TIME DELAY ESTIMATION FOR TDOA SELF-CALIBRATION USING TRUNCATED NUCLEAR NORM REGULARIZATION

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ABSTRACT

Measurements with unknown time delays are common in different applications such as microphone array, radio antenna array calibration, where the sources (e.g. sounds) are transmitted in unknown time instants. In this paper, we present a method for estimating unknown time delays from Time-Difference-of-Arrival (TDOA) measurements. We propose a novel rank constraint on a matrix depending on the measurements and the unknown time delays. The time delays are recovered by solving a truncated nuclear norm minimization problem using alternating direction method of multipliers (ADMM). We show in synthetic experiments that the proposed method recovers the time delays with good accuracy for noisy and missing data.

***Index Terms*—** Time Delay Estimation, Time-Difference-of-Arrival, Nuclear Norm, Self-Calibration

1. INTRODUCTION

Localization based on sound, ultra-sound, radio or signal strength has been studied in numerous applications (beamforming, speaker tracking, crime fighting, indoor localization etc). However, most previous works have been focused on localization of e.g. a sound source using a calibrated detector array, i.e. the positions of either transmitters or receivers are assumed to be known or are measured via external devices [1–5]. In most settings, both the positions of the transmitters and receivers could be difficult to obtain, and the transmitters are not synchronized. In this paper, we focus on the time-difference-of-arrival (TDOA) self-calibration problem, i.e. the problem of determining the positions of a number of receivers and transmitters as well as the unknown time delay for each transmitter, given all receiver-transmitter distances. This is relevant in common settings e.g. microphone arrays, given recordings of sounds (unsynchronized) emitted at unknown locations, to microphones at unknown positions, determine both sound emission positions and microphone locations. Similar scenarios are encountered also for other types of measurements e.g. ultra-sound or radio. Manual synchronization for transmitters can be both difficult and expensive in general. Therefore, efficient and accurate estimation of time delays using only TDOA measurements is of interest.

In this paper we present a method for time delay estimation of TDOA network calibration problem for general dimensions. We utilize a rank constraint on a matrix that is linear in the unknown time delays and propose a new method to estimate the time delays by minimizing a cost function, using nuclear norm minimization. As the rank function is non-convex and discontinuous, some previous work [6] shows that the nuclear norm is the best convex approximation of the rank of a matrix over the unit ball of matrices with norm less than one. So an alternative approach to the rank minimization is to minimize its nuclear norm. However, it is shown in [7] that minimizing the nuclear norm will in general minimize all the singular values simultaneously, which deviates from the goal to minimize the number of non-zero singular values, i.e. the rank of a matrix. To better approximate the rank, one should instead minimize the sum of k smallest singular values by adding a regularizer in the nuclear norm minimization. Singular value thresholding in [8] is a simple yet effective way based on the nuclear norm heuristics and we use that as a subroutine in our optimization scheme.

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2. THE TIME DELAY SELF-CALIBRATION PROBLEM

Let $\{\mathbf{r}_i\}$, $i = 1, \ldots, m$ and $\{\mathbf{s}_j\}$, $j = 1, \ldots, n$ be the spatial coordinates of m receivers and n transmitters, respectively. For measured time of arrival t_{ij} from transmitter $\mathbf{r}_i$ and receiver $\mathbf{s}_j$, we have $v(t_{ij} - t_j) = \|\mathbf{r}_i - \mathbf{s}_j\|_2$, where t_j is the unknown time delay for each transmitter, and v is the speed of measured signals (assumed to be constant). We will in the sequel work with the distance measurements $d_{ij} = vt_{ij}$ and time delays $o_j = vt_j$. The TDOA and TOA calibration problems can then be defined as follows.

Problem 1 *(Time Delay Estimation for Time-Difference-of-Arrival Network) Given relative distance measurements f_{ij} determine the time delays $o_j, j = 1, \ldots, n$ for unknown receiver positions $\mathbf{r}_i$, $i = 1, \ldots, m$ and transmitter positions $\mathbf{s}_j$, $j = 1, \ldots, n$ such that $f_{ij} = \|\mathbf{r}_i - \mathbf{s}_j\|_2 + o_j$.*

Problem 2 *(Time-of-Arrival Network Calibration) Given absolute distance measurements d_{ij} determine receiver positions $\mathbf{r}_i$, $i = 1, \ldots, m$ and transmitter positions $\mathbf{s}_j$, $j = 1, \ldots, n$ such that $d_{ij} = \|\mathbf{r}_i - \mathbf{s}_j\|_2$.*

