HYPERSPECTRAL ABUNDANCE ESTIMATION FOR THE GENERALIZED BILINEAR MODEL WITH JOINT SPARSITY CONSTRAINT

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ABSTRACT

In this paper, we present a novel abundance estimation method for the generalized bilinear model (GBM) via sparse representation for hyperspectral imagery. Because the GBM generalizes the linear mixture model (LMM) by introducing an additional bilinear term, our sparsity-based abundance estimation is performed by utilizing two dictionaries—a linear dictionary containing all the pure endmembers and a bilinear dictionary consisting of all the possible bilinear interaction components. Because the components within the bilinear term are also linearly combined, by employing a composite dictionary made up by the concatenation of the linear and bilinear dictionaries we can reformulate the bilinear problem in a linear sparse regression framework. In this way, the abundance values are estimated from the sparse codes only associated with the linear dictionary. To further improve the estimation performance, we incorporate the joint-sparsity model to exploit the spatial information in the data. The experiments demonstrate the effectiveness of the proposed algorithms on both synthetic and real data.

***Index Terms*—** Abundance estimation, hyperspectral imagery, bilinear model, sparse representation

1. INTRODUCTION

Due to the low spatial resolution of imaging sensors, spectral unmixing, which consists of pure endmember identification and the abundance estimation [1], is a major issue in hyperspectral imagery [2, 3]. The identification problem is usually solved by an automatic endmember extraction algorithm such as the N-Finder [4] or the Vertex Component Analysis (VCA) [5]. Based on the extracted endmembers, the LMM is usually applied for abundance estimation due to its simplicity and analytically tractable solution as in [6, 7]. Nonetheless, as pointed out in [2], due to the wide existence of non-linear effects such as photon scattering, the LMM may not be an appropriate model in many practical situations. To overcome these inherent limitations, the GBM has recently been proposed and well studied in [8] where it generalizes the LMM by introducing a bilinear term counting for the multipath effects.

Motivated by the recent research [9, 10], we present a novel method for solving the GBM based abundance estimation problem via sparse representation. Besides the linear dictionary made up of all pure endmembers, we consider a bilinear dictionary consisting of all second-order endmember interaction components counting for the bilinear term in the GBM. We concatenate the linear and bilinear dictionaries together to form a composite dictionary. Because the components within the bilinear term are also linearly combined, based on the proposed composite dictionary the overall bilinear problem can be reformulated in a linear sparse regression framework. The new sparse recovery problem can still be efficiently solved by an l_1-minimization algorithm such as the alternating direction method of multipliers (ADMM) [11]. Once the sparse code for each single measurement vector (SMV) has been obtained, the abundances are estimated from the sparse codes associated with the linear dictionary. Moreover, as the pixels in hyperspectral images are usually highly correlated in a small neighbourhood, we further introduce a joint-sparsity model [12] to enforce the representations of the multiple measurement vectors (MMV) to share the same support set. The joint-sparsity model further exploits the spatial information in the data, thus it shows an improved performance.

The paper is organized as follows. In Section 2 we discuss about the relation of our proposed method to prior work. The mixture models and the algorithms are presented in Section 3 and 4, respectively. In Section 5, we demonstrate the proposed algorithms with some experiment results. Finally, we concludes the paper in Section 6.

2. RELATION TO PRIOR WORK

Because every endmember does not contribute to all the pixels in the scene, sparseness is an important property of hyperspectral imagery [13]. To utilize this property, Guo et. al. [9] has proposed an l_1-minimization based abundance estimation algorithm. Similar to this method, Iordache et. al. [10] proposed a sparse unmixing algorithm which utilizes a

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pre-selected over-complete USGS spectral library[1]. Because of the l_1-norm regularization, they produce sparser and more stable solutions than the other state-of-the-art algorithms. But these algorithms are developed for the LMM which cannot effectively deal with the non-linearities caused by the multi-layer light scattering effects. In this paper, we extend their work [9, 10] to the GBM based abundance estimation problem to deal with bilinear mixtures. Moreover, the joint sparsity model is further introduced to exploit the spatial information in the data.

3. MIXTURE MODELS

Suppose that $\mathbf{y} \in \mathbb{R}^L$ is an observed mixed pixel of R pure endmembers with L spectral bands. We assume that the dictionary $\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_R]$ is a $L \times R$ matrix in which each column $\mathbf{a}_r \in \mathbb{R}^L$ $(1 \leq r \leq R)$ is a pure endmember vector. Let $\mathbf{x} \in \mathbb{R}^R$ be an abundance vector associated with the observed pixel $\mathbf{y}$.

