE), Unitary ESPRIT (UE), and the deterministic Cramér-Rao bound for strictly SO noncircular sources [20].

Fig. 1 illustrates the RMSE versus the SNR, where N = 20, and the sources have a pair-wise correlation of $\rho = 0.9$ and a uniformly distributed correlation phase in the interval $[0, 2\pi]$. The noncircularity phases contained in Ψ are given by $\varphi_1 = 0$, $\varphi_2 = \pi/2$, and $\varphi_3 = \pi/4$. It is apparent that the analytical results agree with the empirical estimation errors as the SNR increases. This also validates that the asymptotic performance of NC standard ESPRIT and NC Unitary ESPRIT is the same as both coincide with the analytical curve at high SNRs.

In Fig. 2, we depict the RMSE versus the number of snapshots N, where the SNR is fixed at 20 dB and the sources are uncorrelated. The non-circularity phases are given by $\varphi_1 = 0$, $\varphi_2 = \pi/2$, and $\varphi_3 = \pi/8$. The empirical estimation errors of NC standard ESPRIT and NC Unitary ESPRIT match as the sample size increases. Moreover, our analytical results already agree with the empirical ones if N = 10 snapshots are available.



Fig. 1. Analytical and empirical RMSEs versus SNR for M = 10, N = 20, d = 3 correlated sources ($\rho = 0.9$) at $\mu_1 = 0.25, \mu_2 = 0.5, \mu_3 = 0.75$ with non-circularity phases $\varphi_1 = 0, \varphi_2 = \pi/2, \varphi_3 = \pi/4$.



Fig. 2. Analytical and empirical RMSEs versus the snapshots N for M = 10, SNR = 20 dB, d = 3 uncorrelated sources at $\mu_1 = 0.25$, $\mu_2 = 0.5$, $\mu_3 = 0.75$ with non-circularity phases $\varphi_1 = 0$, $\varphi_2 = \pi/2$, $\varphi_3 = \pi/8$.

The simulation results show that our analytical expressions become exact as the SNR or the number of snapshots becomes large.

7. CONCLUSION

In this paper, we have derived a first-order approximation of the analytical performance of 1-D NC standard ESPRIT and 1-D NC Unitary ESPRIT specifically designed for strictly SO non-circular sources. Our results are based on a first-order expansion of the estimation error, which is explicit in the noise perturbation and asymptotic in the effective SNR. We also find generic MSE expressions that only depend on the SO statistics of the noise and merely assume the noise to be zero-mean. Furthermore, we have proven that NC standard ESPRIT and NC Unitary ESPRIT have the same asymptotic performance in the effective SNR and that NC Unitary ESPRIT does not require a centro-symmetric array structure. Thus, it is preferable due to its lower complexity and better performance at low SNRs.

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