OPTIMAL ESTIMATION OF HYBRID MODELS IN STATE-SPACE WITH FIR STRUCTURES

Yuriy S. Shmaliy, Oscar Ibarra-Manzano

Electronics Dept., Guanajuato University, Salamanca, 36855, Mexico

ABSTRACT

An optimal finite impulse response (FIR) estimator is adapted for discrete filtering, smoothing, and prediction of hybrid (continuous/discrete) models over N nearest past measurement points. Its unbiased FIR (UFIR) version ignoring noise and initial errors is also discussed for near optimal estimation when $N \gg 1$. The UFIR estimator is represented with an iterative Kalman-like algorithm efficient for highly oversampled data. An example of applications is given for the Global Positioning System-based measurement of time errors in a crystal clock.

Index Terms— Hybrid state-space model, optimal FIR estimator, iterative UFIR algorithm

1. INTRODUCTION

Measurement is often provided in continuous time employing analog sensors, whereas data are processed in DSP units using computational power. In such cases, the state-space model becomes hybrid (continuous/discrete) and discrete-time estimators are used. The estimator can be designed by combining the Kalman-Bucy [1] and Kalman [2] algorithms which, however, are known to have poor robustness against temporary model uncertainties [3] and high sensitivity to outliers [4]. To improve this performance, the Kalman filter was robustified by Masreliez and Martin in [5]. On the other hand, Jazwinski proposed in [6] using a limited memory filter characterizing it as the only device for preventing divergence in the presence of unbounded perturbations in the system [7]. With decades, the limited memory filters derived within Bayesian and maximum likelihood frameworks [6, 8, 9] have been developed to the finite impulse response (FIR) ones via the convolution [3, 10, 11]. Important distinctive features of FIR estimators are the bounded input/bounded output (BIBO) stability [6, 7], better robustness against temporary uncertainties [3, 11], and low sensitivity to noise and initial errors [10, 12].

In this paper, we adapt the estimators proposed in [10, 13] for p-shift optimal and unbiased FIR filtering ($p = 0$), smoothing ($p < 0$), and prediction ($p > 0$) of hybrid state-space models.

2. STATE-SPACE MODEL

Most generally, the K-state equation describing a linear time-varying system can be written as

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{w}(t)\,, \tag{1}$$

where $\mathbf{x}(t) \in \mathcal{R}^K$ is the state vector, $\mathbf{A}(t) \in \mathcal{R}^{K\times K}$ is the system matrix, and $\mathbf{B}(t) \in \mathcal{R}^{K\times P}$. The system noise vector $\mathbf{w}(t) \in \mathcal{R}^P$, is supposed to have zero mean components, $E\{\mathbf{w}(t)\} = \mathbf{0}$, and the covariance

$$\mathbf{Q}_w(\tau,\theta) = E\{\mathbf{w}(\tau)\mathbf{w}^T(\theta)\}\,. \tag{2}$$

A solution to (1) at discrete point t_n with the initial state $\mathbf{x}(t_{n-1})$ at t_{n-1} can be written as

$$\mathbf{x}(t_n) = \mathbf{\Phi}(t_n,t_{n-1})\mathbf{x}(t_{n-1}) + \int\limits_{t_{n-1}}^{t_n} \mathbf{\Phi}(t_n,\tau)\mathbf{B}(\tau)\mathbf{w}(\tau)\,d\tau\,, \tag{3}$$

where $\mathbf{\Phi}(t,\theta) \in \mathcal{R}^{K\times K}$, $\theta < t$, is the state transition matrix having the properties:

$$\frac{d}{dt}\mathbf{\Phi}(t,\theta) = \mathbf{A}(t)\mathbf{\Phi}(t,\theta)\,, \tag{4}$$

$$\mathbf{\Phi}(t_n,t_m) = \mathbf{\Phi}(t_n,t_{n-1})\mathbf{\Phi}(t_{n-1},t_{n-2})\ldots\mathbf{\Phi}(t_{m+1},t_m)\,, \tag{5}$$

where $m < n$ and $\mathbf{\Phi}(\theta,\theta) = \mathbf{I}$. By $\mathbf{x}_n \triangleq \mathbf{x}(t_n)$, one has

