ON INTRAFERENCE AND ITS IMPLICATIONS IN COMPLEX-VALUED SIGNAL PROCESSING

C. Cheong Took[*], *S. C. Douglas*[†] *and D. Mandic*[‡]

[*]University of Surrey, Department of Computing,GU2 7XH, U.K.
[†] Southern Methodist University, Department of Electrical Engineering, TX 75275-0338, U.S.A
[‡]Imperial College London, Department of Electronic Engineering, SW7 2AZ, U.K.
Emails: c.cheongtook@surrey.ac.uk, douglas@engr.smu.edu, d.mandic@ic.ac.uk

ABSTRACT

We introduce the concept of intra-ference in order to quantify the degree to which the integrity of bivariate (or complex) sources is preserved in applications based on matrix decompositions of bivariate data. This is achieved by examining the pseudocovariance matrix of noncircular complex sources, and by recognising that the pseudocovariance is intrinsically complex valued. We illuminate how the existing decompositions such as the strong uncorrelating transform (SUT) not only decorrelate the bivariate sources from one another, but also decorrelate and scatter the data channels within each bivariate source, thus violating source integrity. Examples showing that the intra-ference arises due to the phase ambiguity in the existing matrix decompositions support the approach.

***Index Terms*—** intraference, pseudocovariance, augmented complex statistics, independent component analysis

1. INTRODUCTION

Many applications based on real- and complex-valued signals require matrix analysis enabled decompositions of multiple observations into the simpler original components. These original sources usually have an associated physical meaning, and so preserving their integrity after matrix manipulations or separation is a prerequisite for meaningful analysis. While for most matrix decompositions for real-valued data it is sufficient to ensure maximum decorrelation of the original components, it is often overlooked that bivariate decompositions of originals from their mixtures are not generic extensions of the real-valued tools. Yet current approaches typically both demix bivariate sources from one another, while at the same time decorrelating the constituent data channels within each separated bivariate source. In this way, not only the order of bivariate sources is not maintained, but also their constitutive data channels are not aligned. Given that the advances in sensor technology have enabled routine recordings of bivariate (and complex) sources, this has increasingly become a major problem in the processing of such signals, as for practical applications the integrity of every recovered source is paramount.

In addition, measures of statistical independence, or at the very least orthogonality of real sources, are routinely used to assess the performance of matrix decompositions in component separation, however, no such metric exists to gauge the integrity of the recovered bivariate (or complex) sources. For convenience, we address the problem in the complex domain, as due to the isomorphism between $\mathbb{C}$ and $\mathbb{R}^2$, the results naturally apply to bivariate real vectors. Complex statistics is characterised by both the standard covariance function $\mathbf{C} = E\{\mathbf{x}\mathbf{x}^H\}$, and the so-called pseudocovariance $\mathbf{P} = E\{\mathbf{x}\mathbf{x}^T\}$. Only their joint consideration allows for a rigorous treatment of real-world bivariate data, which typically exhibit power imbalance or correlation between the data channels within each bivariate (complex) source.

Current matrix decompositions for complex sources typically make use of Takagi's factorisation; the classic result in [1] has been adopted into signal processing through the work of De Lauthawer [2] and Eriksson [3], who coined the term the strongly uncorrelating transform (SUT). More recently, it has been shown that SUT can be replaced by a simpler approximate uncorrelating transform (AUT) [4] for noncircular data (strong intrinsic correlation between the real and imaginary part of the complex signal), while an adaptive version of SUT was proposed in [5]. However, none ensures that the integrity - the order of the data channels within a recovered bivariate source and their degree of correlation - will be preserved.

In this work, we introduce the concept of intra-ference, as a statistical measure of the amount of decorrelation between the two constituent channels within a recovered bivariate source. This is a first and much needed step forward to preserving source integrity, as existing techniques not only permute the order of bivariate sources, but also scatter and decorrelate the data channels within each source. We show that the principal source of intra-ference is the incoherent treatment of the phase information within the pseudocovariance, which is inherently complex-valued. The analysis illuminates how the error caused by intra-ference can be quantified in the context of source separation, while a link between the pseudocovariance and the degree of intra-ference is also established and validated through simulations.

