AN L_1-CONSTRAINED NORMALIZED LMS ALGORITHM AND ITS APPLICATION TO THINNED ADAPTIVE ANTENNA ARRAYS

J. F. de Andrade Jr[⋆] *M. L. R. de Campos*[⋆] *J. A. Apolinário Jr.*[†]

[⋆] Program of Electrical Engineering (PEE), Federal University of Rio de Janeiro (UFRJ)
[†] Department of Electrical Engineering (SE/3), Military Institute of Engineering (IME)
Emails: andrade@dctim.mar.mil.br, mcampos@ieee.org, apolin@ime.eb.br

ABSTRACT

We propose in this work an L_1-norm Linearly Constrained Normalized Least-Mean-Square (L_1-CNLMS) algorithm applied to solve the beamforming problem in Standard Hexagonal Arrays (SHA) and (non-standard) Hexagonal Antenna Arrays (HAA). In addition to the linear constraints present in the CNLMS algorithm, the L_1-CNLMS algorithm takes into account an L_1-norm penalty on the filter coefficients which results in sparse solutions producing Thinned Hexagonal Arrays. The effectiveness of the L_1-CNLMS algorithm is demonstrated by comparing, via computer simulations, its results with those of the CNLMS algorithm. When employing the L_1-CNLMS algorithm to antenna array problems, the resulting effect of the L_1-norm constraint is perceived as a large aperture array with few active array elements.

Index Terms— CNLMS Algorithm, L_1-norm, Sparse Sensor Arrays, Constrained Adaptive Beamforming, Thinned Arrays.

1. INTRODUCTION

In modern telecommunication systems as well as in a number of other applications, the use of adaptive antenna arrays has become an important asset to increase system capacity, improve performance, and attain demanding criteria for quality of service. Interfering signals may impose severe drawbacks and limitations for systems unable to mitigate their effects either due to lack of computational capacity or due to limited power supply (those powered by batteries, for instance). On one hand, algorithm design shall target some specified radiation characteristic, such as 0 dB gain for signals of interest and the ability to mitigate jammers. On the other hand, the number of elements used shall be kept low for a tight control over energy consumption. In general, these design rules are conflicting and a compromise is often the best solution.

In this work, we propose a design criteria embedded in the adaptation algorithm which favors sparse, or thinned, arrays with a good compromise between radiation pattern and the number of array elements in use. Our algorithm is able to turn off elements, or sensors, as deemed appropriate, without violating design linear constraints.

We consider that the possibility of provoking sparsity by a design rule incorporated in the adaptation-algorithm objective function is key to achieving high performance in demanding online and real-time applications. The LASSO [1] and the Sparse LMS [2] algorithms, together with references [3], [4], and [5], offered good starting points to the pursuit of sparse and linearly constrained adaptive filters. Motivated by the preliminary good results presented in [6], the work presented here has focused on a linearly constrained adaptation algorithm with an additional L_1-norm constraint based on the CNLMS [7] algorithm. The new algorithm achieves large degrees of sparsity and yet maintains desired beampattern characteristics. We have also developed a variable step-size to adjust dynamically convergence speed and L_1-norm; this proved to be fundamental to achieving sparsity and performance gain in tandem.

The algorithm is particularly suited for applications where a large number of sensors are needed. Examples are planar arrays employed in mobile satellite communications [8] [9] [10], including those used in tactical military communication systems—operating in bands “X” [11] and “Ku” [12]—and in civilian satellite systems for personal communications.

We have tested the proposed algorithm in two simulation setups involving hexagonal antenna arrays [13] [14] for satellite applications. The first scenario contemplates a standard hexagonal array of 91 elements for mobile receiver systems and, in the second scenario, we use an hexagonal array of 397 elements for geostationary satellite antenna systems.

2. THE CNLMS AND L_1-CNLMS ALGORITHMS

The CLMS algorithm proposed in [15] adjusts, at each iteration, the coefficients of an adaptive linear combiner satisfying linear constraints, such as those determined by beamforming applications. However, the difficulty to choose the step size for a gradient-type algorithm becomes aggravated in constrained scenarios. In [7], the normalized version of the CLMS algorithm was presented for multi-user detection; its use in other applications, such as adaptive beamforming, is straightforward.

