# REFINEMENT IN THE ESTIMATION OF MULTICOMPONENT POLYNOMIAL-PHASE SIGNALS

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## ABSTRACT

In this paper, we consider precise parameter estimation of multicomponent polynomial-phase signals (mc-PPSs). Two estimation refinement methods are proposed and both are initialized by coarse estimates provided by any technique for the mc-PPS estimation. The first method refines components, one by one, by estimating them once again, now from the received signal with all the components suppressed, except for the considered one. In the second method, the recently proposed estimation refinement technique for single PPS based on the filtering and phase unwrapping has been extended to the mc-PPSs. The first method approaches the Cramér-Rao lower bound (CRLB) within a few dBs, whereas the second one practically reaches the CRLB. Both methods are characterized by significantly lower complexity than computationally intensive nonlinear least squares (NLS) methods. Simulations confirm the effectiveness of the proposed methods.

***Index Terms*—** Polynomial phase, parameter estimation, estimation refinement, phase unwrapping.

## 1. INTRODUCTION

Parameter estimation of polynomial-phase signals (PPSs) is an important issue since linear and nonlinear frequency-modulated (FM) signals are found in several practical applications. For example, in synthetic aperture radar (SAR), inverse SAR (ISAR), Doppler radar and sonar imaging, the returns are FM signals [1]. Applications also include biomedicine, seismology, geophysics. The PPS estimation has been extensively dealt with in the literature [1–4]. A very popular approach is based on the high-order ambiguity function (HAF) [2, 3]. The performance of the HAF, however, is deteriorated in the presence of multicomponent PPSs (mc-PPSs). Due to nonlinear structure of the HAF, the components interact with each other giving rise to cross-terms that can mask the desired peaks in the HAF spectrum. In addition, the HAF may fail in identifying components with the same highest order phase coefficients [1].

Barbarossa and co-workers introduced the product high-order ambiguity function (PHAF) [1] that can resolve the components of mc-PPSs and effectively suppress the noise. A fine search algorithm for the PHAF-based mc-PPS parameter estimation is proposed in [5], where the peak frequency in the PHAF spectrum is estimated very accurately, without spectrum oversampling. The PHAF-based estimation is biased and the error-propagation effect makes the estimation of lower order phase coefficients and amplitudes less accurate [1]. A nonlinear least squares (NLS) approach can be used to improve the estimation accuracy [4, 6]. In [4], the Nelder–Mead simplex algorithm (NMSA) is used for minimizing the nonlinear cost function initialized by parameter estimates provided by the PHAF. The method, however, suffers from significant computational complexity that limits its practical applicability. Parameter estimation of product of an FM signal and PPS, found, for example, in radar and sonar applications that involve moving targets with vibrating or rotating parts, is considered in [7] and [8].

In this paper, we propose two methods for the fine parameter estimation of the mc-PPSs. In both methods, the PHAF-based method is used to provide initial (coarse) estimates [5], although any method for the mc-PPS estimation can be used. In the first method, we repeat the estimation of components, one at a time, but now from the received signal where all components, except for the one to be estimated, have been removed. An extension of the refinement method for single PPS estimation [9] to the mc-PPSs constitutes the second method. The former method offers significant improvement over the initial estimate, approaching the Cramér-Rao lower bound (CRLB) within a few dBs, whereas the latter one practically reaches the CRLB. Both methods have significantly lower complexity than the NMSA-based method proposed in [4].

Paper is organized as follows. Section 2 describes the mc-PPS estimation. The proposed estimation refinement methods are presented in Section 3. Simulations and conclusions are given in sections 4 and 5, respectively.

