ADAPTIVE FREQUENCY ESTIMATION OF THREE-PHASE POWER SYSTEMS WITH NOISY MEASUREMENTS

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ABSTRACT

We examine the problem of estimating the frequency of a three-phase power system in an adaptive and low-cost manner when the voltage readings are contaminated with observational error and noise. We assume a widely-linear predictive model for the $\alpha\beta$ complex signal of the system that is given by Clarke's transform. The system frequency is estimated using the parameters of this model. In order to estimate the model parameters while compensating for noise in both input and output of the model, we utilize the notions of total least-squares fitting and gradient-descent optimization. The outcome is an augmented gradient-descent total least-squares (AGDTLS) algorithm that has a computational complexity comparable to that of the complex least mean square (CLMS) and the augmented CLMS (ACLMS) algorithms. Simulation results demonstrate that the proposed algorithm provides significantly improved frequency estimation performance compared with CLMS and ACLMS when the measured voltages are noisy and especially in unbalanced systems.

***Index Terms*—**adaptive frequency estimation, gradient-descent optimization, smart grids, total least-squares, widely-linear modeling.

1. INTRODUCTION

Smart grids collect and act on information regarding the behavior of the consumers and suppliers in an automated manner to enhance the efficiency, reliability, economy, and sustainability of the generation, distribution, and consumption of electrical energy [1].

System frequency is amongst the most important and sensitive parameters to be constantly monitored in the smart grids. Accurate power frequency estimation is crucial to check the health state of the power grid and assures reliable measurement of other system parameters such as voltages, currents, and active and reactive powers. Market economy will presumably drive power systems to operate much closer to their limits necessitating a perfect generation-load balance. Deviation of the system frequency from its rated value faithfully portrays an imbalance between the power generation and load demand. Accordingly, many power-system protection-and-control applications require accurate and fast estimation of the system frequency. An erroneous frequency estimate can result in insufficient load shedding by frequency relays, which in turn may ultimately cause a catastrophic grid failure [2].

Research on frequency estimation of power systems has been conducted for decades generating a copious body of literature (e.g., see [2]-[22] and references therein). Several methods have been proposed to estimate the power system frequency based on zero-crossing technique [4], phase-locked loop [5]-[8], least-squares adaptive filtering [9]-[11], and extended Kalman filter [12]-[14], to name a few. Most of these methods rely on the voltage readings of a single phase of the system. In three-phase systems, none of the single phases can necessarily characterize the whole system and its properties. Therefore, a robust frequency estimator should take into account the information of all three phases [15]-[18]. Applying Clark's transform (also known as $\alpha\beta$ transform), a single complex signal can be used to encompass the three-phase information [19]. It has been shown that the frequency of the three-phase power system can be estimated using a linear predictive model for this complex signal ($\alpha\beta$ signal) [20], [21]. However, since the $\alpha\beta$ signal is improper (its real and imaginary parts have different statistical properties) [22]-[24] when the system is unbalanced (e.g., phases feature different peak voltages), it is better described via a widely-linear model rather than a strictly-linear one [25], [26].

In [21], an algorithm for frequency estimation of three-phase power systems utilizing the $\alpha\beta$ signal is developed based on the widely-linear (augmented) complex least mean square (ACLMS) algorithm [27]. In unbalanced situations, this algorithm significantly outperforms its strictly-linear counterpart proposed in [20], which is based on the complex least mean square (CLMS) algorithm [28], while enjoying the simplicity and numerical stability of the LMS-type algorithms. However, it assumes a noise-free environment,

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i.e., where the voltage measurements are exact and error-free. Such assumptions are often unrealistic since several kinds of error can contaminate the measurements, e.g., sampling, quantization, and instrument errors. Therefore, in practice, this algorithm may have a poor estimation performance because of failing to account for the error in the signals.

Total least-squares (TLS) is a fitting method that improves the accuracy of the least-squares estimation techniques when both the input and output data of a linear system are subject to observational error. TLS minimizes the perturbation in the input and output data that is required to fit the input to the output [29]-[31].