$$\mathbf{x}_n = \mathbf{\Phi}(t_n,t_{n-1})\mathbf{x}_{n-1} + \bar{\mathbf{w}}_n\,, \tag{6}$$

where the zero mean noise vector $\bar{\mathbf{w}}_n$ and its covariance $\mathbf{Q}_{\bar{w}}(i,j) = E\{\bar{\mathbf{w}}_i\bar{\mathbf{w}}_j^T\}$ are, respectively,

$$\bar{\mathbf{w}}_n = \int\limits_{t_{n-1}}^{t_n} \mathbf{\Phi}(t_n,\tau)\mathbf{B}(\tau)\mathbf{w}(\tau)\,d\tau\,, \tag{7}$$

$$\mathbf{Q}_{\bar{w}}(i,j) = \int\limits_{t_{i-1}}^{t_i}\int\limits_{t_{j-1}}^{t_j} \mathbf{\Phi}(t_i,\tau)\mathbf{B}(\tau)\mathbf{Q}_w(\tau,\theta)\mathbf{B}^T(\theta) \times\mathbf{\Phi}^T(t_j,\theta)\,d\theta\,d\tau\,. \tag{8}$$

In discrete time, the M-state measurement equation is

$$\mathbf{y}_n = \mathbf{C}_n\mathbf{x}_n + \mathbf{D}_n\mathbf{v}_n\,, \tag{9}$$

where $\mathbf{y}_n \in \mathcal{Z}^M$ is the measurement vector, $\mathbf{C}_n \in \mathcal{Z}^M$ is the measurement matrix and $\mathbf{D}_n \in \mathcal{Z}^{M\times M}$. The zero mean, $E\{\mathbf{v}_n\} = \mathbf{0}$, measurement noise $\mathbf{v}_n \in \mathcal{Z}^M$, has the covariance

$$\mathbf{Q}_v(i,j) = E\{\mathbf{v}_i\mathbf{v}_j^T\} \tag{10}$$

and it is commonly implied that $E\{\mathbf{w}(t_i)\mathbf{v}_j^T\} = \mathbf{0}$.

In order to provide FIR estimation, (6) and (9) can be expended on a horizon of N nearest past measurement points, from $m = n - N + 1$ to n, as [10]

$$\mathbf{X}_{n,m} = \mathbf{A}_{n,m}\mathbf{x}_m + \mathbf{B}_{n,m}\bar{\mathbf{W}}_{n,m}\,, \tag{11}$$
$$\mathbf{Y}_{n,m} = \mathbf{C}_{n,m}\mathbf{x}_m + \mathbf{G}_{n,m}\bar{\mathbf{W}}_{n,m} + \mathbf{D}_{n,m}\mathbf{V}_{n,m}\,, \tag{12}$$

where $\mathbf{X}_{n,m} \in \mathcal{Z}^{KN}$, $\mathbf{Y}_{n,m} \in \mathcal{Z}^{MN}$, $\bar{\mathbf{W}}_{n,m} \in \mathcal{Z}^{PN}$, and $\mathbf{V}_{n,m} \in \mathcal{Z}^{MN}$ are specified with, respectively,

$$\mathbf{X}_{n,m} = \left[\mathbf{x}_n^T\,\mathbf{x}_{n-1}^T\,\ldots\,\mathbf{x}_m^T\right]^T, \tag{13}$$
$$\mathbf{Y}_{n,m} = \left[\mathbf{y}_n^T\,\mathbf{y}_{n-1}^T\,\ldots\,\mathbf{y}_m^T\right]^T, \tag{14}$$
$$\bar{\mathbf{W}}_{n,m} = \left[\bar{\mathbf{w}}_n^T\,\bar{\mathbf{w}}_{n-1}^T\,\ldots\,\bar{\mathbf{w}}_m^T\right]^T, \tag{15}$$
$$\mathbf{V}_{n,m} = \left[\mathbf{v}_n^T\,\mathbf{v}_{n-1}^T\,\ldots\,\mathbf{v}_m^T\right]^T. \tag{16}$$