## 2. MULTICOMPONENT PPS ESTIMATION

The mc-PPS $x(n), n = 0, \cdots, N-1$, has the following form:

$$x(n) = \sum_{k=1}^{K} A_k e^{j2\pi \sum_{p=0}^{P} \alpha_{k,p}(n\Delta)^p}, \tag{1}$$

where $A_k$ and $\alpha_{k,p}, p = 0, \cdots, P$, are the amplitude and polynomial-phase coefficients of the $k$th component, respectively, $P$ the polynomial order, $K$ the number of components, $\Delta$ the sampling interval and $N$ the signal length. We will assume that $x(n)$ is embedded in zero-mean white Guassian noise $\varepsilon(n)$ with variance $\sigma^2$, i.e., we consider signal

$$y(n) = x(n) + \varepsilon(n). \tag{2}$$

Assuming that the number of components $K$ is known [1, 3], we can estimate the phase coefficients of each component starting from the strongest one as follows [3]:

**Step 1.** Set $k = 1$ and $y_f(n) = y(n)$.

**Step 2.** Estimate the amplitude $A_k$ and phase coefficients $\alpha_{k,p}, p = 0, 1, \cdots, P$, of the strongest component from the PHAF of $y_f(n)$. The obtained estimations are denoted as $\widehat{A}_k$ and $\widehat{\alpha}_{k,p}$. Filter out the estimated component as

$$y_f(n) = y_f(n) - \widehat{A}_k e^{j2\pi \sum_{p=0}^{P} \widehat{\alpha}_{k,p}(n\Delta)^p}. \tag{3}$$

**Step 3.** Set $k = k + 1$. If $k > K$ exit; otherwise, go to Step 2.

Note that, instead of the HAF used in [3], we will use the PHAF since it significantly outperforms the HAF in the presence of mc-PPSs [1]. We can also use the fine search algorithm proposed in [5] to improve the estimation accuracy without adding significant computational burden. The PHAF-based estimation procedure is described in detail in [5] and is not given here due to limited page count.

Two methods for estimation refinement are described in the following section. Both methods use initial parameter estimates obtained by the above described algorithm.

## 3. ESTIMATION REFINEMENT

### 3.1. Method I

In order to refine the estimation of the $k$th component, $k = 1, 2, \cdots, K$, we propose to estimate it once more, now from the received signal from which all the other components have been removed. We start from the strongest component. The algorithm is given below.

**Step 1.** Set $k = 1$ and initial refined estimates as $\widehat{A}_k^r = \widehat{A}_k$ and $\widehat{\alpha}_{k,p}^r = \widehat{\alpha}_{k,p}$ for $k = 1, 2, \cdots, K$ and $p = 0, 1, \cdots, P$.

**Step 2.** Update the refined estimates of the $k$th component, $\widehat{A}_k^r$ and $\widehat{\alpha}_{k,p}^r, p = 0, 1, \cdots, P$ from the PHAF of

$$y^k(n) = y(n) - \sum_{q=1, q\neq k}^{K} \widehat{A}_q^r e^{j2\pi \sum_{p=0}^{P} \widehat{\alpha}_{q,p}^r (n\Delta)^p}. \tag{4}$$

**Step 3.** Set $k = k + 1$. If $k > K$ exit; otherwise, go to Step 2.

In step 2, we update the refined estimation of the $k$th component using $y^k(n)$ given in (4). Note that in $y^k(n)$, all the components except for the $k$th one, have been filtered out. Ideally, if these components have been estimated without error, $y^k(n)$ will contain the $k$th component only plus noise. In reality, however, this is not the case. Nevertheless, the influence of other components on the estimation of the $k$th one has been significantly reduced as it will be shown in simulations.

Note also that the initial refined estimates (step 1) equal the estimates obtained by the PHAF-based procedure (section 2). After refining the $k$th component, its refined estimate is used in the sum in (4), rather than the initial one, in the estimation of the following components.

### 3.2. Method II

The mc-PPS estimation can be refined by extending the single PPS refinement method proposed in [9] (we will refer to it as the O'Shea's method) to the mc-PPS case. We propose an extension of the O'Shea's method as follows:

**Step 1.** Set $k = 1$ and initial refined estimates as $\widehat{A}_k^r = \widehat{A}_k$ and $\widehat{\alpha}_{k,p}^r = \widehat{\alpha}_{k,p}$ for $k = 1, 2, \cdots, K$ and $p = 0, 1, \cdots, P$.