Defining $\epsilon(k) = d(k) - \mathbf{w}^H\mathbf{x}(k)$ as an estimation error associated with coefficient vector $\mathbf{w}$, where $\mathbf{x}(k)$ and $d(k)$ are the input and desired signals, respectively, the CLMS algorithm can be derived [15] [16] starting from the following constrained minimization problem:

$$\min_{\mathbf{w}} E[|\epsilon(k)|^2] \text{ s.t. } \mathbf{C}^H\mathbf{w} = \mathbf{z}, \tag{1}$$

where $\mathbf{C}$ is an $N \times N_C$ constraint matrix and $\mathbf{z}$ is the corresponding constraint vector containing N_C (number of constraints) elements.

Using Lagrange multipliers and an instantaneous estimate of the gradient, the updating equation of the CLMS algorithm at time instant k is expressed as

$$\mathbf{w}(k+1) = \mathbf{P}\left[\mathbf{w}(k) + \mu e^*(k)\mathbf{x}(k)\right] + \mathbf{F}, \tag{2}$$

where $e(k) = d(k) - \mathbf{w}^H(k)\mathbf{x}(k)$ is the *a priori* estimation error, $\mathbf{P} = \mathbf{I}_{N\times N} - \mathbf{C}\left(\mathbf{C}^H\mathbf{C}\right)^{-1}\mathbf{C}^H$ is a projection matrix and $\mathbf{F} = \mathbf{C}\left(\mathbf{C}^H\mathbf{C}\right)^{-1}\mathbf{z}$ is an $N \times 1$ vector.

The authors thank the Brazilian agencies CNPq (contracts 306548/2010-0, 471230/2011-1, and 306749/2009-2), FAPERJ (contract 111.959/2012), and the Brazilian Navy for partially funding this work.

Minimizing the instantaneous *a posteriori* squared error at instant k [7] [16], defined as $e_{ap}^2(k) = [d(k) - \mathbf{w}^H(k+1)\mathbf{x}(k)]^2$, the convergence step in Eq. (2), assuming it time varying, could be replaced by

$$\mu(k) = \frac{\mu_0}{\mathbf{x}^H(k)\mathbf{P}\mathbf{x}(k) + \gamma}, \tag{3}$$

where μ_0 is a fixed convergence factor introduced to control misadjustment and γ is a parameter included to avoid taking large steps when $\mathbf{x}^H(k)\mathbf{P}\mathbf{x}(k)$ becomes too small.

The use of Eq. (3) into Eq. (2) leads to the CNLMS algorithm [7].

2.1. The $\mathbf{L_1}$-CLMS Algorithm

In [6], an L_1-norm penalty is added to the list of constraints in the CLMS algorithm in order to favor coefficient shrinkage towards sparsity:

$$\min_{\mathbf{w}} E[|\epsilon(k)|^2] \text{ s.t. } \begin{cases} \mathbf{C}^H\mathbf{w} = \mathbf{z}, \\ \|\mathbf{w}\|_1 = t. \end{cases} \tag{4}$$

A cost function with the L_1-norm penalty is then defined as

$$\xi(\mathbf{w}) = E[|\epsilon(k)|^2] + \boldsymbol{\lambda}_1^H\big(\mathbf{C}^H\mathbf{w} - \mathbf{z}\big) + \lambda_2\big(\|\mathbf{w}\|_1 - t\big). \tag{5}$$

The instantaneous estimate of the gradient of $\xi(\mathbf{w})$ in Eq. (5) is expressed as

$$\hat{\boldsymbol{\nabla}}_{\mathbf{w}}\xi(\mathbf{w}) = -2e^*(k)\mathbf{x}(k) + \mathbf{C}\boldsymbol{\lambda}_1 + \lambda_2\text{sign}[\mathbf{w}], \tag{6}$$

where $\text{sign}[w] \triangleq w/|w|$ and $\|\mathbf{w}\|_1 = \text{sign}^H[\mathbf{w}]\mathbf{w}$.