In this paper, we focus on solving the time delays as a separate problem to reconstructing the sensor locations i.e. $\{\mathbf{r_i}\}$ and $\{\mathbf{s_j}\}$. Once the time delays $\{o_j\}$ are known or estimated, we can solve the TDOA problems by converting them to TOA problems i.e. by setting $d_{ij} = f_{ij} - o_j$. This strategy is also utilized in [9], where a linear factorization scheme is used to recover the time delays first and the remaining TOA problem is solved independently using [9, 10]. We assume in the following discussion that the dimensionality of the affine spaces spanned respectively by $\{\mathbf{r}_i\}$ and $\{\mathbf{s}_j\}$ are the same and it is denoted by K, for transmitters and receivers in general 3D positions, one has $K = 3$. To understand the intrinsic properties of the TDOA problem, it is important to understand the minimal number of measurements needed to solve the problem. These minimal configurations have previously been determined in [11]. It is relatively straightforward to derive such configurations that satisfy the number of degrees of freedom in the measurements, mn is equal to the number of degrees of freedom in the manifold of unknown parameters, e.g. $4n+3m-6$ for TDOA. In 3D space, TDOA problem has four minimal problems, i.e. $10r/4s$, $7r/5s$, $6r/6s$ and $5r/9s$, cf. [11]. Here we use r and s to denote receivers and transmitters, respectively.

2.1. Time Delays in TDOA

To solve the time delay estimation problem for TDOA measurements, we first introduce the rank constraint on the time delays. Since $(d_{ij})^2 = (f_{ij} - o_j)^2 = (\mathbf{r}_i - \mathbf{s}_j)^T(\mathbf{r}_i - \mathbf{s}_j) = \mathbf{r}_i^T\mathbf{r}_i - 2\mathbf{r}_i^T\mathbf{s}_j + \mathbf{s}_j^T\mathbf{s}_j$, we can construct the vectors

$$\mathbf{R}_i = \begin{bmatrix}1 & \mathbf{r}_i^T & \mathbf{r}_i^T\mathbf{r}_i\end{bmatrix}^T \text{ and}$$
$$\mathbf{S}_j = \begin{bmatrix}\mathbf{s}_j^T\mathbf{s}_j - o_j^2 & -2\mathbf{s}_j^T & 1\end{bmatrix}^T.$$

This gives $f_{ij}^2 - 2f_{ij}o_j = \mathbf{R}_i^T\mathbf{S}_j$. By collecting $\mathbf{R}_i$ and $\mathbf{S}_j$ into matrix $\mathbf{R}$ ($(K+2)\times m$) and $\mathbf{S}$ ($(K+2)\times n$), we have $\mathbf{F} = \mathbf{R}^T\mathbf{S}$, where $\mathbf{F}$ is a matrix in $\mathbb{R}^{m\times n}$ containing $\{f_{ij}^2 - 2f_{ij}o_j\}$. This suggests that matrix $\mathbf{F}$ is at most of rank $K+2$ as we increase m and n. Using this rank constraint and the structure of $\mathbf{R}$, one can exploit linear technique (factorization) to solve for the unknown time delays. This linear technique is proposed in [9] and requires $m = 2(K+2)$ and $n = (K+2)$ measurements, e.g. for $K=3$, one needs $m=10$ receivers and $n=5$ transmitters.