2. BIVARIATE MATRIX DECOMPOSITIONS

Consider the following mixing/unmixing model:

$$\begin{aligned} \text{Mixing:} \quad \mathbf{x}(n) &= \mathbf{A}\mathbf{s}(n) & (1)\\ \text{Unmixing:} \quad \mathbf{y}(n) &= \mathbf{B}\mathbf{x}(n) = \mathbf{B}\mathbf{A}\mathbf{s}(n) & (2)\\ &= \mathcal{P}\mathbf{D}\mathbf{s}(n) = \mathcal{P}\mathbf{\Delta}\mathbf{\Lambda}\mathbf{s}(n) & (3) \end{aligned}$$

where $\mathbf{x}(n)$, $\mathbf{s}(n)$ and $\mathbf{y}(n)$ are respectively the vectors of observed (mixed), original, and recovered bivariate signals, mixing matrix $\mathbf{A}$ models linear mixing, and $\mathbf{B} \sim \mathbf{A}^{-1}$ is the unmixing matrix. In the real domain, both the permutation matrix $\mathcal{P}$ and the scaling matrix $\mathbf{D}$ are trivial ambiguities which can be overlooked in the analysis. In the complex domain, the situation is different, as physically the Euler representation of $\mathbf{D} = \mathbf{\Delta}\mathbf{\Lambda}$, where the diagonal elements $diag(\mathbf{\Delta}) = \delta_{ii} = \exp(\imath\theta_i)$ and $diag(\mathbf{\Lambda}) = \lambda_{ii} = |d_{ii}|$, models the desired phase and magnitude information (and hence the integrity of sources). In other words, from (2), the unmixed output

$$y \quad \propto \quad \delta s = \big(\delta_R s_R - \delta_I s_I\big) + j\big(\delta_R s_I + \delta_I s_R\big) \qquad (4)$$

where δ models the complex-valued scaling ambiguity. Observe that y is a poor estimate of s, since the real part y_R is composed of the real and imaginary parts of s; similar observation applies for the imaginary part y_I. This problem was highlighted in (pp. 384-385 [6] and [7]) stating that it is not possible to solve this phase ambiguity.

Remark$\#1$: The phase ambiguity in the matrix $\mathbf{\Delta}$ is the source of intrinsic mixing within recovered bivariate sources.

3. THE INTRA-FERENCE ANALYSIS

We next introduce the intra-ference and state its properties based on the standard small error assumption [1]. The analysis is motivated by [8], however, our work focuses on the intra-ference in (4), rather than the interference in [8].

3.1. The Error Model

The sources of error in bivariate ICA are: (i) the imperfection of the mixing-unmixing model in (2), (ii) mis-specification of the statistics (assumption of real-valued pseudocovariance). We can express these uncertainties through the error models

$$\mathbf{BA} = \mathbf{I} + \mathbf{\Phi} \qquad \hat{\mathbf{C}} = \mathbf{I} + \boldsymbol{\xi}_c \qquad \hat{\mathbf{P}} = \mathbf{P} + \boldsymbol{\xi}_p \qquad (5)$$

where $\mathbf{\Phi}$ is the "intra-ference matrix", while $\boldsymbol{\xi}_c$ and $\boldsymbol{\xi}_p$ are the errors in estimating the covariance and the pseudocovariance matrices. This allows us to re-formulate the covariance and pseudocovariance of the estimated ICs in the form

$$(\mathbf{I}+\mathbf{\Phi})\hat{\mathbf{C}}(\mathbf{I}+\mathbf{\Phi}^H) = \mathbf{I} \qquad (\mathbf{I}+\mathbf{\Phi})\hat{\mathbf{P}}(\mathbf{I}+\mathbf{\Phi}^T) = \tilde{\mathbf{P}} \quad (6)$$

where $\tilde{\mathbf{P}}$ denotes the pseudocovariance matrix estimated by SUT or AUT. This yields the following approximations