3.1. Linear Mixture Model

The physical assumption underlying LMM is that each incident photon interacts with only one earth surface component, and that the reflected spectra are not scattered. Therefore, they do not mix before entering the HSI sensor. In this case, the model can be described as

$$\mathbf{y} = \sum_{i=1}^{R} \mathbf{a}_i x_i + \mathbf{n} = \mathbf{A}\mathbf{x} + \mathbf{n} \tag{1}$$

where $\mathbf{n} \in \mathbb{R}^L$ is an additive white noise sequence with variance σ^2, denoted as $\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$. To be physically meaningful, the abundance vector $\mathbf{x}$ has to satisfy the abundance non-negative constraint (ANC) and the abundance sum-to-one constraint (ASC)

$$\begin{aligned} &\text{ANC: } x_r \geq 0,\ \forall r \in 1, 2, ..., R, \\ &\text{ASC: } \sum_{r=1}^{R} x_r = 1. \end{aligned} \tag{2}$$

3.2. Generalized Bilinear Mixture Model

Accounting for the presence of multiple photon bounces by introducing an additional bilinear term to the LMM, the generalized bilinear model (GBM) [8] assumes that the observed pixel can be expressed as

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \sum_{i=1}^{R-1} \sum_{j=i+1}^{R} \gamma_{ij} x_i x_j \mathbf{a}_i \odot \mathbf{a}_j + \mathbf{n} \tag{3}$$

[1]http://speclab.cr.usgs.gov/spectral-lib.html

where $0 \leq \gamma_{ij} \leq 1$ is a coefficient that controls the interaction between the i-th and j-th endmembers and the symbol $\odot$ denotes the Hadamard product operation.

The constraint on the abundance $\mathbf{x}$ is the same with LMM. Notice that GBM degenerates to LMM when $\gamma_{ij} = 0, (1 \leq i, j \leq R, i \neq j)$.

4. SPARSE ABUNDANCE ESTIMATION

4.1. Linear Sparse Regression for Abundance Estimation

Recent research in [9, 10] shows that the abundance estimation problem can be formulated as a sparse regression problem. Specifically, with a given dictionary $\mathbf{A}$ and a mixed pixel $\mathbf{y}$, we can formulate the problem as

$$\min_{\mathbf{x}} ||\mathbf{x}||_0,\ s.t.\ ||\mathbf{y} - \mathbf{A}\mathbf{x}||_2 \leq \delta,\ \mathbf{x} \geq \mathbf{0},\ \mathbf{1}^T\mathbf{x} = 1 \tag{4}$$

where $||\cdot||_0$ denotes the l_0-norm which is defined as the number of non-zero entries in the vector of interest, and $\delta \geq 0$ is the error tolerance level due to noises and various modelling errors. This problem is not convex and generally NP-hard to solve. Instead, if the solution is sufficiently sparse, it can be relaxed to a linear programming problem by replacing the l_0-norm with the l_1-norm. However, with the ASC and ANC the l_1-norm of $\mathbf{x}$ remains as a constant equalling to unitary, rendering the entire l_1-minimization problem meaningless. Therefore, in [9, 10], they suggest to relax the ASC and solve problem as follows

$$\min_{\mathbf{x}} \frac{1}{2}||\mathbf{y} - \mathbf{A}\mathbf{x}||_2^2 + \lambda_1||\mathbf{x}||_1,\ s.t.\ \mathbf{x} \geq \mathbf{0} \tag{5}$$

where $||\mathbf{x}||_1 = \sum_{i=1}^{R} |x_i|$ and $\lambda_1 > 0$ is a scalar regularization parameter. This optimization problem is convex and can be solved by the l_1-minimization techniques. In this paper, we adopt the non-negative SMV-ADMM algorithm (named CSUnSAL+ in [10]) for its simplicity, efficiency and robustness.