Here we present a slight modification to the linear scheme in [9]. The idea is to exploit the structure of $\mathbf{S}$ and $\mathbf{R}$. To see this, we first multiplying $\mathbf{F}$ from the right by a $n\times(n-1)$ matrix $\mathbf{C}_n$ of the form $[-\mathbf{1}_{n-1}\ \mathbf{I}_{n-1}]^T$ where $\mathbf{1}_{n-1}$ is a $(n-1)\times 1$ vector with all ones and $\mathbf{I}_{n-1}$ is an $(n-1)\times(n-1)$ identity matrix. This operation subtracts the first column from each column j ($j\geq 2$) of $\mathbf{S}$ and gives a matrix with all zeros at the last row. Equivalently, this gives $\bar{\mathbf{F}} = \mathbf{F}\mathbf{C}_n = \bar{\mathbf{R}}^T\bar{\mathbf{S}}$, where $\bar{\mathbf{F}}$ is a matrix in $\mathbb{R}^{m\times(n-1)}$ with entries $\bar{f}_{ij} = f_{i,j+1}^2 - f_{i1}^2 - 2f_{i,j+1}o_{j+1} + 2f_{i1}o_1$, $\bar{\mathbf{R}}_i = \begin{bmatrix}1 & \mathbf{r}_i^T\end{bmatrix}^T$ and $\bar{\mathbf{S}}_j = \begin{bmatrix}\mathbf{s}_{j+1}^T\mathbf{s}_{j+1} - o_{j+1}^2 - (\mathbf{s}_1^T\mathbf{s}_1 - o_1^2) & -2(\mathbf{s}_{j+1}-\mathbf{s}_1)^T\end{bmatrix}^T$. Since the matrix $\mathbf{C}_n$ removes rows in both $\mathbf{S}$ and $\mathbf{R}$, we call it a *compaction matrix*. This effectively gives a constraint that the matrix $\bar{\mathbf{F}}$ is at most of rank $K+1$. As we have exploited the row of ones in $\mathbf{S}$, we can further utilize the same structure in $\mathbf{R}$. By multiplying $\bar{\mathbf{F}}$ from the left a compaction matrix $\mathbf{C}_m$, correspondingly we have $\hat{\mathbf{F}} = \mathbf{C}_m^T\mathbf{F}\mathbf{C}_n = \hat{\mathbf{R}}^T\hat{\mathbf{S}}$. Here $\hat{\mathbf{R}} = \mathbf{R}\mathbf{C}_m$ and $\hat{\mathbf{S}} = \mathbf{S}\mathbf{C}_n$ and $\hat{\mathbf{F}}$ is in $\mathbb{R}^{(m-1)\times(n-1)}$. It can be easily seen that the first row of $\hat{\mathbf{R}}$ and the last row of $\hat{\mathbf{S}}$ are all zeros. This suggests that further removing the last row of $\hat{\mathbf{R}}$ and the first row of $\hat{\mathbf{S}}$ will preserve $\hat{\mathbf{F}} = \hat{\mathbf{R}}^T\hat{\mathbf{S}}$. We then have $\hat{\mathbf{R}}_i = \begin{bmatrix}(\mathbf{r}_{i+1}-\mathbf{r}_1)^T\end{bmatrix}$, $\hat{\mathbf{S}}_j = \begin{bmatrix}-2(\mathbf{s}_{j+1}-\mathbf{s}_1)^T\end{bmatrix}$ and entries in $\hat{\mathbf{F}}$ are

$$\hat{f}_{ij} = g_{ij} - g_{0j} - g_{i0} + g_{00}, \tag{1}$$

where $g_{ij} = f_{i+1,j+1}^2 - 2f_{i+1,j+1}o_{j+1}$. It is clear that the matrix $\hat{\mathbf{F}}$ is at most of rank K. The formulation and optimization we will describe in the following can be generalized to any dimension K. However, for clarity, we will use the K = 3 case, i.e. all transmitters and receivers are in a 3D space.

3. PROBLEM FORMULATION

For receivers and transmitters in general 3D space, we know from Section 2.1 that the modified measurement matrix $\hat{\mathbf{F}}$ is of rank 3. When we have over-determined case, we can formulate the following optimization problem for finding the time delays $\mathbf{o} = \{o_j\}$

$$\min_{\mathbf{o},\mathbf{X}} \quad \|\mathbf{X}\|_*, \qquad \text{s.t.} \quad \mathbf{B}_0 + \sum_{j=1}^{n} o_j\mathbf{B}_j = \mathbf{X}. \tag{2}$$

Here $\|.\|_*$ is the nuclear norm of a matrix defined as $\|X\|_* = \sum_{i=1}^{min(m,n)}\sigma_i$, where σ_i is the i^{th} singular value of a matrix X in $\mathbb{R}^{m\times n}$. $\mathbf{B}_0,\ldots,\mathbf{B}_n$ are constant matrices in $\mathbb{R}^{(m-1)\times(n-1)}$ derived from (1). Specifically, $\mathbf{B}_0 = \mathbf{C}_m^T\mathbf{F}\mathbf{C}_n$, $\mathbf{B}_1 = (\mathbf{b}_1,\ldots,\mathbf{b}_1)$ where $\mathbf{b}_1 = 2(\{f_{i1} - f_{11}\}_{i\geq 2})$ and $\mathbf{B}_j$ ($j\geq 2$) is a matrix with $(n-2)$ columns of zeros and the j^{th} column is $-2\{f_{ij} - f_{1j}\}_{i\geq 2}$.

