CONVERGENCE RATE OF THE DOMINANT MODE REJECTION BEAMFORMER FOR A SINGLE INTERFERER

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ABSTRACT

The dominant mode rejection (DMR) adaptive beamformer (ABF) is a reduced-rank version of the standard minimum variance distortionless response (MVDR) ABF. DMR uses a structured estimate of the covariance matrix derived from an eigendecomposition of the sample covariance matrix. This paper exploits results from random matrix theory to fit a distribution for the SINR loss of the DMR ABF in the single interferer case. Monte Carlo simulations confirm the accuracy of these expressions. SINR loss quantifies the rate at which the performance of the DMR ABF converges to the performance of the optimal processor. For the single interferer case, the mean SINR loss only depends on the number of snapshots.

Index Terms— adaptive arrays, beamforming, random matrix theory, sample covariance matrix

1. INTRODUCTION

Applications such as passive sonar require detection and localization of quiet sources in the presence of interfering signals and background noise. When the interferers are loud, adaptive processing is typically required since interference can leak through the sidelobes of a conventional beamformer (CBF) and mask a quiet source. Many narrowband adaptive beamformers (ABFs) rely on knowledge of the second-order statistics to determine an appropriate set of weights. For example, the minimum variance distortionless response (MVDR) ABF developed by Capon [1] has the weight vector $\mathbf{w}$:

$$\mathbf{w} = (\mathbf{v}_s^{\mathrm{H}} \mathbf{\Sigma}^{-1} \mathbf{v}_s)^{-1} \mathbf{\Sigma}^{-1} \mathbf{v}_s, \quad (1)$$

where $\mathbf{\Sigma}$ is the ensemble covariance matrix (ECM) of the narrowband data and $\mathbf{v}_s$ is the planewave replica vector associated with the steering direction. In practice the ECM is not available and is replaced by the sample covariance matrix (SCM). Performance depends on the number of snapshots available to estimate the SCM. Abraham and Owsley proposed a reduced-rank version of MVDR called dominant mode rejection (DMR) [2]. The DMR ABF uses a structured estimate of the covariance matrix derived from an eigendecomposition of the SCM. The structured covariance used in DMR models only the loudest discrete interferers, thus it typically requires fewer snapshots to estimate. DMR is related to other eigenspace ABFs, such as those proposed by Hung and Turner [3], Van Veen [4], and Chang and Yeh [5]. The key difference between DMR and these other eigenspace approaches is that DMR assumes its reduced-rank subspace contains only interference, whereas the other approaches assume that the subspace contains both the desired signal and the interference. These differing assumptions lead to fundamental performance differences, particularly when the subspaces must be estimated from the SCM. See Van Trees [6, pp. 556-575] for examples comparing DMR with other eigenspace algorithms. DMR's assumption about the subspace means that it is especially suitable for applications where the source is very quiet compared to the interferers.

It is important to know how many snapshots an ABF requires to achieve a desired level of performance. A metric that is often used to quantify convergence time is the output signal-to-interference and noise ratio (SINR) loss. SINR loss is defined as the ratio of the SINR obtained using sample statistics to the SINR obtained with ensemble statistics. Thus SINR loss quantifies how close an ABF comes to the performance of an optimal processor (one that knows the true statistics). In a classic paper, Reed *et al.* show that the SINR loss of the MVDR ABF is beta-distributed [7]. Their results demonstrate that MVDR requires at least twice as many snapshots as sensors to get within 3 dB of the optimal SINR. No one has derived a comparable theoretical result for the SINR loss of the DMR ABF. Some authors, *e.g.*, Chang and Yeh [5] and Feldman and Griffiths [8], derive results for other eigenspace approaches, but these are not applicable to DMR due to different assumptions about the contents of the reduced-rank subspace.

The goal of this paper is to obtain the statistics of SINR loss for the DMR ABF. The paper focuses on the case of a single interferer since it illustrates the fundamental aspects of the problem and provides useful insights. This study builds on our previous work analyzing the DMR ABF using random matrix theory (RMT) [9, 10, 11]. An important advantage of RMT is that it facilitates analysis when the number of snapshots is less than the number of sensors. Since reduced-rank ABFs are designed to operate with low numbers of snapshots, it is crucial to have valid performance predictions for snapshot-deficient cases. Our previous DMR analyses build on recent RMT results on the accuracy of the eigenvectors of the SCM [12, 13, 14].