In this paper, we develop a frequency estimation algorithm for three-phase power systems assuming noisy phase voltage observations. To this end, we utilize the concepts of TLS fitting and gradient-descent optimization to compute the parameters of a widely-linear predictive model considered for the $\alpha\beta$ signal. The system frequency is then calculated using the model parameters. Simulations testify that the performance of the new algorithm is superior to that of the ACLMS algorithm when the phase voltages are measured in noise while being almost as computationally efficient as ACLMS.

2. PROPOSED ALGORITHM

The voltages of a three-phase power system can be represented as

$$v_n^a = V_n^a \cos(2\pi f \tau n + \phi) + \eta_n^a$$

$$v_n^b = V_n^b \cos\left(2\pi f \tau n + \phi - \tfrac{2\pi}{3}\right) + \eta_n^b$$

$$v_n^c = V_n^c \cos\left(2\pi f \tau n + \phi + \tfrac{2\pi}{3}\right) + \eta_n^c$$

where V_n^a, V_n^b, and V_n^c are the peak values, f is the system frequency, τ is the sampling interval, n is the time index and ϕ is a constant phase while η_n^a, η_n^b, and η_n^c denote the observational errors and noises.

Using Clarke's ($\alpha\beta$) transform [19], i.e.,

$$\begin{bmatrix} v_n^\alpha \\ v_n^\beta \end{bmatrix} = \sqrt{\tfrac{2}{3}} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & \sqrt{3}/2 & -\sqrt{3}/2 \end{bmatrix} \begin{bmatrix} v_n^a \\ v_n^b \\ v_n^c \end{bmatrix},$$

we obtain a complex-valued voltage signal as

$$v_n = v_n^\alpha + j v_n^\beta$$

that can be used for adaptive frequency estimation [32], [33]. Here, $j = \sqrt{-1}$.

A *widely-linear* predictive model for v_n is described as

$$\tilde{v}_{n-1} h + \tilde{v}_{n-1}^* g = \tilde{v}_n$$

or

$$[\tilde{v}_{n-1}, \tilde{v}_{n-1}^*] \begin{bmatrix} h \\ g \end{bmatrix} = \tilde{v}_n$$

where $\tilde{v}_n$ is the noiseless value of v_n, i.e., when $\eta_n^a = \eta_n^b = \eta_n^c = 0$, and superscript $*$ denotes complex-conjugate while h and g are the model parameters that we wish to identify. It is shown in [21] that, using h and g, the system frequency can be estimated as

$$\hat{f} \approx \frac{1}{2\pi\tau} \sin^{-1}\left(\sqrt{\Im^2(h) - |g|^2}\right)$$

where $\Im(\cdot)$ and $|\cdot|$ denote the imaginary part and the absolute value, respectively.

In order to identify h and g at the presence of noise, we utilize an adaptive filter whose tap weights vector, denoted by $\mathbf{w}_n = [w_{n,1}, w_{n,2}]^T$, is taken as an estimate for $[h, g]^T$ at iteration n. We wish to compute $\mathbf{w}_n$ such that it fits the filter input data to the desired filter output data by incurring minimum perturbation:

$$(\mathbf{X}_n^T + \mathbf{\Delta}_n)\mathbf{w}_n = \mathbf{y}_n + \boldsymbol{\delta}_n \tag{1}$$

where

$$\mathbf{X}_n = [\mathbf{x}_1, \dots, \mathbf{x}_{n-1}, \mathbf{x}_n],$$

$$\mathbf{y}_n = [v_1, \dots, v_{n-1}, v_n]^T,$$

$$\mathbf{x}_n = [v_{n-1}, v_{n-1}^*]^T,$$

superscript T stands for transpose, and $\mathbf{\Delta}_n \in \mathbb{C}^{n\times 2}$ and $\boldsymbol{\delta}_n \in \mathbb{C}^{n\times 1}$ denote the minimum input and output perturbations, respectively. Using the singular value decomposition (SVD) of the augmented data matrix, $[\mathbf{X}_n^T, \mathbf{y}_n]$, the total least-squares (TLS) solution for (1) is given by [31]

$$\mathbf{w}_n = -\frac{[z_{n,1}, z_{n,2}]^T}{z_{n,3}} \tag{2}$$

where $[z_{n,1}, z_{n,2}, z_{n,3}]^T$ is the right singular vector corresponding to the smallest singular value of $[\mathbf{X}_n^T, \mathbf{y}_n]$ or the eigenvector corresponding to the smallest eigenvalue of

$$\mathbf{\Psi}_n = \begin{bmatrix} \mathbf{X}_n \\ \mathbf{y}_n^T \end{bmatrix}^* [\mathbf{X}_n^T, \mathbf{y}_n].$$

