DIRECTIONAL HYPERCOMPLEX DIFFUSION

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ABSTRACT

Methods based on partial differential equations (PDE) become increasingly one of the methods of image processing. Recently a diffusion method is appeared, it allows to generalize the diffusion to the complex domain by the injection of a complex number in the heat equation. For small phase angles, the linear process generates the Gaussian and Laplacian pyramids (scale-spaces) simultaneously, depicted in the real and imaginary parts, respectively. The imaginary value serves as a robust edge-detector with increasing confidence in time, thus handles noise well and may serve as a controller for nonlinear processes. In this article we propose to extend this concept by introducing a notion of directionality in such a way as each equation of the system will correspond to a specific direction. It is in our interests to use higher order algebra to adapt the process to the four discrete directions. Then we will focus on the imaginary parts for developing a nonlinear scheme.

Index Terms— PDEs, complex diffusion, directional PDEs, higher order algebra.

1. INTRODUCTION

PDE Take advantages of many mathematical tools and algorithms for signal discretizations. They have a success with good experimental results. PDE-based methods appear in a large variety of image processing and computer vision areas ranging from shape-from-shading and histogram modification to optic flow and stereovision. The simplest PDE method for smoothing images is to apply a linear diffusion process.

$$\partial_t U = \Delta U \tag{1}$$

This equation appears in many physical transport processes. In the context of heat transfer it is called heat equation. In image processing we may identify the concentration with the grey value at a certain location. One of the problems associated with the approach of linear diffusion is that important structural features such as edges are smoothed and blurred. To overcome this problem, Perona and Malik [7] proposed a nonlinear adaptive diffusion process, termed anisotropic diffusion, to reduce the smoothing effect near edges. So, they postulated that the transitions belong to regions of important gradient than those corresponding to the noise. From there, they sought to limit diffusion when the gradient is high:

$$\frac{\partial U(x,t)}{\partial t} = div\left(g\left(|\nabla U(x,t)|\right)\nabla U(x,t)\right). \tag{2}$$

However, Perona and Malik's model have been improved in a work by some of authors [3], [1]. In 2004, G. Gilboa proposed a new approach called complex diffusion that consist to generalize the linear scale spaces in the complex domain [4] . An important observation, supported theoretically and numerically, is that the imaginary part can serve as an edge detector, when the complex diffusion coefficient approaches the real axis. Based on this observation, he developed two nonlinear schemes: a regularized shock filter for image enhancement and a ramp preserving denoising process [6], [5]. Other extensions of Perona-Malik scheme suggest to study directional smoothing strategies. The first diffusion approach developed in [8] concerns textured images which contain orientations ruptures. In our study, we adapt the principle of complex diffusion for several directions using directional PDEs and the concept of hypercomplex numbers. Based on this idea linear and nonlinear diffusion schemes are developped.

The paper is organized as follows: in section 2, we present linear and nonlinear complex schemes. In section 3, we explain the principle of our work and we analyze the case of a system of four PDEs associated with quaternionic number. In section 4, study consists in taking into account both diagonals of the image. This leads to increase the system order. In this part linear and nonlinear approach will also be studied.

2. COMPLEX DIFFUSION

2.1. Isotropic complex diffusion

The complex diffusion equation can be written as follows [4]:

$$\begin{cases} U_t = c \cdot U_{xx}, t > 0, x \in \mathbb{R}, c \in \mathbb{C}, U \in \mathbb{C}, \\ U(x,0) = U_0 \in \mathbb{R}. \end{cases} \tag{3}$$

In the general case, the initial condition U_0 is complex. We take the particular case $U_0 \in \mathbb{R}$, with U_0 the original image.

We write the complex coefficient as follows : $c = re^{(j\theta)}$. When $\theta \to 0$ and $t \geq 0$, the real part of the fundamental solution is assimilated to a Gaussian and the imaginary part tends to the Laplacian of the Gaussian multiplied by t and the magnitude of c [4]:

$$\begin{cases} \lim_{\theta\to 0} \mathrm{Re(U)} = g_\sigma * U_0, \\ \lim_{\theta\to 0} \frac{\mathrm{Im(U)}}{\theta} = tr\Delta g_\sigma * U_0, \end{cases} \tag{4}$$

with Re denotes the real part and Im the imaginary part.