Due to the existence of noise in real measurements, we here assume that the noise are *i.i.d.* zero-mean Gaussian and relax the strict equality constraints on the measurements as

$$\min_{\mathbf{o},\mathbf{X}} \|\mathbf{X}\|_* + \frac{\mu}{2}\|\mathbf{B}_0 + \sum_{j=1}^{n} o_j\mathbf{B}_j - \mathbf{X}\|_F^2, \tag{3}$$

where μ is some positive constant scalar parameter and $\|.\|_F$ is the Frobenius norm. When minimizing the nuclear norm of $\mathbf{X}$, instead of getting a low rank approximation of $\mathbf{X}$, one might minimize the singular values evenly[1]. To avoid this, we apply the same strategy in [7] called *truncated nuclear norm*, where the sum of the $(\min\{m,n\}-K)$ smallest singular values is minimized. To achieve this, the minimization problem in (3) is modified as

$$\min_{\mathbf{o},\mathbf{X},\mathbf{U},\mathbf{V}} \|\mathbf{X}\|_* - \mathrm{Tr}(\mathbf{U}\mathbf{X}\mathbf{V}^T) + \frac{\mu}{2}\|\mathbf{B}_0 + \sum_{j=1}^{n} o_j\mathbf{B}_j - \mathbf{X}\|_F^2. \tag{4}$$

where $\mathbf{U}\in\mathbb{R}^{3\times m}$, $\mathbf{V}\in\mathbb{R}^{3\times n}$, and $\mathbf{U}\mathbf{U}^T = \mathbf{I}_3$ and $\mathbf{V}\mathbf{V}^T = \mathbf{I}_3$. It is shown in [7] that adding the term $-\mathrm{Tr}(\mathbf{U}\mathbf{X}\mathbf{V}^T)$ in the minimization is equivalent to minimizing the truncated nuclear norm. Now (4) is a non-convex optimization problem and an optimization scheme for obtaining local minima is presented in the next section.

4. OPTIMIZATION SCHEME

In the section, we discuss the optimization scheme for the proposed problems. We use a two-step iterative scheme as in [7]. In Step 1, we fix $\mathbf{o}$, $\mathbf{X}$ and solve for $\mathbf{U}$ and $\mathbf{V}$ as an outer loop and in Step 2, with $\mathbf{U},\mathbf{V}$ fixed, we optimize over $\mathbf{o},\mathbf{X}$ iteratively in an inner loop until $\mathbf{X}$ converges. The first step is relatively simple (as shown in Algorithm 4.1) and we will discuss in details the optimization in Step 2.

4.1. Optimization using ADMM

To optimize w.r.t $\mathbf{X}$, $\mathbf{o}$ with $\mathbf{U},\mathbf{V}$ fixed in Step 2, we use the alternating direction methods of multipliers (ADMM). First, by introducing a new variable $\hat{\mathbf{X}}$, we rewrite (3) as

$$\min_{\mathbf{o},\mathbf{X},\hat{\mathbf{X}}} \quad \|\mathbf{X}\|_* - \mathrm{Tr}(\mathbf{U}\hat{X}\mathbf{V}^T) + \frac{\mu}{2}\|\mathbf{B}_0 + \sum_{j=1}^{n} o_j\mathbf{B}_j - \hat{\mathbf{X}}\|_F^2, \qquad \text{s.t.} \quad \mathbf{X} = \hat{\mathbf{X}}. \tag{5}$$

[1]This is indeed what happens for this problem in our initial implementation without the term $-\mathrm{Tr}(\mathbf{U}\mathbf{X}\mathbf{V}^T)$)

Algorithm 4.1 TDOA Nuclear Norm

Input: TDOA measurements $\{f_{ij}\}$ of m receivers and n transmitters, threshold ϵ :

Initialize: Construct $\{\mathbf{B}_j\}$, $j = 0, \ldots, n$ based on (1), set $\mathbf{o} = \mathbf{0}_n$, $\mathbf{X}_1 = \hat{\mathbf{X}}_1 = \mathbf{Y} = \hat{\mathbf{F}}$

Repeat:

1. *Solve for* $\mathbf{U}_l$ *and* $\mathbf{V}_l$ *given* $\mathbf{X}_l$.
   (a) $(\mathbf{A}_l, \mathbf{\Sigma}_l, \mathbf{C}_l) = \mathbf{svd}(\mathbf{X}_l)$ *where* $\mathbf{A}_l = (\mathbf{a}_1, \ldots, \mathbf{a}_m) \in \mathbb{R}^{m \times m}$ *and* $\mathbf{C}_l = (\mathbf{c}_1, \ldots, \mathbf{c}_n) \in \mathbb{R}^{n \times n}$
   (b) $\mathbf{U}_l = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)^T$, $\mathbf{V}_l = (\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3)^T$
2. *Solve* $\{\mathbf{X}_{l+1}, \mathbf{o}_{l+1}\} = \arg\min_{\mathbf{o},\mathbf{X}} \|\mathbf{X}\|_* - \mathrm{Tr}(\mathbf{U}_l \mathbf{X} \mathbf{V}_l^T) + \frac{\mu}{2}\|\mathbf{B}_0 + \sum_{j=1}^n o_j \mathbf{B}_j - \mathbf{X}\|_F^2$