**Step 2.** Update the refined estimates of the $k$th component, $\widehat{A}_k^r$ and $\widehat{\alpha}_{k,p}^r, p = 0, 1, \cdots, P$ from

$$y^k(n) = y(n) - \sum_{q=1, q\neq k}^{K} \widehat{A}_q^r e^{j2\pi \sum_{p=0}^{P} \widehat{\alpha}_{q,p}^r (n\Delta)^p} \tag{5}$$

using the following steps:

**Step 2a.** De-chirp $y^k(n)$ as

$$z(n) = y^k(n) e^{-j2\pi \sum_{p=1}^{P} \widehat{\alpha}_{k,p}^r (n\Delta)^p}. \tag{6}$$

**Step 2b.** Low-pass filter $z(n)$ with a moving average (MA) filter and decimate:

$$z_0(m) = \frac{1}{M} \sum_{n=(m-1)M+1}^{mM} z(n), \quad m = 1, \cdots, Q, \tag{7}$$

where $Q = \lfloor N/M \rfloor$ and $M$ is the MA filter length.

**Step 2c.** Create a vector $\mathbf{V}$ of unwrapped angle of $z_0$, i.e., $\mathbf{V} = \text{unwrap}(\text{angle}(z_0))$. Vector $\mathbf{V}$ is a polynomial in noise with unknown phase coefficients $\mathbf{a} = [\alpha_{k,0}, \delta\alpha_{k,1}, \cdots, \delta\alpha_{k,P}]$, where $\delta\alpha_{k,p} = \alpha_{k,p} - \widehat{\alpha}_{k,p}^r, p = 1, 2, \cdots, P$. Vector $\mathbf{a}$ can be estimated using

$$\widehat{\mathbf{a}} = [\widehat{\alpha}_{k,0}, \widehat{\delta}\alpha_{k,1}, \cdots, \widehat{\delta}\alpha_{k,P}] = (\mathbf{G}^T\mathbf{G})^{-1}\mathbf{G}^T\mathbf{V}, \tag{8}$$

where

$$\mathbf{G} = \begin{bmatrix} 1 & \Delta & \cdots & \Delta^P \\ 1 & 2\Delta & \cdots & (2\Delta)^P \\ \cdots & \cdots & \cdots & \cdots \\ 1 & Q\Delta & \cdots & (Q\Delta)^P \end{bmatrix}. \quad (9)$$

**Step 2d.** Update $\widehat{\alpha}^r_{k,p}$ as follows:

$$\widehat{\alpha}^r_{k,p} = \widehat{\alpha}^r_{k,p} + \frac{\widehat{\delta}\alpha_{k,p}}{M^k}, \quad p = 1, 2, \cdots, P \quad (10)$$

$$\widehat{\alpha}^r_{k,0} = \widehat{\alpha}_{k,0}. \quad (11)$$

**Step 3.** Set $k = k + 1$. If $k > K$ exit; otherwise, go to Step 2.

In step 2a, the $k$th component will be well localized around the DC component of $z(n)$ assuming that the initial estimation has been performed adequately. In the O'Shea's method, the initial parameter estimates can be significantly less accurate than for Newton algorithms [9]. In step 2b, the low-pass filtering is used to increase the signal-to-noise ratio (SNR), thus enabling the use of phase unwrapping and linear least-squares estimation techniques [10]. Since, in general, the polynomial curve fitting in (8) and (9) is ill-conditioned if the signal length or the polynomial order are very large [9, Section IIA], i.e., the process is vulnerable to round-off errors, the decimation takes place in (7).