2.2. Non-linear complex diffusion

Non-linear complex diffusion (Ramp preserving denoising) can rely on the properties of linear complex diffusion. The second derivative (the laplacian) can serve as edges detector, but he suffers from two problems: noise has high second derivative and numerical problem arises when we calculate approximation of the third derivative

[4]. These two problems are solved using the non linear complex diffusion; the imaginary part (divided by θ) is used to control the diffusion process [4]:

$$U_t = div\left(C\left(\mathrm{Im(U)}\right)\nabla U\right), \tag{5}$$

$$C\left(\mathrm{Im(U)}\right) = \frac{e^{j\theta}}{1+\left(\frac{\mathrm{Im(U)}}{k\theta}\right)^2}. \tag{6}$$

For the same reasons discussed in the linear case, the phase θ must be small. Ramp preserving denoising can serve for denoising in illumination changing conditions and avoids effects caused by the gradient especially in the Perona-Malik's model [4]. The complex diffusion employs analysis of different spatial directions. However, The nonlinear directional processes provides an efficient tool and separate the orientation estimation step from the diffusion process. In this paper we propose to develop a filtering based on hypercomplex concept and consider other directions.

3. QUATERNIONIC DIFFUSION

In the previous section, we have seen the improvement achieved by the injection of a complex number in the diffusion equation. The complex diffusion preserves the edges of the image, it may be very consistent for edges with linear variation. We are interested to adapt the principle of diffusion complex for several directions. The number of equations which constitute the system will match to the number of directions. The different parts can be encoded with a higher order algebra . Then the result corresponding to the use of the classical directions can be represented by a quaternionic number. The algebra H, composed of elements in the form $q = (q_0, q_1, q_2, q_3) = (S(q), V(q))$. The real part $S(q)$ and the vector $V(q)$ are commonly called the scalar part and the vector part of the quaternion q.

$$q = q_0 + iq_1 + jq_2 + kq_3. \tag{7}$$

Calculation rules become $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$. with $q_0, q_1, q_2, q_3 \in \mathbb{R}$.

Our approach consist to extend the complex diffusion by associating each direction with orientation analysis of contours. We consider the extension in the linear case. Taking into account that the orientation is defined to be orthogonal to the process direction, we propose the scheme which highlights the classical directions as follow:

$$\begin{cases} \partial_t \mathrm{q}_0 = \frac{\partial^2 \mathrm{q}_0}{\partial x^2} + \frac{\partial^2 \mathrm{q}_0}{\partial y^2} - \theta\frac{\partial^2 \mathrm{q}_1}{\partial x^2} - \theta\frac{\partial^2 \mathrm{q}_2}{\partial y^2}, \\ \partial_t \mathrm{q}_1 = \frac{\partial^2 \mathrm{q}_1}{\partial x^2} + \theta\frac{\partial^2 \mathrm{q}_0}{\partial y^2}, \\ \partial_t \mathrm{q}_2 = \frac{\partial^2 \mathrm{q}_2}{\partial y^2} + \theta\frac{\partial^2 \mathrm{q}_0}{\partial x^2}, \\ IC: q_0 = U_0, q_1 = q_2 = 0. \end{cases} \tag{8}$$

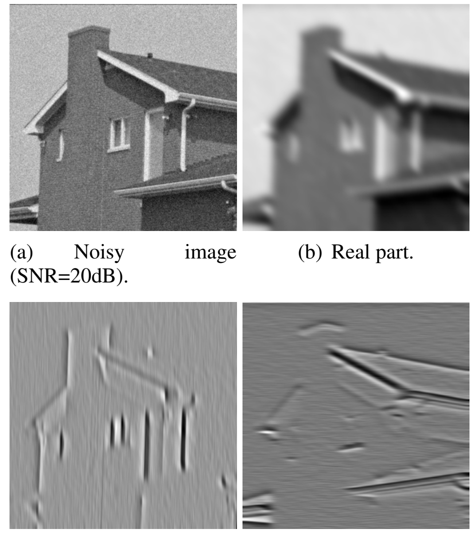
So we obtain a system of differential equations and the image can be represented by quaternion:

$$U = q_0 + i\mathrm{q}_1 + j\mathrm{q}_2. \tag{9}$$

with U_0 the original image.