By (11), $\mathbf{\Phi}(t_m,t_m) = \mathbf{I}$, $\mathbf{\Phi}_{n,m} \triangleq \mathbf{\Phi}(t_n,t_m)$, and

$$\bar{\mathbf{C}}_{n,m} = \mathrm{diag}\Big(\underbrace{\mathbf{C}_n\ \mathbf{C}_{n-1}\ \ldots\ \mathbf{C}_m}_{N}\Big), \tag{17}$$

matrices $\mathbf{A}_{n,m} \in \mathcal{Z}^{KN\times K}$, $\mathbf{B}_{n,m} \in \mathcal{Z}^{KN\times PN}$, $\mathbf{C}_{n,m} \in \mathcal{Z}^{MN\times K}$, $\mathbf{G}_{n,m} \in \mathcal{Z}^{MN\times PN}$, and $\mathbf{D}_{n,m} \in \mathcal{Z}^{MN\times MN}$ attain the forms, respectively,

$$\mathbf{A}_{n,m} = [\,\mathbf{\Phi}_{n,m}^T\ \ \mathbf{\Phi}_{n-1,m}^T\ \ \ldots\ \ \mathbf{\Phi}_{m+1,m}^T\ \ \mathbf{I}\,]^T, \tag{18}$$

$$\mathbf{B}_{n,m} = \begin{bmatrix} \mathbf{I} & \mathbf{\Phi}_{n,n-1} & \ldots & \mathbf{\Phi}_{n,m+1} & \mathbf{\Phi}_{n,m} \\ \mathbf{0} & \mathbf{I} & \ldots & \mathbf{\Phi}_{n-1,m+1} & \mathbf{\Phi}_{n-1,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{I} & \mathbf{\Phi}_{m+1,m} \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{19}$$

$$\mathbf{C}_{n,m} = \bar{\mathbf{C}}_{n,m}\mathbf{A}_{n,m}\,, \tag{20}$$
$$\mathbf{G}_{n,m} = \bar{\mathbf{C}}_{n,m}\mathbf{B}_{n,m}\,, \tag{21}$$
$$\mathbf{D}_{n,m} = \mathrm{diag}\Big(\underbrace{\mathbf{D}_n\ \mathbf{D}_{n-1}\ \ldots\ \mathbf{D}_m}_{N}\Big). \tag{22}$$

3. OPTIMAL FIR ESTIMATE

By the gain matrix $\mathbf{H}_{n,m}(p) \in \mathcal{Z}^{K\times MN}$ applied to (12) and (14), the p-shift FIR estimate $\hat{\mathbf{x}}_{n+p|n}$ can be found at $n+p$ as

$$\hat{\mathbf{x}}_{n+p|n} = \mathbf{H}_{n,m}(p)\mathbf{Y}_{n,m} \tag{23a}$$
$$= \mathbf{H}_{n,m}(p)(\mathbf{C}_{n,m}\mathbf{x}_m + \mathbf{G}_{n,m}\bar{\mathbf{W}}_{n,m} + \mathbf{D}_{n,m}\mathbf{V}_{n,m})\,. \tag{23b}$$

The gain $\tilde{\mathbf{H}}_{n,m}(p)$ will be optimal in the minimum mean square error (MSE) sense by the orthogonality condition [14] leading to

$$\mathbf{0} = E\{[\mathbf{x}_{n+p} - \tilde{\mathbf{H}}_{n,m}(p)(\mathbf{C}_{n,m}\mathbf{x}_m + \mathbf{G}_{n,m}\bar{\mathbf{W}}_{n,m} + \mathbf{D}_{n,m}\mathbf{V}_{n,m})](\mathbf{C}_{n,m}\mathbf{x}_m + \mathbf{G}_{n,m}\bar{\mathbf{W}}_{n,m} + \mathbf{D}_{n,m}\mathbf{V}_{n,m})^T\}\,. \tag{24}$$

Following [10], $\tilde{\mathbf{H}}_{n,m}(p)$ can be found from (24) to be

$$\tilde{\mathbf{H}}_{n,m}(p) = [\bar{\mathbf{H}}_{n,m}(p)\mathbf{Z}_{n,m} + \bar{\mathbf{Z}}_w(p)](\mathbf{Z}_{n,m} + \tilde{\mathbf{Z}}_w + \tilde{\mathbf{Z}}_v)^{-1}\,, \tag{25}$$