4.2. Extension to Generalized Bilinear Mixture Model

A close examination of (3) reveals that GBM can be seen as a LMM with R original and $R^* = \frac{1}{2}R(R-1)$ correlated endmembers. More specifically, by considering each second-order spectral term $\mathbf{a}_i \odot \mathbf{a}_j (1 \leq i < j \leq R)$ as a new spectral components associated with fraction $\gamma_{ij} x_i x_j$, the model can be rewritten as a linear combination of all the spectra

$$\mathbf{y} = \sum_{k=1}^{R} x_k \mathbf{a}_k + \sum_{l=1}^{R^*} e_l \mathbf{b}_l + \mathbf{n} \tag{6}$$

where

$$\begin{aligned} &e_l = \gamma_{ij} x_i x_j,\ \mathbf{b}_l = \mathbf{a}_i \odot \mathbf{a}_j, \\ &l = j + \frac{(2R - i - 2)(i - 1)}{2},\ 1 \leq i < j \leq R. \end{aligned} \tag{7}$$

If we define the bilinear dictionary and its corresponding bilinear abundance representation as $\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{R^*}] \in \mathbb{R}^{L \times R^*}$ and $\mathbf{e} = [e_1, e_2, \ldots, e_{R*}]^T \in \mathbb{R}^{R^*}$, respectively, then we can rewrite (6) as

$$\begin{aligned} \mathbf{y} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{e} + \mathbf{n} \\ &= \mathbf{M}\boldsymbol{\phi} + \mathbf{n} \end{aligned} \tag{8}$$

where $\mathbf{M} = [\mathbf{A}, \mathbf{B}] \in \mathbb{R}^{L \times (R+R^*)}$ is a composite dictionary and $\boldsymbol{\phi} = \left[\mathbf{x}^T, \mathbf{e}^T\right]^T \in \mathbb{R}^{R+R^*}$ is the corresponding composite representation. Therefore, the bilinear problem can be transformed and solved in the linear sparse regression framework by

$$\min_{\boldsymbol{\phi}} \frac{1}{2}||\mathbf{y} - \mathbf{M}\boldsymbol{\phi}||_2^2 + \lambda_1'||\boldsymbol{\phi}||_1, \ s.t.\ \boldsymbol{\phi} \geq \mathbf{0} \tag{9}$$

where $\lambda_1' > 0$. The problem can still be efficiently solved by the non-negative constraint SMV-ADMM algorithm. Once the sparse code $\hat{\boldsymbol{\phi}} = [\hat{\mathbf{x}}^T, \hat{\mathbf{e}}^T]^T$ is obtained, the abundance can be estimated from the vector $\hat{\mathbf{x}}$. Hence, we can effectively get rid of the small annoying bilinear components in GBM and accurately predict the abundances for the linear combinations.

4.3. Joint Sparsity Model for Abundance Estimation

Here, we introduce some more notations first. Let us define $\mathbf{Y} = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_N] \in \mathbb{R}^{L \times N}$ be a observation matrix which contains N adjacent bilinear mixed pixels, and let $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N] \in \mathbb{R}^{R \times N}$ and $\mathbf{E} = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_N] \in \mathbb{R}^{R^* \times N}$ be the linear and bilinear abundance matrices associated with the dictionaries $\mathbf{A}$ and $\mathbf{B}$, respectively.

As the neighbouring pixels usually have similar endmembers, we seek a joint sparse representation on the abundance matrix $\boldsymbol{\Phi} = [\mathbf{X}^T, \mathbf{E}^T]^T \in \mathbb{R}^{(R+R^*) \times N}$, so that all its columns have the same support set and satisfy $\mathbf{Y} = \mathbf{M}\boldsymbol{\Phi}$. Similar to the case of SMV, the joint sparsity problem can be formulated as

$$\min_{\boldsymbol{\Phi}} \frac{1}{2}||\mathbf{M}\boldsymbol{\Phi} - \mathbf{Y}||_F^2 + \lambda_2||\boldsymbol{\Phi}||_{1,2}, \ s.t.\ \boldsymbol{\Phi} \geq \mathbf{0} \tag{10}$$

where $\lambda_2 > 0$ is an appropriate balancing parameter, $||\boldsymbol{\Phi}||_{1,2} = \sum_{j=1}^{R+R^*} ||\boldsymbol{\phi}^i||_2$ and $\boldsymbol{\phi}^i \in \mathbb{R}^N$ is the i-th row of the matrix $\boldsymbol{\Phi}$. Once the composite sparse representation $\hat{\boldsymbol{\Phi}} = [\hat{\mathbf{X}}^T, \hat{\mathbf{E}}^T]^T$ is obtained, the abundances can be obtained from $\hat{\mathbf{X}}$.