Fig.1 shows the behavior of the quaternionic isotropic diffusion in the case of noisy image (SNR = 20dB). We observe that the results depicted smoothing and edges detection which provides a consistent linear process Fig.1(c), Fig.1(d).

Then we propose to extend this method by adapting process to both directions which form the diagonal.



(a) Noisy image (SNR=20dB). (b) Real part.

(c) First imaginary part. (d) Second imaginary part.

Fig. 1. Results of quaternionic isotropic diffusion

4. EXTENSION OF THE STUDY TO HYPERCOMPLEX

We generalize this principle to be suitable to both directions which form the diagonal. Therefore we will get four parts dealing simultaneously processing allong the four directions. In this case, these parts can be combined to give rise to anisotropic smoothing.

4.1. Isotropic model

Our objective is to adapt the process to the diagonal. Taking into account the digital format of image, it is easy to implement smoothing which is made along the both vectors which form the diagonal. To realize a smoothing along these directions we must introduce the notion of directional derivative.

The notion of directional derivative quantifies the local change of a function depending on several variables and at a specific point along a specific direction in space of these variables.The derivative of f at the point u along the vector v is, if it exists, the derivative at (0) of the function of the real variable $t \rightarrow f(u + tv)$:

$$D_h f(u) = \overset{t\rightarrow 0|t\neq 0}{\rightarrow} \frac{f(u+tv) - f(u)}{t}. \tag{10}$$

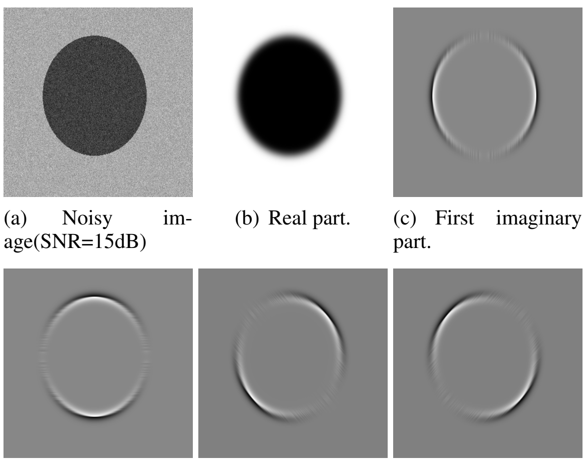
In our case, study consists in highlighting the vectors: $(1, 0), (0, 1)$ which constitute the basic element and $v_1(1, 1), v_2(-1, 1)$ are unit vectors which depict the diagonal. To remain in the context of the study we argue from analogy and we obtain the model represented by the extended PDE system:

$$\begin{cases} \partial_t q_0 = \partial_x^2 q_0 + \partial_y^2 q_0 + D_{v_1}^2 q_0 + D_{v_2}^2 q_0 \\ -\theta\partial_x^2 q_1 - \theta\partial_y^2 q_2 - \theta D_{v2}^2 q_3 - \theta D_{v1}^2 q_4, \\ \partial_t q_1 = \partial_x^2 q_1 + \theta\partial_y^2 q_0, \\ \partial_t q_2 = \partial_y^2 q_2 + \theta\partial_x^2 q_0, \\ \partial_t q_3 = D_{v_1}^2 q_3 + \theta D_{v_2}^2 q_0, \\ \partial_t q_4 = D_{v_2}^2 q_4 + \theta D_{v_1}^2 q_0. \end{cases} \tag{11}$$

We obtain four imaginary parts which correspond to the images that contain the details following classical discrete directions on the one hand, and a real part which is characterized by a linear smoothing which incorporates a diagonal diffusion on the other. The image can be depicted by an Hypercomplex number:

$$U = q_0 + e_1 q_1 + e_2 q_2 + e_3 q_3 + e_4 q_4. \tag{12}$$

For introduction to hypercomplex algebra see for example [2]. Now watching the effect in the case of the isotropic filtering on a synthetic image *circle* with the different images on the Fig.2. On the Fig.2(b), the real part provides an homogeneous regularization along the edges and does not produces widening. However, in other parts (Fig.2(c)-2(f)), we get images that contain the details along the orthogonal direction of the orientation of the filtering. For example on Fig.2(c) the study is done in the horizontal direction, on the other side the filter applied disappears details horizontal and keep the vertical details. Generally more we increase the number of iterations and the edges are well presented. After studying the isotropic case, we propose to extend this principle in the nonlinear case, this can be done by using secondary parts.