Applying the steepest descent method [16], we obtain

$$\mathbf{w}(k+1) = \mathbf{w}(k) - \frac{\mu}{2}\Big\{-2e^*(k)\mathbf{x}(k) + \mathbf{C}\boldsymbol{\lambda}_1 + \lambda_2\text{sign}[\mathbf{w}(k)]\Big\}. \tag{7}$$

It is possible to solve for $\boldsymbol{\lambda}_1$ premultiplying Eq. (7) by $\mathbf{C}^H$ such that

$$\boldsymbol{\lambda}_1 = (\mathbf{C}^H\mathbf{C})^{-1}\mathbf{C}^H\Big\{2e^*(k)\mathbf{x}(k) - \lambda_2\text{sign}[\mathbf{w}(k)]\Big\}. \tag{8}$$

Due to the constraints in the minimization problem stated in Eq. (4), the approximation $\text{sign}^H[\mathbf{w}(k)]\mathbf{w}(k+1) \approx t$ is proposed; this shall be valid, at least near convergence, as $\mathbf{w}(k)$ and $\mathbf{w}(k+1)$ are expected to be in the same hyper-quadrant.

Defining the L_1-norm $t_k = \text{sign}^H[\mathbf{w}(k)]\mathbf{w}(k)$ and taking into account that $N = \text{sign}^H[\mathbf{w}(k)]\text{sign}[\mathbf{w}(k)]$, it is possible to premultiply Eq. (7) by $\text{sign}^H[\mathbf{w}(k)]$ and eliminate vetor $\mathbf{w}(k+1)$:

$$\begin{aligned} t = t_k - \frac{\mu}{2}\Big\{&-e^*(k)\text{sign}^H[\mathbf{w}(k)]\mathbf{x}(k) + \text{sign}^H[\mathbf{w}(k)]\mathbf{C}\boldsymbol{\lambda}_1 \\ &+\lambda_2\text{sign}^H[\mathbf{w}(k)]\text{sign}[\mathbf{w}(k)]\Big\}. \end{aligned} \tag{9}$$

After isolating λ_2 from Eq. (9) and defining the L_1-norm error $e_{L_1}(k) = t - t_k$, we have

$$\begin{aligned} \lambda_2 = &\left(\frac{-2}{N\mu}\right)e_{L_1}(k) + \frac{2}{N}e^*(k)\text{sign}^H[\mathbf{w}(k)]\mathbf{x}(k) \\ &- \frac{1}{N}\text{sign}^H[\mathbf{w}(k)]\mathbf{C}\boldsymbol{\lambda}_1. \end{aligned} \tag{10}$$

The Lagrange multipliers $\boldsymbol{\lambda}_1$ and λ_2 can be calculated solving the system of equations stated by Eqs. (8) and (10).

The update equation for the L_1-CLMS algorithm can be obtained from the update equation presented in [6]:

$$\mathbf{w}(k+1) = \mathbf{P}\left[\mathbf{w}(k) + \mu\bar{\mathbf{P}}(k)\mathbf{x}(k)e^*(k)\right] + \mathbf{F} + \mathbf{F}_{L1}(k) \tag{11}$$

with

$$\begin{cases} \bar{\mathbf{P}}(k) = \left[\mathbf{I} - \left(\frac{\mathbf{P}\text{sign}[\mathbf{w}(k)]}{\|\mathbf{P}\text{sign}[\mathbf{w}(k)]\|^2}\right)\text{sign}^H[\mathbf{w}(k)]\right]\mathbf{P}, \\ \mathbf{F}_{L1}(k) = \mu_{L1}\left(\frac{\mathbf{P}\text{sign}[\mathbf{w}(k)]}{\|\mathbf{P}\text{sign}[\mathbf{w}(k)]\|^2}\right)e_{L_1}(k), \\ e_{L_1}(k) = t - \text{sign}^H[\mathbf{w}(k)]\mathbf{w}(k). \end{cases}$$