To solve (10), we propose a non-negative constraint MMV-ADMM algorithm. We first introduce an auxiliary matrix variable $\mathbf{Z} \in \mathbb{R}^{(R+R^*) \times N}$ and transform the problem as

$$\begin{aligned} &\min_{\boldsymbol{\Phi}, \mathbf{Z}} \frac{1}{2}||\mathbf{M}\boldsymbol{\Phi} - \mathbf{Y}||_F^2 + \lambda_2||\mathbf{Z}||_{1,2}, \\ &s.t.\ \boldsymbol{Z} \geq \mathbf{0},\ \boldsymbol{\Phi} - \mathbf{Z} = \mathbf{0}. \end{aligned} \tag{11}$$

Thus, the augmented Lagrangian function of Eq. (11) is

$$\begin{aligned} L_\mu(\boldsymbol{\Phi}, \mathbf{Z}, \mathbf{T}) &= \frac{1}{2}||\mathbf{M}\boldsymbol{\Phi} - \mathbf{Y}||_F^2 + \lambda_2||\mathbf{Z}||_{1,2} \\ &+ < \mathbf{T}, \boldsymbol{\Phi} - \mathbf{Z} > + \frac{\mu}{2}||\boldsymbol{\Phi} - \mathbf{Z}||_F^2 \end{aligned} \tag{12}$$

where $\mu > 0$ is the penalty parameter, $\mathbf{T} \in \mathbb{R}^{(R+R^*) \times N}$ is the Lagrangian multiplier and we let $\boldsymbol{\Lambda} = \mathbf{T}/\mu$. We minimize the augmented Lagrangian function iteratively by fixing one variable and update the other, the entire algorithm is summarized in Algorithm 1 below.

Algorithm 1 Non-negative constraint MMV-ADMM Algorithm for Jointly Sparse Abundance Estimation

Input: the scalar λ_2, the matrix $\mathbf{Y}$ and the dictionary $\boldsymbol{\Phi}$;
Output: The estimated abundance $\hat{\mathbf{X}}$;

1: Initialize: $\boldsymbol{\Phi}^0, \mathbf{Z}^0, \boldsymbol{\Lambda}^0, \mu, k = 0$;
2: **while** not converged **do**
3: Fix $\mathbf{Z}$ and update $\boldsymbol{\Phi}$ by:
$$\boldsymbol{\Phi}^{k+1} = (\mathbf{M}^T\mathbf{M} + \mu\mathbf{I})^{-1}(\mathbf{M}^T\mathbf{Y} + \mu(\mathbf{Z}^k - \boldsymbol{\Lambda}^k))$$
4: Fix $\boldsymbol{\Phi}$ and update $\mathbf{Z}$ by:
$$\mathbf{Z}^{k+1} = \max\left[S_{\lambda_2/\mu}(\boldsymbol{\Phi}^{k+1} + \boldsymbol{\Lambda}^k), 0\right]$$
5: Update the Lagrangian Multiplier $\boldsymbol{\Lambda}$:
$$\boldsymbol{\Lambda}^{k+1} = \boldsymbol{\Lambda}^k + \boldsymbol{\Phi}^{k+1} - \mathbf{Z}^{k+1}$$
6: Update k: $k = k + 1$.
7: **end while**
8: **return** $[\hat{\mathbf{X}}^T, \hat{\mathbf{E}}^T]^T = \mathbf{Z}^k$.

In Algorithm 1, Step 3 is essentially a ridge regression problem. Though the problem in Step 4 is non-smooth, it has a closed-form solution by a soft row-shrinkage thresholding operator $S_\kappa(\cdot)$ described in [14].

5. EXPERIMENT

In the first experiment, the proposed SMV-ADMM algorithm is evaluated on two synthetic images of size 50×50 generated by LMM and GBM, respectively. The endmember dictionary $\mathbf{A} \in \mathbb{R}^{224 \times 12}$ is constructed by randomly choosing 12 endmembers from a selected USGS library[2]. For each pixel generated by both models, we randomly choose three endmembers from $\mathbf{A}$ and mix them by Dirichlet distribution with both ANC and ASC. The parameter γ_{ij} in (3) for GBM is generated by uniform distribution on the interval $[0.5, 1]$. All the pixels are corrupted by a white noise with SNR=40dB. We compare our algorithm with four other algorithms, that is, FCLS [6], CSUnSAL+ [10], KFCLS [15] and GDA [8]. The first two algorithms are developed for the LMM, the latter are for non-linear models. The estimation results are shown in Table 1, where the signal-to-reconstruction error (SRE) and the reconstruction error (RE) are introduced in [10, 8], respectively. From the results, we can see that *(i)* FCLS shows the best performance on the LMM, but it does not perform well