(a) Noisy image(SNR=15dB) (b) Real part. (c) First imaginary part.

(d) Second imaginary part. (e) Third imaginary part. (f) Fourth imaginary part.

Fig. 2. Results of the isotropic directional hypercomplex diffusion

4.2. Anisotropic case

The approach seen in the case of ramp preserving denoising provides a better approximation of the solution and avoids errors due to discretization. We rely on this basic principle to make filters seen previously non-linear. These filters allow the extraction of edges along the horizontal vertical and diagonal directions. We assume that through this study, all edges are presented separately. Firstly we propose the basic scheme represented by the following equations:

$$\begin{cases} \partial_t q_0 = \partial_x(C_2 \partial_x q_0) + \partial_y(C_1 \partial_y q_0) + D_{v1}(C_4 D_{v1} q_0) \\ +D_{v2}(C_3 D_{v2} q_0) - \theta \partial_x(C_1 \partial_x q_1) - \theta \partial_y(C_2 \partial_y q_2) \\ -\theta D_{v1}(C_3 D_{v1} q_3) - \theta D_{v2}(C_4 D_{v2} q_4), \\ \partial_t q_1 = \partial_x(C_1\ \partial_x q_1) + \theta \partial_y(C_1\ \partial_y q_0), \\ \partial_t q_2 = \partial_y(C_2\ \partial_y q_2) + \theta \partial_x(C_2\ \partial_x q_0), \\ \partial_t q_3 = D_{v1}(C_3\ D_{v1} q_3) + \theta D_{v2}(C_3\ D_{v2} q_0), \\ \partial_t q_4 = D_{v2}(C_4\ D_{v2} q_4) + \theta D_{v1}(C_4\ D_{v1} q_0), \end{cases} \tag{13}$$

with C_i function of nonlinearity depending obviously on imaginary parts $C_i = f(|q_1|, |q_2|, |q_3|, |q_4|)$. Regarding the anisotropic definition two methods seem obvious. The first uses a combination of different imaginary parts, while the second is purely directional.

4.2.1. *First anisotropic model*

In eq.(13) we put $C = C_1 = C_2 = C_3 = C_4$. We need to find a combination that fits within pattern of nonlinear diffusion and can control the anisotropic diffusion in such a way as it can slow down the process near the edges. So the combination should be able to present the contour in any direction. We propose to choose the maximum of imaginary parts $C = C(\max(|q_1|, |q_2|, |q_3|, |q_4|))$. This choice can be justified by the fact that it can always take the maximum value of the edges shown in each imaginary part.

$$C = \frac{1}{1 + (\frac{\max(|q_1|,|q_2|,|q_3|,|q_4|)}{k\theta})^2}. \tag{14}$$

4.2.2. *Second anisotropic model*

The main idea was to bring up the components that have a directional diffusion, therefore, we can simply implement a nonlinear diffusion scheme. In order to do so, we use the model of Perona-Malik for each of secondary part to reduce the processing near edges and to maintain it also. Having said that, secondary parts are injected into the first part, which contains the result of the diffusion, in order to control the anisotropic process like the complex diffusion. This scheme is obtained by introducing in eq.(11) the following function of diffusion: $C_i = \frac{1}{1+(\frac{q_i}{k\theta})^2}$.
Now, we propose to compare the non-linear schemes.

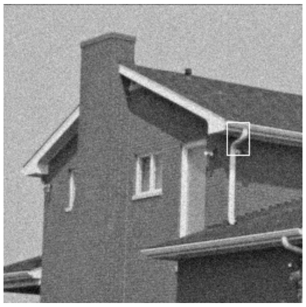
4.2.3. *Results and discussion*

Fig.3 displays an experiment comparing the anisotropic process applied to denoising image with $SNR = 20dB$. The first approach does not removes noise allong edges and produces blurring zones (Fig.3(f), Fig.3(e)). This undesirable phenomenon is due to the fact that the maximum does not match always to the best smoothing direction. The process according to the diagonal direction can be observed in the result of the second approach. That restore the edge as well as the complex diffusion which produces more distortion but it enhance most of contours (Fig.3(d),Fig.3(h)). A slight bluring neighborhood edges can be observed on Fig.3(h).