2.2. The New $\mathbf{L_1}$-CNLMS Algorithm

In order to obtain a normalized version of the L_1-CLMS algorithm, a time-varying convergence factor shall be calculated minimizing the instantaneous *a posteriori* squared error at instant k [16], i.e., we calculate μ, at instant k, which minimizes $e_{ap}^2(k)$:

$$\mu(k) = \frac{\mu_0\left[e^*(k) - \mathbf{x}^H(k)\mathbf{F}_{L1}\right]}{e^*(k)\mathbf{x}^H(k)\bar{\mathbf{P}}(k)\mathbf{x}(k) + \gamma}. \tag{12}$$

The factors μ_0 and μ_{L1} are fixed convergence factors introduced to control misadjustment. If μ_0 is chosen equal to 1, the instantaneous *a posteriori* squared error is equal to zero, as expected for a normalized algorithm.

The proposed algorithm, comprising Eqs. (11) and (12), produces sparse solutions as a consequence of the L_1-norm shrinkage, yielding more economical arrays. Comparing the update equations for the L_1-CNLMS and the CNLMS algorithms, it is possible to infer they have the same order of computational complexity, i.e., $\mathcal{O}(N \times N_c)$. In this work, matrix $\mathbf{C}$ and vector $\mathbf{z}$ define the linear constraints to assure nulls in the directions of the jammers and a constant gain in the direction of interest, while the L_1-norm constraint is controlled by parameter t.

3. THE ANTENNA ARRAY

Consider a Standard Hexagonal Array (SHA) composed of N receiving antennas (sensors) and N_C receiving narrowband signals coming from different azimuths (θ_i) and zeniths (ϕ_i) [13] defined mathematically by $(\theta_1, \phi_1), \cdots, (\theta_{N_C}, \phi_{N_C})$. The samples observed by N sensors during M snapshots, assuming an analytic signal in the discrete domain, are denoted by $\mathbf{x}(k)$, with k from 1 to M. The $N \times 1$ signal vector is modeled as

$$\mathbf{x}(k) = \sum_{i=1}^{N_C}\mathbf{a}(\theta_i, \phi_i)s_q(k) + \boldsymbol{\eta}(k), \;\; k = 1, 2, \cdots, M. \tag{13}$$

Using matrix notation, the last expression can be written as $\mathbf{A}\mathbf{s}(k) + \boldsymbol{\eta}(k)$ and an $N \times M$ input data matrix $\mathbf{X}$ can be defined and expressed as

$$\mathbf{X} = [\mathbf{x}(1) \;\; \mathbf{x}(2) \;\; \cdots \;\; \mathbf{x}(M)] = \mathbf{A}\mathbf{S} + \boldsymbol{\mathcal{N}}, \tag{14}$$

where $\mathbf{S}$ is the $N_c \times M$ complex signal envelope matrix at the array center, $\mathbf{A} = [\mathbf{a}(\theta_1, \phi_1), \cdots, \mathbf{a}(\theta_{N_C}, \phi_{N_C})]$ is the $N \times N_c$ steering matrix and $\boldsymbol{\mathcal{N}}$ is the $N \times M$ noise matrix. The steering matrix columns are defined as:

$$\begin{aligned} \mathbf{a}_i = \exp\Big\{-j\frac{2\pi}{\lambda}\Big[&\cos(\theta_i)\sin(\phi_i) \otimes \mathbf{d}_u \\ &+ \sin(\theta_i)\sin(\phi_i) \otimes \mathbf{d}_v\Big]\Big\}, \quad i = 1, \cdots, N_C, \end{aligned} \tag{15}$$

where $\mathbf{d}_u$ and $\mathbf{d}_v$ are $N \times 1$ vectors containing the Cartesian u (abscissa) and v (ordinate) components, respectively, denoting position for each array element in Cartesian coordinate plane. The symbol $\otimes$ represents the Kronecker product [13] and λ is the signal wavelength.