Until: $\sqrt{\|\mathbf{X}_{l+1} - \mathbf{X}_l\|_F^2 + \|\mathbf{o}_{l+1} - \mathbf{o}_l\|_F^2} < \epsilon$

We can see that the augmented Largrange function of (5) is as follows

$$L(\mathbf{o}, \mathbf{X}, \hat{\mathbf{X}}, \mathbf{Y}) = \|\mathbf{X}\|_* - \mathrm{Tr}(\mathbf{U}\hat{X}\mathbf{V}^T) + \frac{\mu}{2}\|\mathbf{B}_0 + \sum_{j=1}^{n} o_j \mathbf{B}_j - \hat{\mathbf{X}}\|_F^2 + \frac{\lambda}{2}\|\mathbf{X} - \hat{\mathbf{X}}\|_F^2 + \mathrm{Tr}\left((\mathbf{Y}^T(\mathbf{X} - \hat{\mathbf{X}})\right), \tag{6}$$

where λ is a positive scalar. With the schemes in ADMM, we alternate the optimization on certain set of variables by fixing the rest of the variables. Specifically, starting with initial values that $\mathbf{o}_1 = \mathbf{0}_n$, $\mathbf{X}_1 = \hat{\mathbf{X}}_1 = \mathbf{Y} = \hat{\mathbf{F}}$, we have the following three iterative steps for iteration $k+1$:

Computing $\mathbf{X}_{k+1}$: Given $\mathbf{o}_k$, $\hat{\mathbf{X}}_k$, and $\mathbf{Y}_k$, we minimize $L(\mathbf{o}_k, \mathbf{X}, \hat{\mathbf{X}}_k, \mathbf{Y}_k, \lambda)$ over $\mathbf{X}$,

$$\mathbf{X}_{k+1} = \mathrm{argmin}_{\mathbf{X}}\Big(\|\mathbf{X}\|_* - \mathrm{Tr}(\mathbf{U}\hat{X}_k\mathbf{V}^T) + \frac{\mu}{2}\|\mathbf{B}_0 + \sum_{j=1}^{n} o_{j_k}\mathbf{B}_j - \hat{\mathbf{X}}_k\|_F^2 + \frac{\lambda}{2}\|\mathbf{X} - \hat{\mathbf{X}}_k\|_F^2 + \mathrm{Tr}\left((\mathbf{Y}_k^T(\mathbf{X} - \hat{\mathbf{X}}_k)\right)\Big), \tag{7}$$

which is equivalent to the following by ignoring the constants:

$$\mathbf{X}_{k+1} = \arg\min_{\mathbf{X}} \Big(\|\mathbf{X}\|_* + \frac{\lambda}{2}\|\mathbf{X} - (\hat{\mathbf{X}}_k - \frac{1}{\lambda}\mathbf{Y}_k)\|_F^2\Big). \tag{8}$$

This can be solved by Singular Value Thresholding theorem [8].

Computing $\mathbf{o}_{k+1}, \hat{\mathbf{X}}_{k+1}$: Fix $\mathbf{X}_{k+1}$ and $\mathbf{Y}_k$, we can calculate $\mathbf{o}_{k+1}$ and $\hat{\mathbf{X}}_{k+1}$ as follows:

$$\{\mathbf{o}_{k+1}, \hat{\mathbf{X}}_{k+1}\} = \arg\min_{\mathbf{o},\hat{\mathbf{X}}}\big(\frac{\mu}{2}\|\mathbf{B}_0 + \sum_{j=1}^{n} o_j\mathbf{B}_j - \hat{\mathbf{X}}\|_F^2 + \frac{\lambda}{2}\|\mathbf{X}_{k+1} - (\hat{\mathbf{X}} - \frac{1}{\lambda}\mathbf{Y}_k)\|_F^2\big), \tag{9}$$

which is sum of two quadratic functions and can be solved by finding $\{\mathbf{o}, \hat{\mathbf{X}}\}$ such that $\partial L(\mathbf{o}, \mathbf{X}_{k+1}, \hat{\mathbf{X}}, \mathbf{Y}_k) = 0$.