$$\begin{aligned} \mathbf{I} &\approx \mathbf{I} + \mathbf{\Phi} + \mathbf{\Phi}^H + \boldsymbol{\xi}_C \\ \boldsymbol{\xi}_C &\approx -(\mathbf{\Phi} + \mathbf{\Phi}^H) & (7)\\ \tilde{\mathbf{P}} &\approx \mathbf{P} + \mathbf{\Phi}\mathbf{P} + \mathbf{P}\mathbf{\Phi}^T + \boldsymbol{\xi}_P \\ \tilde{\boldsymbol{\xi}}_P &\approx -(\mathbf{\Phi}\mathbf{P} + \mathbf{P}\mathbf{\Phi}^T) & (8) \end{aligned}$$

where the term $\tilde{\boldsymbol{\xi}}_P = \boldsymbol{\xi}_P - (\tilde{\mathbf{P}} - \mathbf{P})$ is related to the degree of the intra-ference, and is used in the analysis in Section 3.3.

3.2. Definition of Intra-ference

Using the error model in (5) and assuming no permutation, we define the intra-ference for the ith bivariate source through the deviation of diagonal term $[i,i]$ of $\mathbf{BA}$ from unity

$$E\{\Phi^2[i,i]\} = E\{|(\mathbf{BA})[i,i] - 1|^2\} \qquad (9)$$

In practice, to circumvent the permutation ambiguity, sources are sorted in a descending order of the absolute value of their pseudocovariance (noncircularity). This is logical because singular value decomposition sorts the independent components in a descending order of their singular values, when Takagi factorisation is employed.

3.3. Analysis of the Intra-ference

To derive the intra-ference $\mathbf{\Phi}[i,i]$ from (7)-(8), consider all ith diagonal elements of the matrices in (7)-(8), to give

$$\boldsymbol{\varepsilon}_i \approx -\mathbf{\Psi}_i\boldsymbol{\theta}_i \qquad (10)$$

$$\begin{bmatrix} \varepsilon_{CR}[i,i] \\ \tilde{\varepsilon}_{PR}[i,i] \\ \tilde{\varepsilon}_{PI}[i,i] \end{bmatrix} \approx - \begin{bmatrix} 1 & 1 & 0 \\ P_R[i,i] & 0 & -P_I[i,i] \\ P_I[i,i] & 0 & P_R[i,i] \end{bmatrix} \begin{bmatrix} \Phi_R[i,i] \\ \Phi_R[i,i] \\ \Phi_I[i,i] \end{bmatrix}$$

The vector $\boldsymbol{\varepsilon}_i$ can be rewritten as complex-valued

$$\boldsymbol{\varepsilon}_i = \frac{1}{2}\mathbf{\Omega}\boldsymbol{\epsilon}_i$$

$$\begin{bmatrix} \varepsilon_{CR}[i,i] \\ \tilde{\varepsilon}_{PR}[i,i] \\ \tilde{\varepsilon}_{PI}[i,i] \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -j & \imath \end{bmatrix} \begin{bmatrix} \xi_C[i,i] \\ \tilde{\xi}_P[i,i] \\ \tilde{\xi}_P^*[i,i] \end{bmatrix} \qquad (11)$$

As $E\{\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i^T\} = \frac{1}{4}\mathbf{\Omega}E\{\boldsymbol{\epsilon}_i\boldsymbol{\epsilon}_i^H\}\mathbf{\Omega}^H$, we have[1]

$$E\{\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i^T\} = \begin{bmatrix} \xi_C^2 & \xi_C\tilde{\xi}_{PR} & \xi_C\tilde{\xi}_{PI} \\ \xi_C\tilde{\xi}_{PR} & \tilde{\xi}_{PR}^2 & \tilde{\xi}_{PR}\tilde{\xi}_{PI} \\ \xi_C\tilde{\xi}_{PI} & \tilde{\xi}_{PR}\tilde{\xi}_{PI} & \tilde{\xi}_{PI}^2 \end{bmatrix} \qquad (12)$$