Considering a plane wave with wavelength λ incident from direction (θ, ϕ) that propagates across a planar array of N isotropic antennas arranged on the same plane according to the grid $\mathbf{D} = [\mathbf{d}_u, \mathbf{d}_v]_{N\times 2}$, the beampattern for a given direction (θ, ϕ) of an array processor having coefficient vector $\mathbf{w}$ can be calculated by

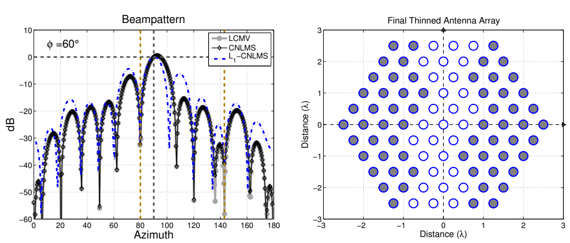
$$B(\theta,\phi) = \mathbf{w}^H \exp\left\{-j\frac{2\pi}{\lambda}\left[\cos(\theta)\sin(\phi) \otimes \mathbf{d}_u + \sin(\theta)\sin(\phi) \otimes \mathbf{d}_v\right]\right\}. \quad (16)$$

4. SIMULATION RESULTS

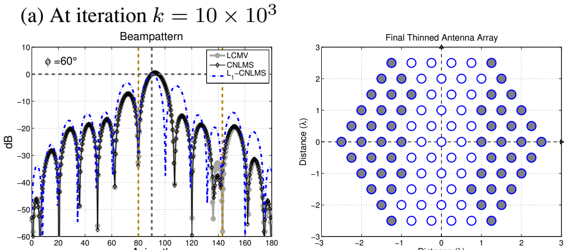
In this section, results from two simulations are presented. We compare the results of the L_1-CNLMS algorithm with those of the CNLMS algorithm.

4.1. SHA for *X*-band Geostationary Military Satellite System

The first simulation was carried out considering an SHA suitable for receiving geostationary military satellite signals in the X-band [13] [17]. This SHA employs 6 elements on each edge, resulting in a total of 91 elements as can be observed in Figure 1. Three narrowband analytical QPSK signals are considered in this simulation, i.e., a signal of interest and two interfering signals (jammers). All parameters used in this simulation are listed in Table 1.

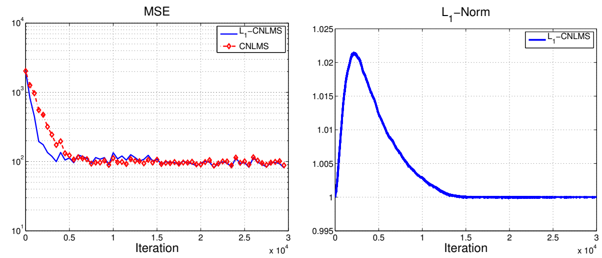


(a) At iteration $k = 10 \times 10^3$



(b) At iteration $k = 30 \times 10^3$

Fig. 1: Thinned SHA and respective beampattern for the L_1-CNLMS and CNLMS algorithms, and the LCMV solution. The elements represented by white circles are those turned off by the L_1-CNLMS algorithm. In Cartesian coordinate plane, the azimuth angle begins in the first quadrant, abscissa axis, and increases in counterclockwise direction.



(a) MSE (b) L_1-norm

Fig. 2: MSE and L_1-norm for coefficient vector during shrinkage process applied to a 91-elements SHA.

Table 1: Parameter Values for Simulations I and II

Parameter	**I**	**II**
L_1-**CNLMS** Step-size (μ_0)	2×10^{-2}	2×10^{-2}
CNLMS Step-size (μ_0)	1×10^{-2}	1×10^{-2}
Elements' separation	$\frac{\lambda}{2}$	λ
L_1-norm constraint (t)	1.00	1.04
Signals' Frequencies	8 GHz	12 GHz
Signal of Interest (θ_S, ϕ_S)	(90°, 60°)	(90°, 45°)
Signal SNR	20 dB	20 dB
Jamming 1 (θ_1, ϕ_1)	(80°, 60°)	(87.50°, 45°)
Jamming 2 (θ_2, ϕ_2)	(143°, 60°)	(88.75°, 47.50°)
Jamming 3 (θ_3, ϕ_3)	-	(91.25°, 47.50°)
Jamming 4 (θ_4, ϕ_4)	-	(92.50°, 45°)
Jamming 1, 2, 3, 4 SNR	40 dB	40 dB

The results presented in Figure 1 show a thinned adaptive beamformer designed iteratively by the L_1-CNLMS algorithm with only 57% of the total number of elements in use. Performance in terms of beampattern and MSE, as depicted in Figures 1 and 2, respectively, show negligible penalty incurred by sparsity.