The rest of the paper is organized as follows. The following section defines notation and provides background on narrowband beamforming using DMR. Sec. 3 reviews the relevant results from our previous RMT analysis of the DMR ABF. Sections 4 and 5 present the new results for the mean SINR loss and the distribution of SINR loss, respectively. Sec. 6 concludes the paper.

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2. BACKGROUND

Consider the narrowband planewave beamforming problem. Assume that the measured sensor data vector $\mathbf{p}$ consists of a single loud planewave interferer plus spatially-white noise: $\mathbf{p} = \tilde{b}_1\mathbf{v}_1 + \mathbf{n}$, where $\tilde{b}_1$ is the random complex amplitude of the interferer, $\mathbf{v}_1$ is the interferer replica, and $\mathbf{n}$ is the noise vector. $\tilde{b}_1$ is a complex circular Gaussian random variable and $\mathbf{n}$ is a vector of independent complex circular Gaussian random variables. Using weight vector $\mathbf{w}$, the beamformer output is $\mathbf{w}^{\mathrm{H}}\mathbf{p}$, where $^{\mathrm{H}}$ denotes the Hermitian transpose. The remainder of this section provides the necessary background on the DMR ABF and develops simplified expressions for the SINR in the single interferer case.

2.1. Dominant Mode Rejection Adaptive Beamformer

As noted in the introduction, the DMR ABF relies on a structured estimate of the covariance matrix that is based on the eigendecomposition of the SCM. Consider the SCM estimated using L snapshots and its eigendecomposition:

$$\mathbf{S} = \frac{1}{L}\sum_{l=1}^{L}\mathbf{p}_l\mathbf{p}_l^{\mathrm{H}} = \underbrace{\sum_{n=1}^{N} g_n\mathbf{e}_n\mathbf{e}_n^{\mathrm{H}}}_{\text{eigendecomposition}}. \tag{2}$$

The array has N sensors. g_n is the nth sample eigenvalue and $\mathbf{e}_n$ is the nth sample eigenvector. DMR assumes that the eigenvectors associated with the D largest eigenvalues define the interference subspace. It approximates the covariance matrix as the sum of the interference subspace plus an estimate of the noise subspace, *i.e.*,

$$\mathbf{S}_{\mathrm{DMR}} = \underbrace{\sum_{n=1}^{D} g_n\mathbf{e}_n\mathbf{e}_n^{\mathrm{H}}}_{\text{largest e-vals}} + \sum_{n=D+1}^{N} s_w^2\mathbf{e}_n\mathbf{e}_n^{\mathrm{H}} \tag{3}$$

where s_w^2 is the estimated noise power:

$$s_w^2 = \left(\frac{L}{L-1}\right)\left(\frac{1}{N-D}\right)\sum_{n=D+1}^{N} g_n. \tag{4}$$

Note that s_w^2 is the average of the lowest eigenvalues scaled by $\frac{L}{L-1}$, which accounts for the inherent bias in the sample variance calculation. The DMR ABF weight vector has the same form as the MVDR weight vector defined in (1) with $\mathbf{\Sigma}$ replaced by $\mathbf{S}_{\mathrm{DMR}}$, *i.e.,*

$$\mathbf{w}_{\mathrm{DMR}} = \left(\mathbf{v}_s^{\mathrm{H}}\mathbf{S}_{\mathrm{DMR}}^{-1}\mathbf{v}_s\right)^{-1}\mathbf{S}_{\mathrm{DMR}}^{-1}\mathbf{v}_s, \tag{5}$$

where $\mathbf{v}_s$ is the planewave replica associated with the steering direction.

Computing the DMR weight vector requires knowledge of the rank D of the interference subspace. In practice D is estimated from data using standard criteria such as those described by Wax and Kailath [15] and Nadakuditi and Edelman [16]. For the single interferer case considered in this paper, assume that $D = 1$ is known.