which can be used to compute the covariances of interest, that is, $E\{\boldsymbol{\theta}_i\boldsymbol{\theta}_i^T\} = \mathbf{\Psi}_i^{-1}E\{\boldsymbol{\varepsilon}_i\boldsymbol{\varepsilon}_i^T\}\mathbf{\Psi}_i^{-T}$, where

$$\mathbf{\Psi}_i^{-1} = \frac{-1}{P_R^2[i,i] + P_I^2[i,i]} \times$$

$$\begin{bmatrix} 0 & P_R[i,i] & P_I[i,i] \\ -(P_R^2[i,i] + P_I^2[i,i]) & -P_R[i,i] & -P_I[i,i] \\ 0 & -P_I[i,i] & P_R[i,i] \end{bmatrix} \qquad (13)$$

[1] For clarity, in the sequel we omit the index $[i,i]$.

To measure the intra-ference, only the knowledge of $E\{\Phi_R^2[i,i]\}$ and $E\{\Phi_I^2[i,i]\}$ is required (just the diagonal elements [1,1] and [3,3] of $E\{\boldsymbol{\theta}_i\boldsymbol{\theta}_i^T\}$), this is achieved by

$$E\{\Phi_R^2\} = \frac{\left[P_R\tilde{\xi}_{PR} + P_I\tilde{\xi}_{PI}\right]^2}{\left[P_R^2 + P_I^2\right]^2} \quad \text{when } \xi_C = 0 \quad (14)$$

$$E\{\Phi_I^2\} = \frac{\left[P_R\tilde{\xi}_{PI} - P_I\tilde{\xi}_{PR}\right]^2}{\left[P_R^2 + P_I^2\right]^2} \quad (15)$$

The intra-ference can now be defined as

$$\textbf{Intraference}: E\{\Phi^2[i,i]\} = E\{\Phi_R^2[i,i]\} + E\{\Phi_I^2[i,i]\}$$
$$E\{\Phi^2[i,i]\} = E\{|\tilde{\xi}_P|^2\}/|P[i,i]|^2 \quad (16)$$

This simplification for $E\{\Phi_R^2\}$ in (14) is valid, as the mis-specification the covariance error $\xi_C[i,i] = 1-(\sum_N s_i s_i^*)/N$ can be overlooked due to the scaling ambiguity.

Remark$\#$2: The phase ambiguity in the demixing of complex sources arises due to the error in the estimation of the pseudocovariance matrix and is not related to the errors in the estimation of the covariance matrix.

Remark$\#$3: The intra-ference, as defined in (16), arises from both the mis-specification error of the pseudocovariance of the ith independent component and the estimation error during the demixing process.

To elaborate on the significance of Remark#1, consider a perfect scenario where the estimated singular values of the independent components correspond to their actual values, that is, $|P[i,i]| = |\hat{P}[i,i]|$. Then, the error $\tilde{\xi}_P[i,i]$ can be expressed in the Euler form as

$$\tilde{\xi}_P[i,i] \approx \big|P[i,i]\big| E\big\{\big[\exp(j\theta_i) - \exp(j\hat{\theta}_i)\big]\big\} \quad (17)$$

where the angle $\hat{\theta}$ is obtained from the (complex valued) pseudocovariance matrix of the estimated independent components. This allows us to simplify (16) into

$$E\{\Phi^2[i,i]\} \approx E\{|\exp(j\theta_i) - \exp(j\hat{\theta}_i)|^2\} \quad (18)$$

which illustrates that the intra-ference measure is invariant to the magnitude of the pseudocovariance when $|P[i,i]| = |\hat{P}[i,i]|$. However, in practice $|P[i,i]| \neq |\hat{P}[i,i]|$ and therefore the expression in (16) is better suited to measure the intra-ference.

Remark$\#$4: For the standard SUT, $\hat{\theta}_i = 0$ always holds. Thus, if $\theta_i \neq 2\pi i$, for any integer i, the SUT is unconditionally biased, since in this case the intra-ference is given by $E\{\Phi^2[i,i]\} \approx 2[1 - E\{\cos(\theta_i)\}] \neq 0$.