4.2. HAA for Ku-band Communication Satellite

The second simulation was carried out considering a hexagonal antenna array (HAA) for Ku-band Communication Satellite. This HAA employs 12 elements on each edge, resulting in a total of 397 elements. The scheme proposed is based on a real satellite communication system able to generate 64 beam spots [18] with average center-to-center spot angle equal to $2.5°$. Although the frequency reuse of 1:4 imposes that no adjacent spot shares the same frequency channel, the short separation between the spots of same frequency group causes mutual interference (see Figure 3). Due to the high gains required by satellite antenna systems, it is sometimes not possible to use SHA in which the distance between adjacent array elements is half wavelength. In this simulation, ths distance is equal to one wavelength. Five narrowband analytical QPSK signals are considered in this simulation, i.e., a signal of interest and four interfering signals (jammers) disposed in a footprint according to the lattice shown in Figure 3.

This simulation intends to shown the narrowband interference between channels using the same frequency. Only one spot is considered, although this result can be extended to the total number of spots.

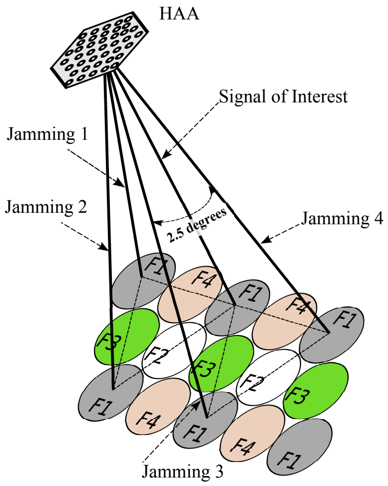
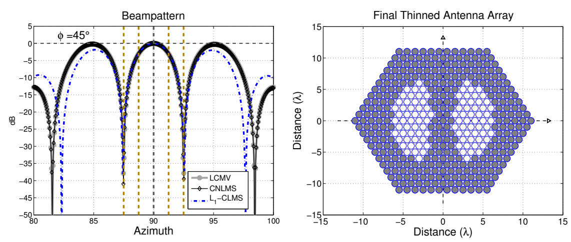
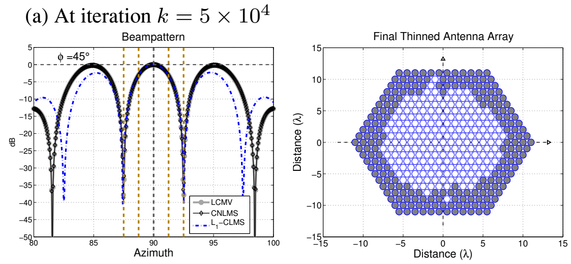


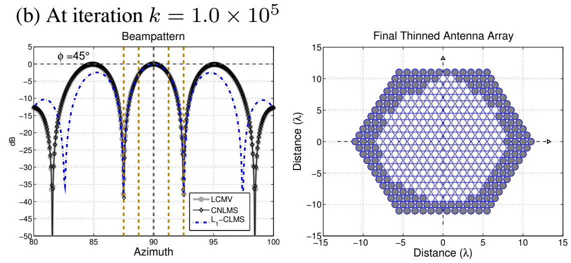
Fig. 3: Spot footprint.