Similar to other ABF algorithms, DMR performs poorly if the interferer moves inside the mainlobe. For this reason most DMR implementations have a test to exclude sample eigenvectors that are close to the steering direction. Cox and Pitre suggest removing an eigenvector when the generalized cosine squared between it and the steering replica is greater than 0.5 [17], meaning that it lies within the -3 dB points of the mainlobe.

2.2. SINR for the Single Interferer Case

The output SINR for a planewave source with power σ_s^2 is defined as

$$\mathrm{SINR} \triangleq \frac{\sigma_s^2|\mathbf{w}^{\mathrm{H}}\mathbf{v}_s|^2}{\mathbf{w}^{\mathrm{H}}\mathbf{\Sigma}_{\mathrm{I+N}}\mathbf{w}}, \tag{6}$$

where $\mathbf{v}_s$ is the replica associated with the source and $\mathbf{\Sigma}_{\mathrm{I+N}}$ is the ensemble interference-plus-noise (I+N) covariance matrix. Note that while the ensemble I+N matrix is used for the SINR calculation, the weight vector in (6) is computed using the estimated DMR covariance $\mathbf{S}_{\mathrm{DMR}}$. For the case of a single interferer in spatially-white noise, the ensemble covariance is defined as $\mathbf{\Sigma}_{\mathrm{I+N}} = \sigma_1^2\mathbf{v}_1\mathbf{v}_1^{\mathrm{H}} + \sigma_w^2\mathbf{I}$, where σ_1^2 is the interferer power, $\mathbf{v}_1$ is the interferer replica, and σ_w^2 is the noise power. Substituting the ECM into (6) and rearranging yields

$$\mathrm{SINR} = \frac{\frac{\sigma_s^2}{\sigma_w^2}|\mathbf{w}^{\mathrm{H}}\mathbf{v}_s|^2}{\frac{\sigma_1^2}{\sigma_w^2}|\mathbf{w}^{\mathrm{H}}\mathbf{v}_1|^2 + \mathbf{w}^{\mathrm{H}}\mathbf{w}} = \frac{\mathrm{SNR}\cdot|\mathbf{w}^{\mathrm{H}}\mathbf{v}_s|^2}{\mathrm{INR}\cdot|\mathbf{w}^{\mathrm{H}}\mathbf{v}_1|^2 + \mathbf{w}^{\mathrm{H}}\mathbf{w}}, \tag{7}$$

where SNR and INR are the signal-to-noise ratio and interference-to-noise ratio, respectively. Assuming that the ABF is steered towards the true source direction, $|\mathbf{w}^{\mathrm{H}}\mathbf{v}_s|^2$ equals 1. In this case the SINR can be written as a function of the beamformer notch depth (ND) and white noise gain (WNG):

$$\mathrm{SINR} = \frac{\mathrm{SNR}}{\mathrm{INR}\cdot\mathrm{ND} + 1/\mathrm{WNG}} = \frac{\mathrm{SNR}\cdot\mathrm{WNG}}{\mathrm{INR}\cdot\mathrm{ND}\cdot\mathrm{WNG} + 1}. \tag{8}$$

ND is the absolute value squared of the beampattern in the direction of the interferer:

$$\mathrm{ND} \triangleq |B(\theta_1)|^2 = \left|\mathbf{w}^{\mathrm{H}}\mathbf{v}_1\right|^2, \tag{9}$$

and WNG is the improvement in SNR provided by the beamformer when the noise is spatially white:

$$\mathrm{WNG} \triangleq \frac{1}{\mathbf{w}^{\mathrm{H}}\mathbf{w}}. \tag{10}$$

For a given scenario (fixed INR and SNR), (8) shows that SINR reduces to a simple function of ND and WNG for the single interferer case. The next section reviews relevant results about WNG and ND for the DMR ABF.

3. REVIEW OF PRIOR RESULTS ON DMR NOTCH DEPTH AND WHITE NOISE GAIN

In previous work we analyzed the ensemble performance of the DMR ABF [11], used numerical simulations to characterize WNG [11] for the DMR ABF implemented using the SCM, and applied RMT to develop and validate a model for mean ND [9, 10]. This section provides a brief summary of these results so they can be used in the analysis of SINR loss.