Computing $\mathbf{Y}_{k+1}$: $\mathbf{Y}$ can be updated as

$$\mathbf{Y}_{k+1} = \mathbf{Y}_k + \lambda(\mathbf{X}_{k+1} - \hat{\mathbf{X}}_{k+1}). \tag{10}$$

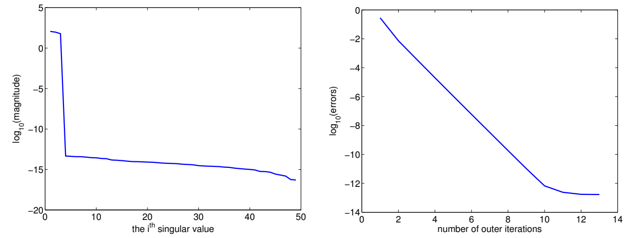

Fig. 1. Synthetic TDOA measurements with no noise ($m = n = 50$). Left: Singular values of the matrix $\mathbf{X}$ after optimization; Right : Speed of convergence ($\|\mathbf{o} - \mathbf{o}_{gt}\|_F$) for the ADMM algorithm ($\mu = 10,\ \lambda = 1, \epsilon = 10^{-12}$).

We iterate the steps described above in an inner loop until $\mathbf{X}$ converges. Then we feed the updated $\mathbf{X}$ to the outer loop as in Step 1 of the above algorithm to update $\mathbf{U}$, $\mathbf{V}$. The outer loop runs until $\sqrt{\|\mathbf{X}_{l+1} - \mathbf{X}_l\|_F^2 + \|\mathbf{o}_{l+1} - \mathbf{o}_l\|_F^2} < \epsilon$ or a maximum number of iterations is reached.

Missing Data The previous iterative steps can be easily modified for TDOA measurements with missing data. In this case, we only need to replace the term in the above $\|\mathbf{B}_0 + \sum_{j=1}^n o_j\mathbf{B}_j - \hat{\mathbf{X}}\|_F^2$ by $\|\mathbf{B}_{0\Omega} + \sum_{j=1}^n o_j\mathbf{B}_{j\Omega} - \hat{\mathbf{X}}_\Omega\|_F^2$ where Ω is set of measurements that have been observed[2]. This modification will only change slightly the iterative updating step for $\mathbf{o}_{k+1}, \hat{\mathbf{X}}_{k+1}$ and the other steps remain the same.

5. EXPERIMENTS

In this section, we present experimental results of our method on synthetic data. It is of interest to see the performance of the method on synthetic data regarding speed of convergence, accuracy and its sensitivity to noise. We simulate the positions of the transmitters and receivers by drawing independently from the standard normal distribution, i.e. $r_{ik} \sim \mathcal{N}(0,1)$ and $s_{jk} \sim \mathcal{N}(0,1)$ for $k = 1, ..., K$. The time delays are also sampled from standard normal distribution, i.e. $o_{ij} \sim \mathcal{N}(0,1)$. First of all, we study the convergence of the algorithm for noise free cases. For well-constrained over-determined cases (large m and n), we can see that in Figure 1 (left) that the ADMM algorithm converges to a rank-3 matrix $\mathbf{X}$ and the ground truth time delays $\mathbf{o}_{gt}$ in a few outer iterations. It runs typically on a MacBook Air with 1.8 GHz Intel Core i5 and 8 GB memory in around 2s for $m = n = 50$.

To further understand the effects of increasing measurements, we run the method on noise-free data for different fixed number of receivers m and vary the number of transmitters n. For this experiment, we set $\epsilon = 10^{-12}$ and the maximum number of iterations to 5000. We observe that in Figure 2 (left) for small m i.e. m = 5, the method does not converges to a reliable initial solution for the time delays. For larger m's ($m \geq 6$), we can see that the relative errors of recovered time delays decreases as we have more transmitters, suggesting the benefits of increasing number of measurements. Due to the fact that the function we are optimizing are non-convex, over-constrained case with more measurements might reduce the number of local minima, thus gives better convergence behavior for random

[2]Here we can assume that there is one receiver that has complement measurements of the n transmitters, and one transmitter that is measured by all m receivers.