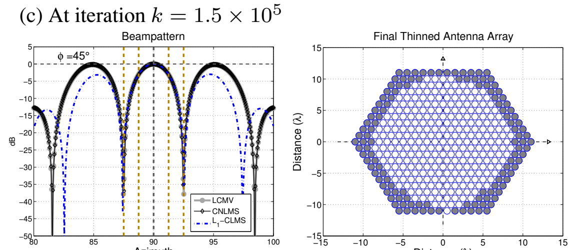
(a) At iteration $k = 5 \times 10^4$



(b) At iteration $k = 1.0 \times 10^5$



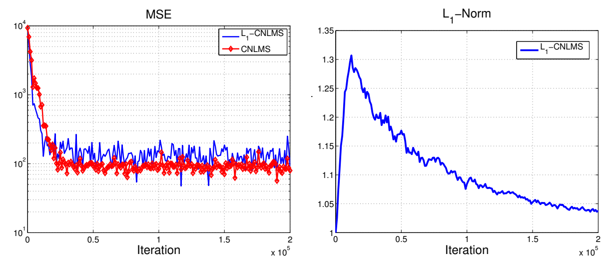
(c) At iteration $k = 1.5 \times 10^5$



(d) At iteration $k = 2.0 \times 10^5$

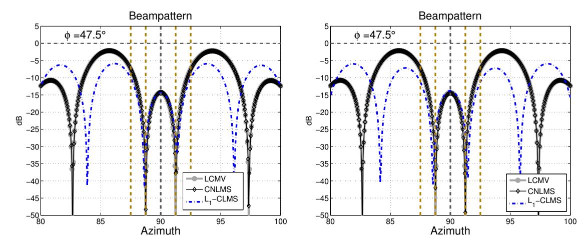
Fig. 4: Thinned HAA and respective beampattern for the L_1-CNLMS and CNLMS algorithms, and the LCMV solution. The elements represented by white circles are those turned off.

In this simulation, the thinned beamformer designed recursively with the new L_1-CNLMS algorithm uses, after convergence, only 32% of the total number of elements without compromising the beampattern response or MSE convergence, as shown in Figures 4, 5 and 6. A typical value for L_1-norm constraint t equal to 1 was used, this value is the minimal L_1-norm value feasible according to constraints stated in Eq. (4).

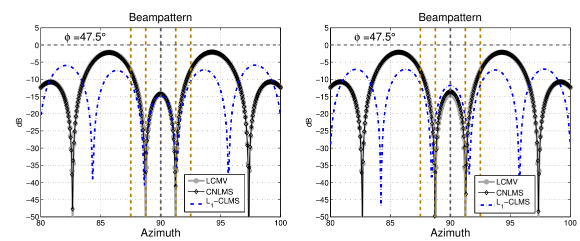


(a) MSE (b) L_1-norm

Fig. 5: MSE and L_1-norm for coefficient vector during shrinkage process applied to a 397-elements HAA.



(a) At iteration $k = 5.0 \times 10^4$ (b) At iteration $k = 1.0 \times 10^5$



(c) At iteration $k = 1.5 \times 10^5$ (d) At iteration $k = 2.0 \times 10^5$

Fig. 6: HAA beampattern at zenith $\phi = 47.5°$ for the L_1-CNLMS and CNLMS algorithms, and the LCMV solution.

5. CONCLUSIONS

The results of the simulations presented in Section 4 show that the L_1-CNLMS algorithm, proposed in this paper, provides a suitable adaptive algorithm for mobile satellite communication systems where power supply requirements are critical and the use of adaptive antenna array systems became a necessity.

The ability of the proposed algorithm to produce sparse solutions, with subsequent shutting down of a large number of system's Low Noise Blocks (LNB), allows flexibility in the energy consumption of satellite communications systems. Similar results can also be obtained in mobile communication systems that employ adaptive antenna arrays.

6. REFERENCES

[1] Robert Tibshirani, "Regression shrinkage and selection via the LASSO," *Journal of the Royal Statistical Society. Series B (Methodological)*, vol. 58, no. 1, pp. 267–288, 1996.

[2] Yilun Chen, Yuantao Gu, and Aalfred O. Hero III, "Sparse LMS for system identification," in *Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP 2009)*, Ann Arbor, USA, April 2009.

[3] Scott Shaobing Chen, David L. Donoho, Michael, and A. Saunders, "Atomic decomposition by basis pursuit," *SIAM Journal on Scientific Computing*, vol. 20, pp. 33–61, 1998.