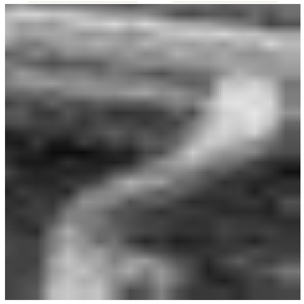
On Fig.4 we give illustration of anisotropic diffusion schemes seen previously used to remove noise that result from a low quality JPEG compression. The filter of the diffusion of Perona and Malik tend to create large flat zones and boundaries inside smooth regions but it enhance most of the dges (Fig.4(c)). This effect does not appear in other non linear process. The first approach removes the noise along with edges (Fig.4(e)). Whereas the second approach does not enhance edges as the Perona-Malik process but removes most of noise and does not produce staircasing effect (Fig.4(f)). Compared to the complex diffusion, the second approach is more adapted to the debloking (Fig.4(d)).

5. CONCLUSION

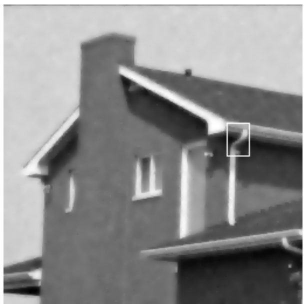
We employ the hypercomplex number to set up 2D smoothing scheme using directional strategy. We studies a different manners by combining edges informations.The experiment results show that the introduction of directionality preserves edges and considers directional analysis. The adaptive strategy to any direction is able to improve the behavior of our process. However we necessite a consistent discretization schemes.



(a) Noisy image(SNR=20dB).

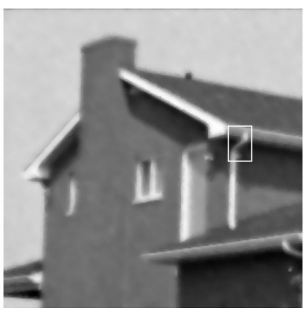


(b) Zoom on the noisy image.

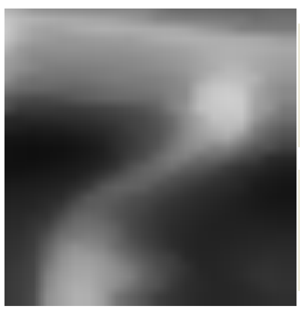


(c) Complex diffusion (k=5,SNR=18.24dB).

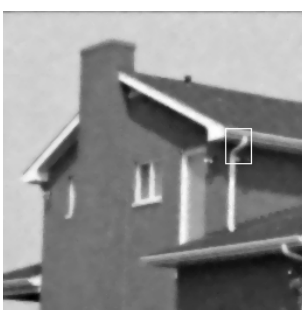
(d) Zoom on the complex diffusion.



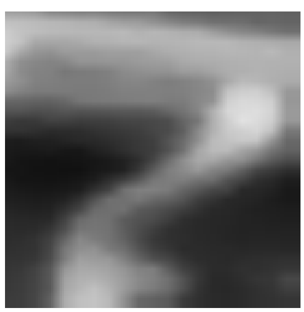
(e) Hypercomplex (first approach) (k=6,SNR=17.07dB).



(f) Zoom on the hypercomplex (first approach).

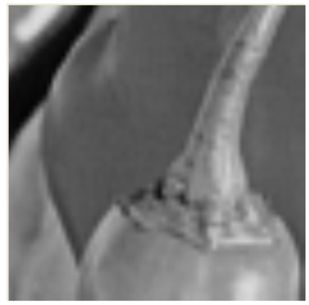


(g) Hypercomplex (second approach) (k=1,SNR=19.12dB).



(h) Zoom on the hypercomplex (second approach).

Fig. 3. Comparaison of the anisotropic diffusion process



(a) Original image.

(b) Noisy image (low quality JPEG).

(c) Perona and Malik ($k = 15$).

(d) Complex diffusion ($k = 4$).

(e) Hypercomplex (first approach) ($k = 6$).

(f) Hypercomplex (second approach) ($k = 1$).

Fig. 4. Experiment results of anisotropic diffusion process

Acknowledgement

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