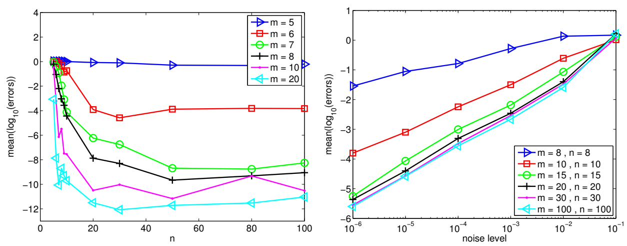

Fig. 2. Synthetic experiments - average errors $\|\mathbf{o} - \mathbf{o}_{gt}\|_F / \|\mathbf{o}_{gt}\|_F$ on 100 random synthetic TDOA measurements. Left: Noise-free data - varying n for different fixed m; Right: Noisy data - varying m and n for different levels of noise

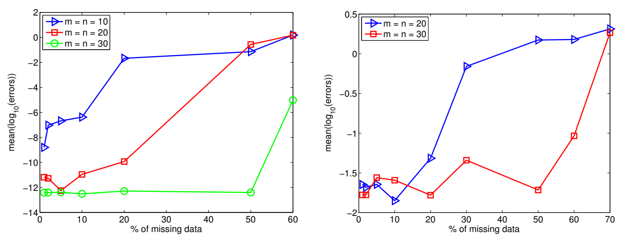

Fig. 3. Synthetic TDOA measurements with varying percentage of missing data. Left : noisy-free data for different m and n; Right : noisy data with Gaussian noise of standard deviation 10^{-2}.

initialization. It is also noted that for a fixed m, when n is larger than 20, the convergence of the method does not change much.

We also test the method under different levels of noise with increasing number of measurements. We see that in Figure 2 (right) the error of recovered time delays decreases as the number of measurement increases. Up to certain number of measurements, we see that the method does not gain better performance e.g. $m = n = 30$ gives similar errors as $m = n = 100$. It is also observed that when the noise level reaches standard deviation of 10^{-1}, the method perform poorly no matter how much the number of measurements increases.

While the previous experiments assume that measurements are complete, it would be of interest to see how the algorithm performs with missing data. In this experiment, we run the algorithm on noise-free synthetic TDOA example where we randomly remove certain percentage of entries in the measurements. We vary the percentage of the missing entries in the measurement matrix, we can see that in Figure 3 for relatively large m and n, the algorithm only breaks down when more than 50% of the measurements are missing ($m = n = 30$). For smaller m and n ($m = n = 10$), the method fails to provide good estimate when there are more than 20% missing entries . For noisy measurements with missing data, with noise level as 10^{-2}, which is typical in practice, for $m = n = 20$, the algorithm works well until there are more than 30% missing data (Fig.3, right). For more over-constrained cases $m = n = 30$, it is more robust to increasing number of missing entries (up to 50%) and it gives fairly good initial guess of the time delays even under noise.

6. PRIOR WORKS

Time delay estimation has been studied extensive before. In the most related work [9] a different constraint is used, which corresponds to the matrix F in Section 2.1. This makes it possible to solve for the time delays for at least 10 receivers and at least 5 transmitters. For special configuration where the receivers lie on a line, [12] presented minimal solvers for both TOA and TDOA measurements. In [13,14], the TOA self-calibration is studied for the special case of one receiver being in the same position as one transmitter. Unlike all these previous works, we focus on general 3D TDOA problem and utilize the rank-3 constraint on TDOA measurements, and formulate the problem as optimization using truncated nuclear norm. The method can easily handle missing data, which could be difficult for previous works.

Other previous work that is less related to the present paper are (i) [11], where solutions to the TOA self-calibration problem of three transmitters and three receivers in the plane is given. (ii) [15] where a TDOA setup is used for indoor navigation based on non-linear optimization, but the method can get stuck in local minima and is dependent on initialization, (iii) [16] and refined in [17] where a far field approximation was utilized to initialize the TOA and TDOA case with good results.

Nuclear norm minimization for minimizing rank has been used previously in many different applications. Examples of this is in [7] a matrix completion algorithm using truncated nuclear norm regularization is proposed and validated in application of filling in the missing pixels of a given image. Also in [18] the low-rank constraint is explored in the non-rigid structure from motion problem. And the rank minimization problem is further relaxed to a nuclear norm minimization. In particular, it is equivalent to a trace minimization as the target matrix is positive-definite.

7. CONCLUSIONS

In this paper, we study the problem of determining the unknown time delays in the TDOA self-calibration problem. We explore a rank constraint of a matrix that is linear in the time delays and propose a method based on truncated nuclear norm minimization for recovering the time delays for TDOA measurements. The method is general for cases where the receivers and transmitters respectively span a K-dimensional affine space [3]. We show experimentally that this method gives good estimates for the time delays with noise and missing data. They can be used as initial solutions for converting TDOA measurements to TOA measurements. As future work, it is of interest to improve the robustness by using $\mathbf{L}_1$-norm $\left\|\mathbf{B}_0 + \sum_{j=1}^{n} o_{j_k} \mathbf{B}_j - \hat{\mathbf{X}}_k\right\|_1$ in the cost function. We are also investigating modeling the problem as finding a matrix of exactly rank 3 instead of minimizing the truncated nuclear norm. Another direction is to extend this work for unsynchronized TOA problem with unknown time delays for both receivers and transmitters [19]. Solving that is of great advantage than [19] when far-field approximation is required.