where the auxiliary matrices are

$$\mathbf{Z}_{n,m} = \mathbf{C}_{n,m}\mathbf{R}_m\mathbf{C}_{n,m}^T\,, \tag{26}$$
$$\tilde{\mathbf{Z}}_w \triangleq \tilde{\mathbf{Z}}_w(n,m) = \mathbf{G}_{n,m}\bar{\mathbf{\Psi}}_w\mathbf{G}_{n,m}^T\,, \tag{27}$$
$$\tilde{\mathbf{Z}}_v \triangleq \tilde{\mathbf{Z}}_v(n,m) = \mathbf{D}_{n,m}\mathbf{\Psi}_v\mathbf{D}_{n,m}^T\,, \tag{28}$$
$$\bar{\mathbf{Z}}_w(p) \triangleq \bar{\mathbf{Z}}_w(n,m,p) = \bar{\mathbf{B}}_{n+p,m}\bar{\mathbf{\Psi}}_w(p)\mathbf{G}_{n,m}^T\,, \tag{29}$$

the mean square initial state is $\mathbf{R}_m = E\{\mathbf{x}_m\mathbf{x}_m^T\}$, the noise covariance function matrices are given by

$$\bar{\mathbf{\Psi}}_w(p) \triangleq \bar{\mathbf{\Psi}}_w(n,m,p) = E\{\bar{\mathbf{W}}_{n+p,m}\bar{\mathbf{W}}_{n,m}^T\}\,, \tag{30}$$
$$\mathbf{\Psi}_v \triangleq \mathbf{\Psi}_v(n,m) = E\{\mathbf{V}_{n,m}\mathbf{V}_{n,m}^T\}\,, \tag{31}$$

and the unbiased gain

$$\bar{\mathbf{H}}_{n,m}(p) = \mathbf{\Phi}_{n+p,m}(\mathbf{C}_{n,m}^T\mathbf{C}_{n,m})^{-1}\mathbf{C}_{n,m}^T \tag{32}$$

satisfies the unbiasedness condition

$$E\{\tilde{\mathbf{x}}_{n|n}\} = E\{\mathbf{x}_n\}\,. \tag{33}$$

The mean square initial state function $\mathbf{Z}_{n,m}$ can be found by solving the discrete algebraic Riccati equation [10],

$$\mathbf{0} = \mathbf{Z}_{n,m}(\tilde{\mathbf{Z}}_w + \tilde{\mathbf{Z}}_v)^{-1}\mathbf{Z}_{n,m} + 2\mathbf{Z}_{n,m} + \tilde{\mathbf{Z}}_w + \tilde{\mathbf{Z}}_v - \mathbf{Y}_{n,m}\mathbf{Y}_{n,m}^T(\tilde{\mathbf{Z}}_w + \tilde{\mathbf{Z}}_v)^{-1}\mathbf{Z}_{n,m}\,, \tag{34}$$

which solution can be found either following [15] or numerically.

3.1. Unbiased FIR Estimate

Utilizing (32), the unbiased FIR estimate can be written as

$$\bar{\mathbf{x}}_{n+p|n} = \bar{\mathbf{H}}_{n,m}(p)\mathbf{Y}_{n,m} \tag{35a}$$
$$= \mathbf{\Phi}_{n+p,m}(\mathbf{C}_{n,m}^T\mathbf{C}_{n,m})^{-1}\mathbf{C}_{n,m}^T\mathbf{Y}_{n,m} \tag{35b}$$

It has been shown in [16] that (35a) becomes virtually optimal when $N \gg 1$ that makes it attractive for engineering applications. On the other hand, large N implies the computational problem that can be circumvented with the iterative Kalman-like algorithm proposed in [13]:

$$\bar{\mathbf{x}}_{l+p|l} = \mathbf{\Phi}_{l+p,l+p-1}\bar{\mathbf{x}}_{l+p-1|l-1} + \mathbf{\Phi}_{l+p,l+p-1}\mathbf{\Upsilon}_l^{-1}(p)\mathbf{F}_l\mathbf{C}_l^T \times[\mathbf{y}_l - \mathbf{C}_l\mathbf{\Upsilon}_l(p)\bar{\mathbf{x}}_{l+p-1|l-1}]\,, \tag{36}$$