## 4. SIMULATIONS

The proposed methods are evaluated for a three-component PPS signal embedded in white Gaussian noise with zero mean and variance $\sigma^2$. All the signal components are third-order PPSs with phase coefficients given in Table 1. Without loss of generality, we adopted the zero initial phase for all the three components. Components I, II and III are characterized by the SNR of 18 dB, 16 dB and 14 dB, respectively. The SNR is defined by SNR $= 20\log_{10}(A_k/\sigma)$, where $A_k$ is the amplitude of the $k$th component, $k = 1, 2, 3$. In addition, the signal length is $N = 512$ and $\Delta = 1/N$.

For both proposed refinement methods, the initial (coarse) estimation is performed using the PHAF [5, Section 2]. The PHAF peaks are maximized with the dichotomous approach [5, Section 3]. Thus the estimation accuracy cannot be further improved using the PHAF domain only. In our simulations, we used the same setup in the PHAF calculation as in the simulations section in [5]. In the second proposed method, the MA filter length is $M = 5$.

Figure 1 presents the mean squared error (MSE) curves versus noise variance obtained in the estimation of each coefficient from Table 1. The noise variance is varied from 2 to 20 in steps of 2. The MSE is calculated according to

$$\text{MSE} = 10\log_{10}\frac{1}{N_{sim}}\sum\nolimits_{n=1}^{N_{sim}}\left(\widehat{\alpha}^n_{k,p} - \alpha_{k,p}\right)^2, \quad (12)$$

**Table 1**. Phase coefficients of considered three-component PPS

| Component I | Component II | Component III |
|---|---|---|
| $a_{1,1} = \frac{\pi^2}{37}N$ | $a_{2,1} = \frac{\pi^2}{53}N$ | $a_{3,1} = -\frac{\pi^2}{47}N$ |
| $a_{1,2} = -\frac{\pi^2}{29}N$ | $a_{2,2} = -\frac{\pi^2}{67}N$ | $a_{3,2} = \frac{\pi^2}{27}N$ |
| $a_{1,3} = \frac{\pi^2}{41}N$ | $a_{2,3} = -\frac{\pi^2}{33}N$ | $a_{3,3} = \frac{\pi^2}{85}N$ |

where $\widehat{\alpha}^n_{k,p}$ is the estimation of the $p$th coefficient ($p = 1, 2, 3$) of the $k$th component in the $n$th simulation and $N_{sim}$ is the number of Monte Carlo simulations. In our simulations, $N_{sim} = 300$. The left three subplots correspond to the estimation of Component I, the middle three subplots to Component II and the right three subplots to the estimation of Component III.

The initial PHAF estimation is depicted by dashed line with squares, the Method I refinement by dotted line, the Method II refinement by solid line with circles, the CRLB by dashdot line. We also added the results obtained by the NMSA-based method [4] (dashed line with points).

Clearly, the proposed refinement methods provide improvement over the initial estimate. Method I approaches the CRLB for all the considered coefficients with the bias of a couple of dBs. On the other hand, Method II practically reaches the CRLB, which is also the case for the NMSA-based method. However, the complexity of the NMSA-based method significantly exceeds that of the other two methods since a 9-dimensional search is performed. In our simulations, the execution time of the NMSA exceeded the proposed methods' time approximately 30 times. The NMSA is implemented in Matlab by the `fminsearch` function.

Figure 1 also shows that the PHAF-based estimates are sufficiently accurate in the sense that Method II reaches the CRLB with such initial estimates. In other words, more accurate initial estimates would not yield any improvement in accuracy of Method II. Finally, note that the difference between the initial and refined estimates of the third (weakest) component is smaller than with the other two components. This is due to removing the strongest two components in (3) prior to estimating the third one.

## 5. CONCLUSIONS

Coefficients of a mc-PPS can be estimated very accurately without using time-consuming NLS-based methods. To that end, we proposed two methods. In the first method, the estimation of each component is refined using the received signal from which all other components have been filtered out. In the second method, filtering and phase unwrapping approach was used. Filtering increases the SNR which enables the use of the phase unwrapping approach which reaches the CRLB at higher SNRs. The execution time is substantially reduced compared to the NMSA-based method without degrading the estimation accuracy.