4. SIMULATIONS

The analysis in (16)-(18) shows that the intraference becomes prominent when the associated phase of a source is not an integer of π. Therefore, in the simulations our aim was to illustrate the effect of intraference on signals with nonvanishing phase angle, a typical case in practical applications.

To that end, we controlled the degree of correlation, ρ, between the real and imaginary part of a complex-Gaussian zero mean signal $x = x_R + jx_I$, thus giving rise to a complex-valued pseudocovariance. This was achieved by combining two zero mean uncorrelated variables x_1 and x_2 as

$$x_R(t) = x_1(t), \quad x_I(t) = \rho x_1(t) + x_2(t)\sqrt{1-\rho^2} \quad (19)$$

In this way, a full range of correlation between x_R and x_I was introduced, ranging from uncorrelated ($\rho = 0$) to fully correlated ($\rho = 1$).

The effects of intra-ference were studied comprehensively over two case studies: (a) performance for a varying number of sources, and (b) performance under additive noise. To ensure that the complex-valued signal is noncircular and is statistics contains both the variance c and pseudocovariance $p = p_R + jp_I$ (where $|p_R| + |p_I| = 1$ for consistency), the correlation coefficient ρ was calculated via the pseudocovariance, as

$$\rho = p_I/(2E\{x_1^2\}) \quad (20)$$

since the pseudocovariance $p = E\{x_R^2 - x_I^2\} + 2jE\{x_R x_I\}$. For rigour the variances of the real and the imaginary parts have to be chosen in such a way so as to satisfy the constraint $p \leq c$, which has to be reflected in the powers of the real and imaginary channel, through $p_R = E\{x_R^2 - x_I^2\}$ and $c = E\{x_R^2 + x_I^2\}$. In our experiments, we set the covariance to $c = 1.1$; then for example, if $p = -0.1 + j0.9$, representing a phase angle of 0.45 radians, the following properties of the bivariate channels follow:

$$E\{x_R^2\} = E\{x_1^2\} = 0.5, \quad E\{x_I^2\} = 0.6$$
$$E\{x_2^2\} = \frac{1.1 - E\{x_1^2\}(1+\rho^2)}{1-\rho^2} \quad (21)$$

The unit variance was not selected in the simulations, in order to avoid singular cases: for instance, when the pseudocovariance is to $p = -0.1 + j0.9$. In this case, the conditions $E\{x_R^2\} = 0.45$ and $E\{x_I^2\} = 0.55$ need to be satisfied so that $p_R = -0.1$, however, this also means that $E\{x_2^2\} \to \infty$ in (21), as $\rho = 1$ - clearly, $E\{x_2^2\} \to \infty$ cannot be implemented.

Finally, the sources were generated from the MA model

$$y(n) = x(n) + 0.9x(n-1) + 0.95x(n-2) \quad (22)$$

with real coefficients, as a complex filtering process would unnecessarily alter the statistics of the signal x. The degree of intra-ference presented in the simulations was taken as the mean value, that is, $\frac{1}{N}\sum_{i=1}^{N}\Phi^2[i,i]$, where the $\Phi^2[i,i]$ was calculated from (9). In the simulations, the theoretical value of intra-ference (broken line) was obtained as a first order approximation of the actual intraference, and follows very well the dynamics of the actual intraference.

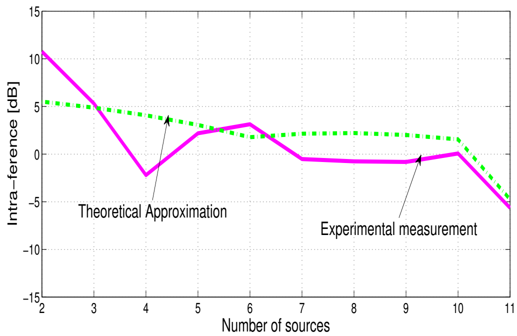

Fig. 1. Intraference as a function of the number of sources.