[2]http://www.lx.it.pt/ bioucas/code/sunsal_demo.zip

Table 1. Comparison of five abundance estimation algorithms on LMM and FM

Criteria	Model	Estimation Algorithm				
		FCLS [6]	KFCLS [15]	CSUnSAL+[10]	GDA [8]	Proposed
SRE (dB)	LMM	**41.4160**	8.7507	38.9119	41.3122	37.8415
	GBM	11.6985	4.0998	13.1667	10.6583	**22.4512**
RE ($\times 10^{-2}$)	LMM	0.0684	N.A.	0.0733	**0.0674**	0.1031
	GBM	3.5478	N.A.	0.9300	3.7415	**0.5513**

Table 2. Comparison of Reconstruction Error for Fig. 1

Algorithm	RE ($\times 10^{-3}$)
FCLS	7.2142
CSUnSAL+	6.8127
GDA	6.6532
SMV-ADMM	6.2779
MMV-ADMM	**6.0615**

on the GBM; *(ii)* though KFCLS is designed for non-linear mixture, it shows even worse performance on GBM; *(iii)* although CSUnSAL+ is for the LMM, it shows some robustness on the GBM where small non-linearities occur; *(iv)* the proposed algorithm shows competitive results on the LMM and it performs much better on the GBM than the other algorithms.

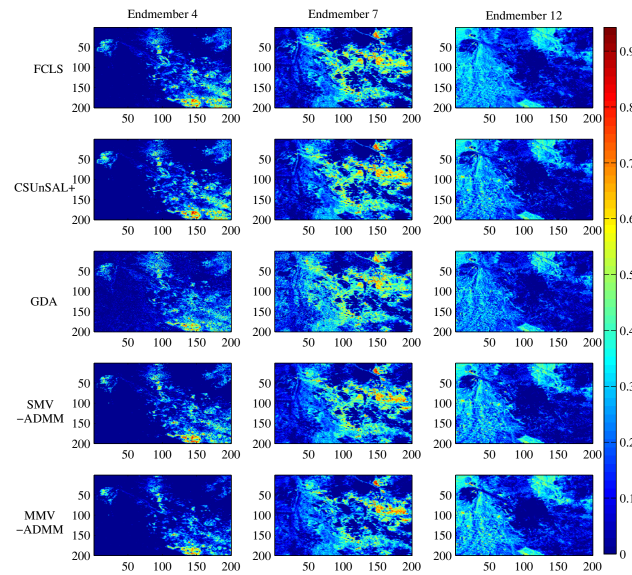
In the second experiment, we evaluate the performance of the proposed SMV-ADMM and MMV-ADMM algorithms on the well-known AVIRIS Cuprite image over the Cuprite mining region in Nevada, USA, which is available online in reflectance units[3]. The portion we use in this experiment is the 512×614 image of the sector labelled as "f970619t01p02_r02_sc02.a.rfl". Before analysis, we get rid of the water absorption bands leaving 188 bands in total [10]. First, 12 pure endmembers are automatically extracted from the image by VCA. Though the VCA is designed under the assumption of the LMM, as explained in [8], we can still apply it to the GBM where only small non-linearities occur. We compare our proposed SMV-ADMM and MMV-ADMM algorithms with the FCLS, CSUnSAL+ and GDA algorithms on a sub-image of size 200×200. The estimated abundances are displayed in Fig. 1, and the results of reconstruction errors are shown in Table 2. From the results, we can conclude that *(i)* the proposed SMV-ADMM algorithm shows much lower RE compared with the FCLS, CSUnSAL+ and GDA algorithms; *(ii)* with the spatial information employed, the proposed MMV-ADMM algorihtm shows further improved results.

6. CONCLUSION

In this paper, we have proposed a simple but very effective method for bilinear abundance estimation by transforming the problem into a linear sparse recovery task. The reformulated problem can still be efficiently solved by non-negative constraint l_1-minimization techniques such as the SMV-ADMM algorithm. Furthermore, as the adjacent pixels usually contain similar endmembers, we employ the joint sparsity model to take advantage of the spatial information by the proposed MMV-ADMM algorithm. The simulation results demonstrate our proposed method with significant improvements both on synthetic and real data sets.

[3]http://aviris.jpl.nasa.gov/html/aviris.freedata.html

Fig. 1. Comparison of Abundance Estimation on Real Data



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