The solution of (2) is optimal. However, obtaining it comes at the expense of updating and performing eigendecomposition of the 3×3 matrix, $\mathbf{\Psi}_n$, at each iteration. In the light of the analysis of [31], a computationally more efficient alternative approach can be devised by minimizing the following cost function over $\mathbf{w} \in \mathbb{C}^{2\times 1}$:

$$\mathcal{J}_n(\mathbf{w}) = \frac{\left\| [\mathbf{X}_n^T, \mathbf{y}_n] \begin{bmatrix} \mathbf{w} \\ -1 \end{bmatrix} \right\|^2}{\left\| \begin{bmatrix} \mathbf{w} \\ -1 \end{bmatrix} \right\|^2},$$

which can also be written as

$$\mathcal{J}_n(\mathbf{w}) = \frac{\|\mathbf{X}_n^T\mathbf{w} - \mathbf{y}_n\|^2}{\|\mathbf{w}\|^2 + 1} = \frac{\sum_{i=1}^{n} |\mathbf{x}_i^T\mathbf{w} - v_i|^2}{\|\mathbf{w}\|^2 + 1} \tag{3}$$

where $\|\cdot\|$ denotes the Euclidean norm.

Assuming that the noises are stationary correlation-ergodic and the sampling interval is considerably smaller than the voltage period, i.e., $\tau \ll 1/f$, we can conclude that $\mathbf{x}_i^T\mathbf{w} - v_i$, $i = 1, \dots, n$, are also correlation-ergodic. Consequently, we may replace the time-average in (3) with ensemble-average and obtain the following alternative cost function:

$$J(\mathbf{w}) = \frac{E[|\mathbf{x}_n^T\mathbf{w} - v_n|^2]}{\|\mathbf{w}\|^2 + 1}.$$

Observe that $J(\mathbf{w})$ is in fact the Rayleigh quotient of

$$\mathbf{\Phi} = E\left\{\begin{bmatrix}\mathbf{x}_n \\ v_n\end{bmatrix}^* [\mathbf{x}_n^T, v_n]\right\}$$

with argument $[\mathbf{w}^T, -1]^T$ and reaches its minimum value $\lambda_{\min}$ (the smallest eigenvalue of $\mathbf{\Phi}$) when $[\mathbf{w}^T, -1]^T$ is the eigenvector corresponding to $\lambda_{\min}$ [34]-[36].

It is known that the critical points of the Rayleigh quotient cost function, $J(\mathbf{w})$, are the eigenvectors of $\mathbf{\Phi}$ and the critical values of $J(\mathbf{w})$ are the eigenvalues of $\mathbf{\Phi}$. Moreover, $\lambda_{\min}$ is the only stable critical value (local and global minimum) of $J(\mathbf{w})$. As a result, the minimizer of $J(\mathbf{w})$ is unique and the global minimum of $J(\mathbf{w})$ can be reached using the gradient-descent method from any initial point given the choice of an appropriate step-size [34]-[37].

The gradient of the cost function is calculated as

$$\begin{aligned}\nabla J(\mathbf{w}) &= \frac{\partial J(\mathbf{w})}{\partial \mathbf{w}^H} \\ &= \frac{-2E[\mathbf{x}_n^*(v_n - \mathbf{x}_n^T\mathbf{w})](\|\mathbf{w}\|^2 + 1) - 2\mathbf{w}^*E[|v_n - \mathbf{x}_n^T\mathbf{w}|^2]}{(\|\mathbf{w}\|^2 + 1)^2} \\ &\approx \frac{-2\mathbf{x}_n^*(v_n - \mathbf{x}_n^T\mathbf{w})(\|\mathbf{w}\|^2 + 1) - 2\mathbf{w}^*|v_n - \mathbf{x}_n^T\mathbf{w}|^2}{(\|\mathbf{w}\|^2 + 1)^2}\end{aligned}$$