4.1. Performance analysis for a varying number of sources

In the first set of simulations, the source separation problem was considered beyond the 2×2 case. Fig. 1 illustrates that the performance of SUT was unconditionally biased, highlighting the problems arising from not addressing the intra-ference. The theoretical approximation of the intra-ference followed quite closely the actual intra-ference; the difference between the theoretical approximation and the actual intra-ference decreased as the number of sources increased. The estimation of the phase angle for the ith source was calculated by taking the average of the ith column of the inverse of the unmixing matrix; this average improved as the number of elements (sources) within the ith column increased, leading to a decrease in the error. The theoretical intra-ference was approximated using the small error assumption, and approached the actual intraference as the number of sources increased.

4.2. Performance analysis in the presence of noise

Fig. 2 illustrates the robustness of the intra-ference measure in the presence of noise, over a range of signal to noise ratios (SNR), for white Gaussian noise, highlighting the inadequacy of bivariate source separation when not taking into account the phase ambiguity. The discrepancy between the theoretical approximation and the actual intra-ference, even at high signal-to-noise ratios, can be explained by the small error assumption in the theoretical approximation. The general trend in these experiments indicates zero intra-ference, because the variances of the real and imaginary parts of the sources were not significantly different.

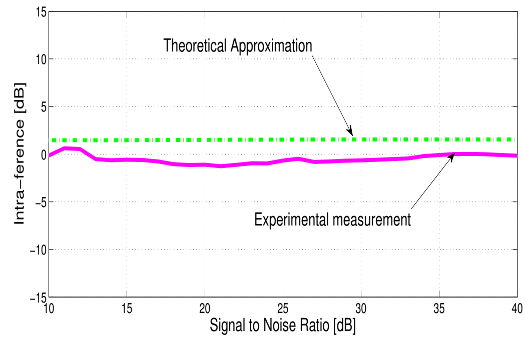

Fig. 2. Performance in terms of signal-to-noise ratio.

5. CONCLUSIONS

We have shown that the physical meaning of the phase error in independent component analysis of bivariate signals corresponds to the inadequacy of current matrix decomposition techniques to perform a correct estimate of the pseudocovariance of complex valued sources. We have shown that phase ambiguity manifests itself in the mixing between the real and imaginary parts of the reconstructed source. To quantify this phenomenon, we have introduced the concept of intra-ference between the real and imaginary parts and have proposed theoretical estimators of intra-ference. Simulations, under the small error assumption, have validated the analysis, and have highlighted the caveats in current techniques not considering the complex-valued nature of the pseudocovariance.

6. REFERENCES

[1] R. A. Horn, *Matrix Analysis*, Cambridge University Press, 1990.

[2] L. De Lathauwer and B. De Moore, "On the blind separtion of non-circular source," *in Proceedings of the 11th European Signal Processing Conference (EUSIPCO2002)*, vol. 2, pp. 99–102, Sep. 2002.

[3] J. Eriksson and V. Koivunen, "Complex Random Vectors and ICA Models: Identifiability, Uniqueness, and Separability," *IEEE Transactions on Information Theory*, vol. 52, no. 3, pp. 1017–1029, 2006.

[4] C. Cheong Took, S. C. Douglas, and D. P. Mandic, "On Approximate Diagonalisation of Correlation Matrices in Widely Linear Signal Processing," *IEEE Transactions on Signal Processing*, vol. 60, no. 3, pp. 1469 – 1473, 2012.

[5] S. C. Douglas, J. Eriksson, and V. Koivunen, "Adaptive estimation of the strong uncorrelating transform with applications to subspace tracking," *IEEE International Conference on Acoustics, Speech and Signal Processing*, pp. 941–944, 2006.

[6] A. Hyvärinen, J. Karhunen, and E. Oja, *Independent Component Analysis*, John Wiley & Sons, INC, 2001.

[7] P. Comon, "Independent Component Analysis, a New Concept?," *Signal Processing*, vol. 36, no. 3, pp. 287–314, 1994.

[8] A. Yeredor, "Performance analysis of the strong uncorrelating transformation in blind separation of complex-valued sources," *IEEE Transactions on Signal Processing*, vol. 60, no. 1, pp. 478–483, 2012.