8. REFERENCES

[1] S. T. Birchfield and A.. Subramanya, "Microphone array position calibration by basis-point classical multidimensional scaling," *IEEE transactions on Speech and Audio Processing*, vol. 13, no. 5, 2005.

[2] D. Niculescu and B. Nath, "Ad hoc positioning system (aps)," in *GLOBECOM-01*, 2001.

[3]The codes are available for download at http://www2.maths.lth.se/vision/downloads/.

[3] E. Elnahrawy, Xl. Li, and R. Martin, "The limits of localization using signal strength," in *SECON-04*, 2004.

[4] V. C. Raykar, I. V. Kozintsev, and R. Lienhart, "Position calibration of microphones and loudspeakers in distributed computing platforms," *IEEE transactions on Speech and Audio Processing*, vol. 13, no. 1, 2005.

[5] J. Sallai, G. Balogh, M. Maroti, and A. Ledeczi, "Acoustic ranging in resource-constrained sensor networks," in *eCOTS-04*, 2004.

[6] Benjamin Recht, Maryam Fazel, and Pablo A. Parrilo, "Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization," Tech. Rep., 2007.

[7] D. Zhang, Y. Hu, J. Ye, X. Li, and X. He, "Matrix completion by truncated nuclear norm regularization," in *CVPR*, 2012, pp. 2192–2199.

[8] J.F. Cai, E.J. Candès, and Z. Shen, "A singular value thresholding algorithm for matrix completion," *Siam Journal of Optimization*, vol. 20, no. 4, pp. 1956–1982, 2010.

[9] M. Pollefeys and D. Nister, "Direct computation of sound and microphone locations from time-difference-of-arrival data," in *Proc. of International Conference on Acoustics, Speech and Signal Processing*, 2008.

[10] Yubin Kuang, Simon Burgess, Anna Torstensson, and Kalle Åström, "A complete characterization and solution to the microphone position self-calibration problem," in *Proc. of International Conference on Acoustics, Speech and Signal Processing*, 2013.

[11] H. Stewénius, *Gröbner Basis Methods for Minimal Problems in Computer Vision*, Ph.D. thesis, Lund University, APR 2005.

[12] Erik Ask, Simon Burgess, and Kalle Åström, "Minimal structure and motion problems for toa and tdoa measurements with collinearity constraints," in *ICPRAM*, 2013.

[13] M. Crocco, A. Del Bue, M. Bustreo, and V. Murino, "A closed form solution to the microphone position self-calibration problem," in *37th International Conference on Acoustics, Speech, and Signal Processing (ICASSP 2012), Kyoto, Japan*, March 2012.

[14] Marco Crocco, Alessio Del Bue, and Vittorio Murino, "A bilinear approach to the position self-calibration of multiple sensors," *Trans. Sig. Proc.*, vol. 60, no. 2, pp. 660–673, Feb. 2012.

[15] Thomas Janson, Christian Schindelhauer, and Johannes Wendeberg, "Self-localization based on ambient signals," in *Proceedings of the 6th international conference on Algorithms for Sensor systems, Wireless Adhoc Networks, and Autonomous Mobile Entities*, Berlin, Heidelberg, 2010, ALGOSENSORS'10, pp. 176–188, Springer-Verlag.

[16] S. Thrun, "Affine structure from sound," in *Proceedings of Conference on Neural Information Processing Systems (NIPS)*, Cambridge, MA, 2005, MIT Press.

[17] Y. Kuang, E. Ask, S. Burgess, and K. Åström, "Understanding toa and tdoa network calibration using far field approximation as initial estimate," in *ICPRAM*, 2012.

[18] Y. Dai, H. Li, and M. He, "A simple prior-free method for nonrigid structure-from-motion factorization," in *Computer Vision and Pattern Recognition (CVPR), 2012 IEEE Conference on*. IEEE, 2012, pp. 2018–2025.

[19] Simon Burgess, Yubin Kuang, and Kalle Åström, "Node localization in unsynchronized time of arrival sensor networks," in *Proceedings of the 21st International Conference on Pattern Recognition*, 2012.