where superscript H denotes complex-conjugate transpose. Subsequently, a gradient-descent total least-squares estimate of the sought-after parameters can be iteratively achieved as

$$\begin{aligned}\mathbf{w}_n &= \mathbf{w}_{n-1} - \frac{\mu}{2}\nabla J(\mathbf{w}_{n-1}) \\ &= \mathbf{w}_{n-1} + \mu\epsilon_n(\mathbf{x}_n + \epsilon_n\mathbf{w}_{n-1})^*\end{aligned}$$

where

$$\epsilon_n = \frac{v_n - \mathbf{x}_n^T\mathbf{w}_{n-1}}{\|\mathbf{w}_{n-1}\|^2 + 1}$$

and $\mu > 0$ is the step-size parameter. Accordingly, the system frequency is adaptively estimated as

$$\hat{f}_n \approx \frac{1}{2\pi\tau}\sin^{-1}\left(\sqrt{\Im^2(w_{n,1}) - |w_{n,2}|^2}\right).$$

We call the new algorithm, which is based on a widely-linear model, *augmented gradient-descent total least-squares* (AGDTLS).

3. NUMERICAL STUDIES

We consider a three-phase power system where $f = 50$, $\tau = 2 \times 10^{-4}$, and the noises (η_n^a, η_n^b, and η_n^c) are zero-mean white Gaussian with variance σ_η^2. We use two metrics to quantify and compare the steady-state frequency estimation performance of different algorithms, the *bias* that is defined as $\left|E\left[\hat{f}_n\right] - f\right|$ and the *root mean-squared error (RMSE)* that is defined as

$$\sqrt{E\left[\left(\hat{f}_n - f\right)^2\right]}.$$

We evaluate the expectations by averaging over 10^3 steady-state values and ensemble-averaging over 10^3 independent trials.

In Figs. 1 and 2, we compare the performance of the complex least mean square (CLMS) [20], augmented complex least mean square (ACLMS) [21], and AGDTLS algorithms by plotting the bias and RMSE against σ_η^2 when the system is balanced, i.e., $V_n^a = V_n^b = V_n^c = 1$. We adjust the step-sizes to yield identical initial convergence rates for all the algorithms. Fig. 1 shows that CLMS has a smaller bias than the widely-linear algorithms when dealing with a balanced system in a noisy environment. This can be attributed to the fact that widely-linear algorithms *overmodel* the problem when the system is balanced. They have one extra parameter that only serves to amplify the noise, hence deteriorate the performance when the system is balanced and v_n is proper. However, in this case, AGDTLS is superior to ACLMS and outperforms CLMS in terms of RMSE.

In Figs. 3 and 4, we compare the performance of the algorithms when the system is unbalanced such that $V_n^a = 1.2$, $V_n^b = 0.8$, and $V_n^c = 1$. The other parameters are the same as before. Figs. 3 and 4 show that, similar to the balanced case, AGDTLS significantly outperforms ACLMS when the system is unbalanced. Moreover, we observe that for an unbalanced system, widely-linear modeling generally results in a better estimation performance, especially when the noises have relatively small variances, viz., less than 0.002 for ACLMS and less than 0.01 for AGDTLS in the simulated scenario. When the variance of the noises is higher than these values, the strictly-linear algorithm, CLMS, approaches and surpasses the widely-linear algorithms in terms of bias performance. Nevertheless, at such points, all the considered algorithms exhibit a poor and

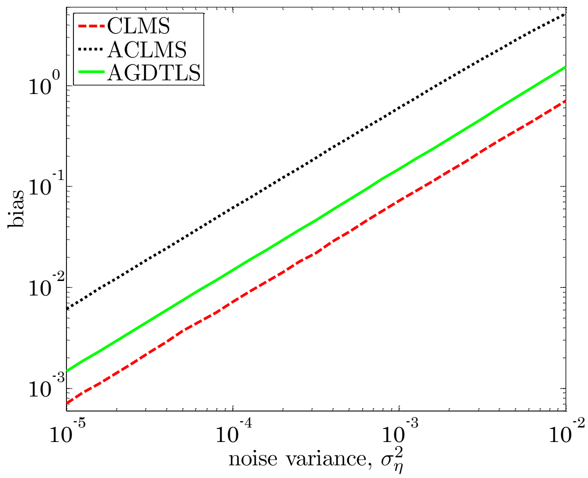

Fig. 1. The bias of different algorithms when the system is balanced.