[4] Bradley Efron, Trevor Hastie, Iain Johnstone, and Robert Tibshirani, "Least angle regression," *Annals of Statistics*, vol. 32, pp. 407–499, 2004.

[5] M. L. R. de Campos and J. A. Apolinário, Jr., "Shrinkage methods applied to adaptive filters," in *Proceedings of the International Conference on Green Circuits and Systems*, Shangai, June 2010, IEEE, pp. 41–45.

[6] José F. de Andrade Jr., Marcello L. R. de Campos, and José A. Apolinário Jr., "An L1-norm linearly constrained LMS algorithm applied to adaptive beamforming," in *Proceedings of the 7th IEEE Sensor Array and Multichannel Signal Processing Workshop (SAM 2012)*, Hoboken, USA, June 2012.

[7] José A. Apolinário Jr., Stefan Werner, Paulo S. R. Diniz, and Timo I. Laakso, "Constrained normalized adaptive filtering for CDMA mobile communications," in *Proceedings of the 9th European Signal Processing Conference (EUSIPCO 1998)*, Rhodes, Greece, September 1998.

[8] A. V. Shishlov, "Vehicular antennas for satellite communications," in *Proceedings of the VIII International Conference on Antenna Theory and Techniques (ICATT 2011)*, Kyev, Ukraine, September 2011.

[9] Roberto V. Gatti, Luca Marcaccioli, Elisa Sbarra, and Roberto Sorrentino, "Flat array antenna for Ku-band mobile satellite terminals," in *Proceedings of the 5th European Conference on Antennas and Propagation (EuCAP 2011)*, Rome, Italy, April 2011.

[10] M. Tripodi, F. DiMarca, T. Cadili, C. Mollura, F. DiMaggio, and M. Russo, "Ka band active phased array antenna system for satellite communication on the move terminal," in *Proceedings of the 5th European Conference on Antennas and Propagation (EuCAP 2011)*, Rome, Italy, April 2011.

[11] Len P. Calabretta, Matthew Ryerse, and Lawrence N. Goeller, "X-band, K_a-band, and future Milsatcom requirements: an outing in trade space," in *Proceedings of the IEEE Military Communications Conference (MILCOM 1998)*, Boston, USA, October 1998.

[12] Richard S. Wexler, Richard Hoffmann, Robert Collins, and Phil Moran, "Successful development and test of SATCOM On-The-Move (OTM) Ku-band Ka-band Systems for the Army's Warfighter Information Network-Tactical (WIN-T)," in *Proceedings of the IEEE Military Communications Conference (MILCOM 2006)*, Washington, USA, October 2006.

[13] Harry L. Van Trees, *Optimum Array Processing (Detection, Estimation, and Modulation Theory, Part IV)*, Wiley-Interscience, 1st edition, March 2002.

[14] Luiz C. Kretly, Arismar Cerqueira S. Jr., and A. Tavora A.S., "A hexagonal antenna array prototype for adaptive system application," in *Proceedings of the 5th International Symposium on Wireless Personal Multimedia Communications (WPMC 2002)*, Honolulu, USA, October 2002.

[15] Otis L. Frost III, "An algorithm for linearly constrained adaptive array processing," *Proceedings of the IEEE*, vol. 60, no. 8, pp. 926–935, August 1972.

[16] Paulo S. R. Diniz, *Adaptive Filtering: Algorithms and Practical Implementation*, Springer, 3rd edition, October 2010.

[17] G. Liang, W. B. Gong, H. J. Liu, and J. P. Yu, "Development of 61-channel digital beamforming (DBF) transmitter array for mobile satellite communication," *Progress In Electromagnetics Research (PIER)*, vol. 97, pp. 177–195, 2009.

[18] M.C. Vigano, G. Toso, S. Selleri, C. Mangenot, P. Angeletti, and G. Pelosi, "GA optimized thinned hexagonal arrays for satellite applications," in *Proceedings of 2007 IEEE Antennas and Propagation Society International Symposium*, Honolulu, USA, June 2007.