where

$$\begin{aligned}
\mathbf{F}_l &= \mathbf{\Phi}_{l,l-1}(\mathbf{\Xi}_l + \mathbf{F}_{l-1}^{-1})^{-1}\mathbf{\Phi}_{l,l-1}^T\,, && (37)\\
\mathbf{\Xi}_l &= \mathbf{\Phi}_{l,l-1}^T\mathbf{C}_l^T\mathbf{C}_l\mathbf{\Phi}_{l,l-1}\,, && (38)\\
\mathbf{F}_s &= \mathbf{\Phi}_{s,m}(t_s,t_m)\mathbf{\Lambda}\mathbf{\Phi}_{s,m}^T(t_s,t_m)\,, && (39)\\
\bar{\mathbf{x}}_{s+p|s} &= \mathbf{\Phi}_{s+p,m}\mathbf{\Lambda}\mathbf{C}_{s,m}^T\mathbf{Y}_{s,m}\,, && (40)\\
\mathbf{\Lambda} &= (\mathbf{C}_{s,m}^T\mathbf{C}_{s,m})^{-1}\,, && (41)\\
\mathbf{\Upsilon}_l(p) &= \mathbf{\Phi}_{l,l-|p|-1}\ \text{, if } p<0 \text{ (smoothing)} && (42)\\
&= \mathbf{\Phi}_{l,l-1}\ \text{, if } p=0 \text{ (filtering)}\\
&= \mathbf{I}\ \text{, if } p=1 \text{ (1-step prediction)}\\
&= \mathbf{\Phi}_{l+p-1,l}^{-1}\ \text{, if } p>1 \text{ (prediction)}\,.
\end{aligned}$$

$s = l_{\min} - 1$, $m = n - N + 1$, and an iterative variable l ranges from $l_{\min}$ to n. To avoid singularities, one can set $l_{\min} \geqslant m + K$. The true estimate is taken at $l = n$.

4. APPLICATION TO CLOCK MODEL

It is known from [17] that the clock time interval error $x(t)$ is mostly caused by the clock oscillator instabilities and can be modeled with the finite Taylor series expansion as

$$x(t) = x_{10} + x_{20}t + \frac{x_{30}}{2}t^2 + w_x(t)\,, \tag{43}$$

where $x_{10} = x_1(0)$, $x_{20} = x_2(0)$, and $x_{30} = x_3(0)$ are the clock states at zero, namely the time error (first state), fractional frequency offset (second state), and linear frequency drift rate (third state), respectively. Noise $w_x(t) = \varphi(t)/2\pi\nu_{\text{nom}}$ is defined by the oscillator random phase deviation $\varphi(t)$ and nominal frequency ν_{nom} in Hz.

In state-space, (43) is represented with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{44}$$

and $\mathbf{B}$ identity and can be translated to discrete time with

$$\mathbf{\Phi}_{n,n-1} = \begin{bmatrix} 1 & \tau & \frac{\tau^2}{2} \\ 0 & 1 & \tau \\ 0 & 0 & 1 \end{bmatrix} \tag{45}$$

and the time-invariant noise covariance matrix [16]

$$\mathbf{Q}_w(\tau) = \tau \begin{bmatrix} q_1 + \frac{q_2\tau^2}{3} + \frac{q_3\tau^4}{20} & \frac{q_2\tau}{2} + \frac{q_3\tau^3}{8} & \frac{q_3\tau^2}{6} \\ \frac{q_2\tau}{2} + \frac{q_3\tau^3}{8} & q_2 + \frac{q_3\tau^2}{3} & \frac{q_3\tau}{2} \\ \frac{q_3\tau^2}{6} & \frac{q_3\tau}{2} & q_3 \end{bmatrix}, \tag{46}$$

in which q_1, q_2, and q_3 are the diffusion parameters [18] associated with the clock white noise. Note that clock has also colored Gaussian noise components.

Measurement is commonly described with (9) having $\mathbf{C} = [\,1\ 0\ 0\,]$ and $\mathbf{D}$ identity. In our experiment, it was organized for the crystal clock imbedded in the Stanford Frequency Counter SR620. Another SR620 was used to measure each second the time difference between the clock and GPS SynPaQ III Timing Sensor. To obtain the reference trend, simultaneous measurement was provided for the Symmetricom Cesium Frequency Standard CsIII. In such a set, the GPS timing sensor induces the sawtooth noise v_n uniformly distributed from -50 ns to 50 ns with the variance $Q_v = 50^2/3$ ns^2 in the presence of the GPS time temporary uncertainty. Following [19], the optimal averaging interval was found to be $N_{\text{opt}} = 3500$ and the 3-state Kalman-like filter (36)–(42) used as an optimal filter with $p = 0$. For the 3-state Kalman algorithm, (46) was specified following [18] via the Allan deviation available for the clock investigated.