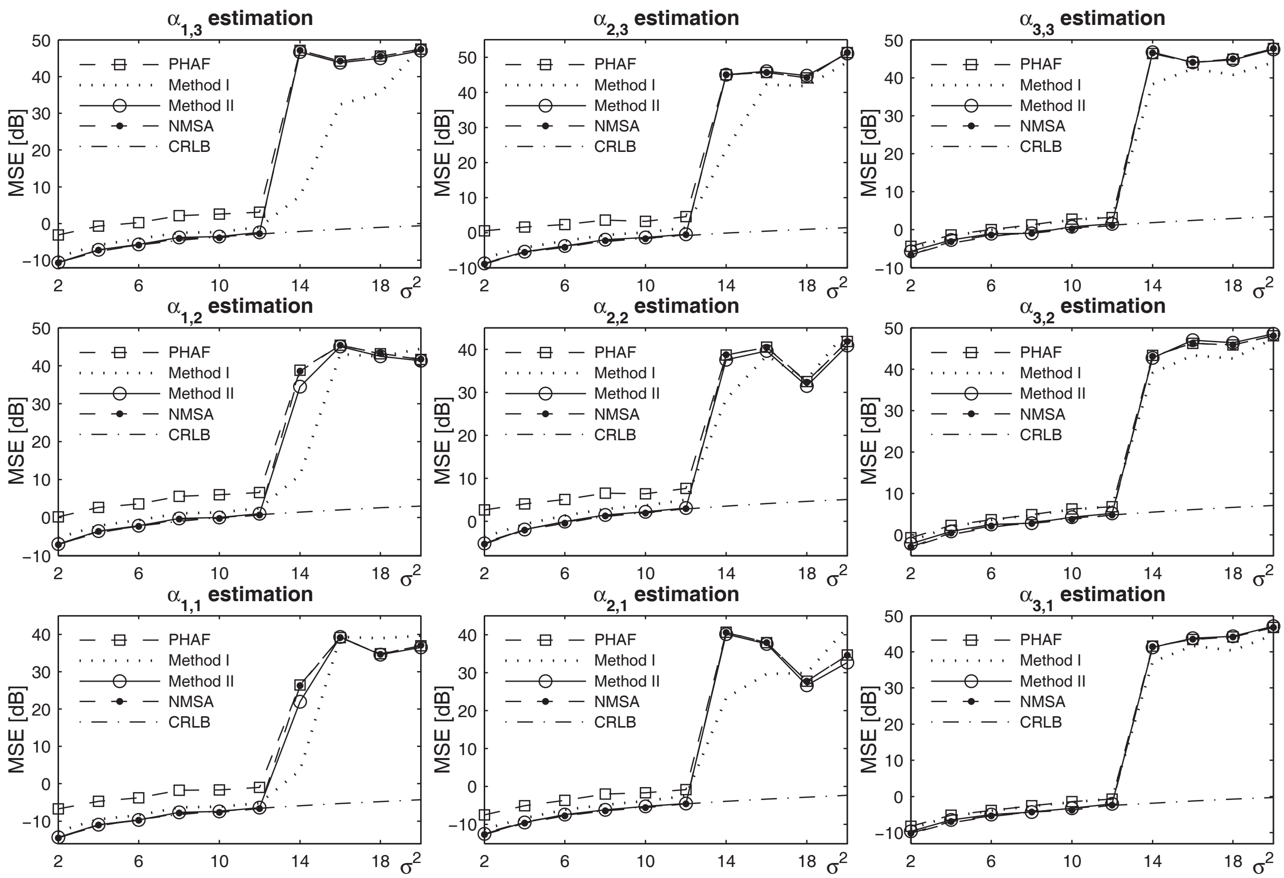


**Fig. 1**. MSE versus noise variance $\sigma^2$. *Left column:* Component I estimation MSE; *Middle column:* Component II estimation MSE; *Right column:* Component III estimation MSE.

Future research will include the analysis of statistical properties of the proposed estimators and how close the initial estimates need to be.

## 6. REFERENCES


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