First, the analysis of the ensemble case in [11] shows that for high INR, the WNG of the ensemble DMR ABF is equal to:

$$\mathrm{WNG}_{\mathrm{ens}} = \frac{N}{1 + \cot^2(\mathbf{v}_1, \mathbf{v}_s)}, \tag{11}$$

where $\cot^2$ is defined as the square of the ratio of the generalized cosine and sine between the interferer replica and the steering replica. The factor of $\frac{1}{1+\cot^2}$ predicts the loss in WNG of the ensemble DMR ABF as compared to the optimal WNG of N. This loss is the cost of placing a notch in the beampattern in the direction of the interferer. Note that in the ensemble case, the loss factor is only a function of the interferer location relative to the look direction and not a function of the INR. For interferers outside the mainlobe, $\cot^2$ is small, thus $\text{WNG}_{\text{ens}} \approx N$.

The second result of the analysis in [11] is that the ensemble ND is guaranteed to be deep enough that the ensemble SINR depends only on the white noise component, *i.e.*,

$$\text{SINR}_{\text{ens}} \approx \text{SNR} \cdot \text{WNG}_{\text{ens}} \approx \text{SNR} \cdot N, \tag{12}$$

where the final expression uses the approximation for WNG_{ens} discussed above.

In addition to analyzing the ensemble performance, Reference [11] presents a detailed empirical study of the single interferer case for a 50-element array. The environment includes a single interferer located near the peak sidelobe of the CBF and spatially-white noise. The simulation data includes 3000 Monte Carlo trials for each set of parameters. INR varies between -40 dB and +40 dB and the number of snapshots varies between 2 and 50,000. The numerical results presented in Sections 4-5 are based on the same simulation data set. One important result of the empirical study is that, for interferers located outside the mainlobe, the WNG of the sample DMR ABF is concentrated around the ensemble value. Thus, it is appropriate to assume that $\text{WNG} \approx N$ for all values of INR and L.

The final result needed for the SINR loss analysis is the mean ND. Using recent results from RMT, Reference [9] develops a model for mean ND as a function INR and the number of snapshots. A companion paper [10] validates the ND model using data from a recent deep water tomography experiment. Similar to a Bode plot for frequency response magnitude, the RMT ND model produces piecewise linear asymptotes on a log-log plot. While the model works over a large range of INRs and interferer locations, the SINR loss derivation below focuses on loud interferers located outside the mainlobe. For these interferers, the model predicts that the mean ND is inversely proportional to the number of sensors, the INR, and the number of snapshots:

$$\mathcal{E}(\text{ND}) \approx \frac{1}{N \cdot \text{INR} \cdot L}. \tag{13}$$

4. MEAN SINR LOSS

As discussed in Sec. 1, the SINR loss ρ is defined as the ratio of the SINR obtained using sample statistics to the SINR obtained with ensemble statistics. For the single interferer case, ρ is

$$\rho \triangleq \frac{\text{SINR}}{\text{SINR}_{\text{ens}}} = \frac{\dfrac{\text{SNR} \cdot \text{WNG}}{\text{WNG} \cdot \text{INR} \cdot \text{ND} + 1}}{\text{SINR}_{\text{ens}}}. \tag{14}$$

Using the approximations for SINR_{ens} and WNG discussed in the previous section, SINR loss reduces to a function of ND and the parameters N and INR:

$$\rho \approx \frac{\dfrac{\text{SNR} \cdot N}{N \cdot \text{INR} \cdot \text{ND} + 1}}{\text{SNR} \cdot N} = \frac{1}{N \cdot \text{INR} \cdot \text{ND} + 1}. \tag{15}$$

Assuming that ND is sufficiently concentrated about its mean, $\mathcal{E}(\rho)$ can be written [18, pp. 112-113]:

$$\mathcal{E}(\rho) \approx \frac{1}{N \cdot \text{INR} \cdot \mathcal{E}(\text{ND}) + 1}. \tag{16}$$