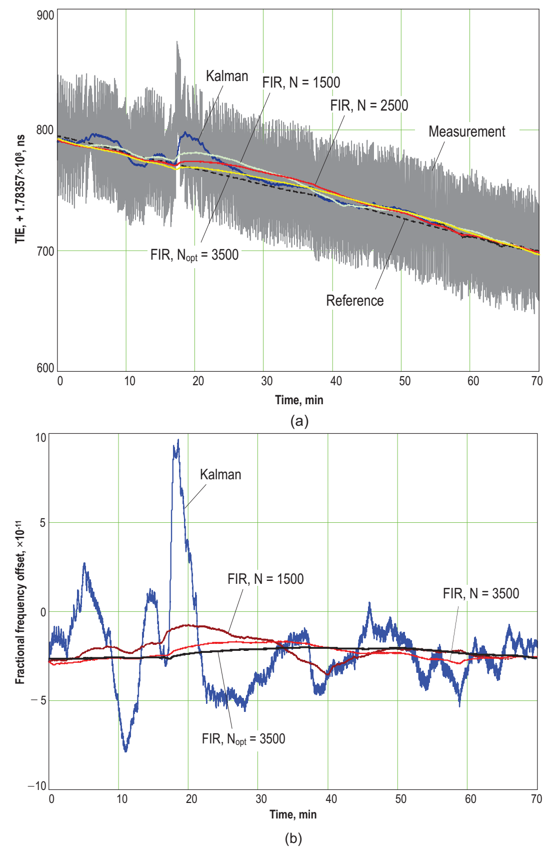

Fig. 1. Typical GPS-based TIE measured and states estimated of the OCXO-based clock: (a) first state and (d) second state.

The x_{1n} and x_{2n} estimated are sketched in Fig. 1a and Fig. 1b, respectively, in line with the reference measurement (dashed). An analysis reveals that the white Gaussian approximation of $\mathbf{Q}_w(\tau)$ by (46) is unsuccessful and the Kalman filter produces the worst estimates. It is especially neatly seen in the estimates of the second state (Fig. 1b). Just on the contrary to the Kalman filter, the Kalman-like one ignores noise and initial errors and relies only on N_{opt}. Provided $N_{\text{opt}} = 3500$, this filter shows better robustness against the

Table 1. Errors of the OCXO-based Clock State Estimation with the UFIR Kalman-Like and Kalman Algorithms

Filter	Stdev, ns	Bias, ns	RMSE, ns	EB, ns
$N = 1500$	5.236	5.582	7.654	4.473
$N = 2500$	4.504	3.292	5.579	3.464
$N = 3500$	**3.401**	**1.431**	**3.690**	2.928
Kalman	6.287	5.774	8.537	

GPS time uncertainties and produces much smaller errors, especially for the second state (Fig. 1b). It works better even with lower values of $N = 2500$ and $N = 1500$.

Table 1 gives us statistics for the Kalman and Kalman-like estimates in line with the error bound (EB) calculated following [20]. Although the Kalman-like algorithm certainly works better, neither of these algorithms fits the EB specialized for white Gaussian noise. This is due to the clock colored noise and the GPS time temporary uncertainties in the measurement.

5. CONCLUSION

The p-shift optimal FIR estimator was adapted for discrete-time filtering, smoothing, and prediction of hybrid (continuous/discrete) state-space models over N nearest past measurement points. As a special case, we have considered the UFIR one ignoring noise and initial errors and becoming near optimal when $N \gg 1$. For fast computation, the latter was represented with the iterative Kalman-like form. As an example of applications, we have exploited the Kalman-like UFIR and Kalman algorithms for state estimation in an ovenized crystal clock via the GPS-based measurements of time errors. It has been shown that the clock colored noise and GPS time temporary uncertainties force the Kalman filter to produce large errors. In contrast, the UFIR filter demonstrates better robustness, lower excursions, and smaller random noise at the output.

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