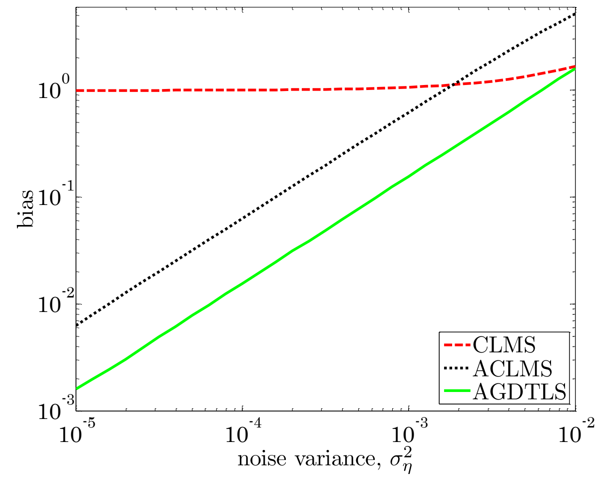

Fig. 3. The bias of different algorithms when the system is unbalanced.

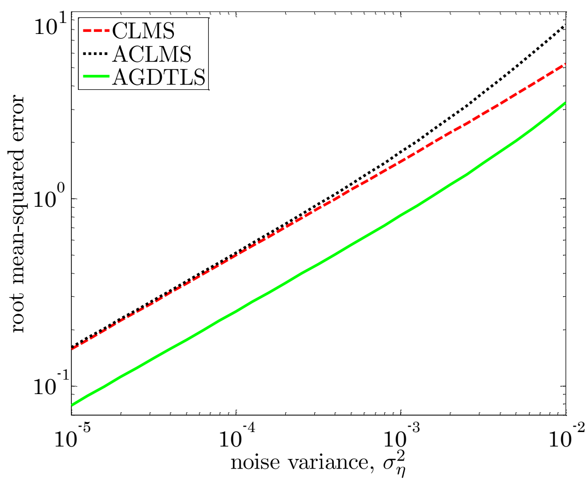

Fig. 2. The root mean-squared error of different algorithms when the system is balanced.

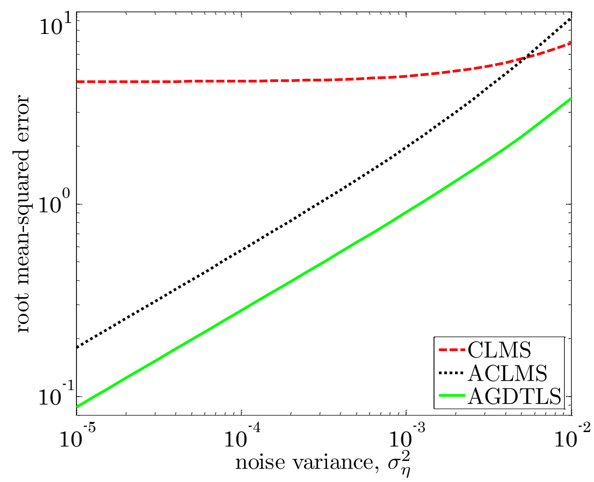

Fig. 4. The root mean-squared error of different algorithms when the system is unbalanced.

unreliable performance soliciting more sophisticated and accurate methods such as the widely-linear complex Kalman filter [38].

4. CONCLUSION

We have developed an adaptive frequency estimation algorithm for three-phase power systems by assuming a widely-linear predictive model for the system's $\alpha\beta$ complex signal generated by Clarke's transform and finding a total least-squares fit via the gradient-descent method. The proposed algorithm has the same order of computational complexity as the complex least mean square (CLMS) and the augmented complex least mean square (ACLMS) algorithms and, as verified by the simulation results, outperforms CLMS and ACLMS when the system voltage signals are observed in noise.

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