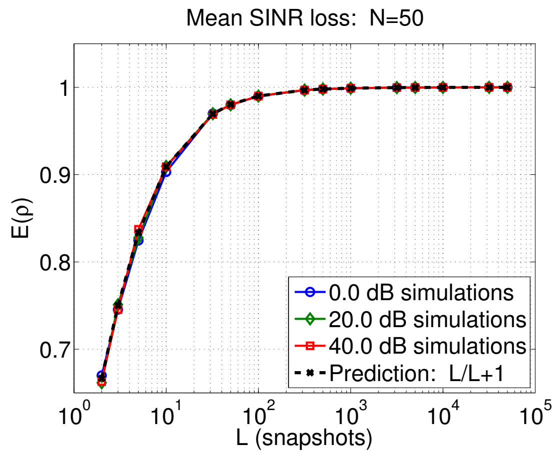

Fig. 1. Mean SINR loss as a function of snapshots for a 50-element linear array with a single interferer located near the peak sidelobe of the conventional beamformer. The plot compares the prediction from (17) with estimates from simulations (3000 Monte Carlo trials) for three different INRs: 0 dB, 20 dB, and 40 dB.

Substituting the mean ND from the RMT model in (13) and simplifying yields the following expression for mean SINR loss:

$$\mathcal{E}(\rho) \approx \frac{L}{L+1}. \tag{17}$$

According to (17), $\mathcal{E}(\rho)$ is independent of N and INR. It is important to note that this result assumes that the array has the aperture required to keep the interferer out of the mainlobe of the look direction.

Fig. 1 compares the mean SINR loss prediction to simulation results. The simulations are for a 50-element linear array with a single interferer located near the peak sidelobe of the CBF. The figure shows results for three INRs between 0 dB and 40 dB. There is very good agreement between the RMT prediction and the simulation results over the entire range of snapshot values. Note that $\mathcal{E}(\rho)$ is greater than 0.5 for all values of L, which indicates that for a single interferer, the DMR ABF only requires 2 snapshots to achieve a mean output SINR within 3 dB of the optimal. This is consistent with the conventional wisdom that reduced-rank ABFs require the number of snapshots to be proportional to the number of interferers, rather than the number of sensors [5, 8]. For a single interferer, DMR requires two snapshots, as opposed to the $2N$ snapshots that MVDR requires [7].

5. DISTRIBUTION OF SINR LOSS

As stated in Sec. 1, the goal of the paper is to obtain the distribution of SINR loss for the DMR ABF. Based on (15) the distribution of ρ depends on the distribution of ND. While our previous RMT analysis [9] provides an accurate prediction of mean ND, it does not derive the associated probability distribution function (PDF). Fortunately, intution about the PDF can be obtained from histograms of simulation data. Fig. 2 shows the histogram of ND (on a linear scale) for the case of a single interferer with 40 dB INR. The weight vector for the 50-element array was generated using $L = 32$ snapshots. Based on this histogram the distribution appears to be exponential. This hypothesis was tested using a large set of simulations for the

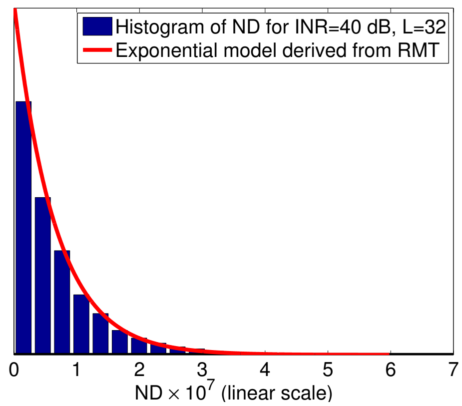

Fig. 2. Histogram of ND on a linear scale obtained for a single interferer with 40 dB INR and 32 snapshots. The red solid line is the exponential PDF with the mean parameter derived from the RMT ND model.

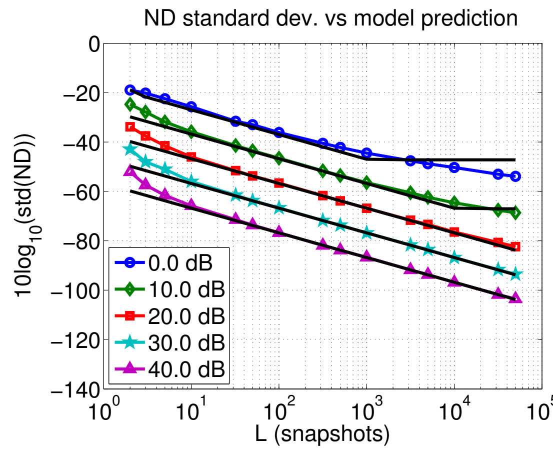

Fig. 3. Standard deviation of ND versus number of snapshots on a log-log scale. The colored lines and symbols show the average results for 3000 Monte Carlo trials. The black lines overlaid on the plot are the RMT model predictions.

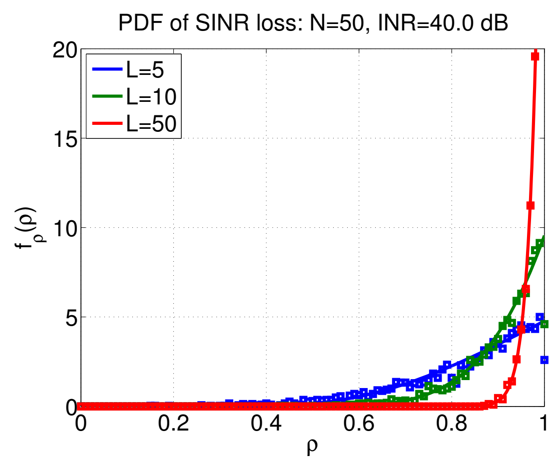

Fig. 4. PDF of SINR loss for the case of a single interferer with 40 dB INR. Solid lines are the model predictions derived from (19) and symbols indicate the histogram data based on the simulations.

50-element array. The simulations are for a single interferer located near the peak sidelobe of the CBF. The simulations considered snapshot values between 2 and 50,000 and INRs between 0 and 40 dB. Kolmogorov-Smirnov (K-S) tests [18] of this data set indicate that the ND is exponentially-distributed when $L \geq 32$ and INR ≥ 30 dB. For lower INRs, ND is exponentially-distributed for some snapshot cases, but not all.

Based on the results of the K-S tests, it appears that ND is exponentially-distributed for sufficiently high INR. Recall that the exponential PDF has a single parameter a [18]:

$$f_{\mathrm{ND}} \approx \begin{cases} ae^{-a(\mathrm{ND})} & \mathrm{ND} \geq 0 \\ 0 & \mathrm{ND} < 0 \end{cases}, \tag{18}$$

where a is the mean of the exponential random variable. According to the RMT ND model, a should be $1/(N \cdot \mathrm{INR} \cdot L)$. Since the standard deviation of an exponential random variable is equal to its mean, the RMT model should predict the spread of ND. To check this prediction, Fig. 3 plots the standard deviation of ND as a function of snapshots for the simulation data set. The black lines overlaid on the plot are the predicted standard deviation based on the RMT model. The agreement is very good for a wide range of L values.

Assuming that the ND is an exponential with known mean, it is straightforward to derive the cumulative distribution function for ρ using (15) and differentiate to obtain the PDF. The resulting PDF is f_ρ:

$$f_\rho(\rho) \approx \frac{Le^L}{\rho^2} e^{-L/\rho}. \tag{19}$$

Fig. 4 compares the analytical PDF of SINR loss with histograms of simulation data. The solid lines show the model prediction for three snapshot cases ($L = 5, 10, 50$), and the symbols show the histograms for the same cases. The agreement between the analytical predications and the simulation data is quite good.

6. SUMMARY

This paper presents new theoretical results for the SINR loss of the DMR ABF in the single interferer case. Monte Carlo simulations confirm the accuracy of the predictions for the mean and the distribution. Assuming that the array has sufficient aperture to keep the interferer out of the mainlobe, the mean SINR loss only depends on the number of snapshots, and is not a function of array size. The derived distribution of SINR loss indicates that the DMR ABF is likely to achieve performance within 3 dB of the optimal beamformer with only a few snapshots when there is a single interferer. Future work will focus on extending these results to the multiple interferer case.

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