EFFECT OF INITIAL PHASE IN TWO TONE SEPARATION USING EMPIRICAL MODE DECOMPOSITION

Jiexin Gao, Dimitrios Hatzinakos

The Edward S. Rogers Sr. Department of Electrical and Computer Engineering,
University of Toronto,
10 King's College Road, Toronto, ON, Canada, M5S 3G4

ABSTRACT

Empirical mode decomposition (EMD) is an adaptive method for nonlinear and nonstationary signal processing. Although the algorithm is easy to implement and widely deployed, its theoretical background and limitations remain uncertain. This paper investigates the performance of EMD in two tone separation problem, especially for the transition region between perfect separation and failure, with emphasis on the effect of the initial phase. Relationships between amplitude ratio, frequency ratio, initial phase and performance are derived.

Index Terms— Empirical Mode Decomposition, Tone Separation

1. INTRODUCTION

Empirical mode decomposition is an adaptive signal decomposition method proposed by Huang et al [1]. Due to its empirical and algorithmic nature, there is currently no complete theoretical foundation for this method. To better interpret the underlying principle, Rilling [2] studied two tone separation problem using EMD and derived analytical constraints under which EMD may succeed in separation or fail. Delechelle [3] computed the mean envelop using a 4^{th} order PDE and proved its equivalence to the sifting operation. Meignen [4] re-formulated the sifting process by minimizing the squared derivative of mean envelop.

This paper investigates the performance of EMD in two tone separation problem, with emphasis in the transition region between perfect separation and failure. The paper is organized as follows. Section 2 summarizes the main results from [2]. After defining the signal model and performance measure in Section 3, analysis and derivation of the relationship between amplitude ratio, frequency ratio, initial phase and performance is presented in Section 4. Finally a comparison between the estimated and simulation performance is given in Section 5.

This work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

2. RELATED WORKS

Two tone Separation using EMD has been investigated by Rilling [2] by constructing a simplified model that is parameterized by the amplitude ratio, frequency ratio and phase difference of the two tones:

$$x(t; a, f, \varphi) = \cos(2\pi t) + a\cos(2\pi ft + \varphi), t \in \mathbb{R}$$

where a is the amplitude ratio and f is the frequency ratio between the two tones. Without loss of generality, they further required that $f \in (0, 1)$ so that the first component in the model is always of higher frequency. To analyze the performance of EMD for tone separating, the following performance measure was utilized:

$$c^{(n)}(a, f, \varphi) \triangleq \frac{\left\| d_1^{(n)}(t; a, f) - \cos(2\pi t) \right\|_{L^2(T)}}{\left\| a\cos(2\pi ft + \varphi) \right\|_{L^2(T)}}$$

where $d_1^{(n)}(t; a, f)$ stands for the first Intrinsic Mode Function(IMF) extracted with n sifting iterations and $\|\bullet\|_{L^2(T)}$ is the Euclidean norm on functions defined over $[0, T]$. This performance measure is close to 0 when the two components are correctly separated, and is close to 1 when EMD fails to extract the two tones. By measuring and averaging it over all possible φ, they acquired results similar to Figure 1. It was observed that the ability of EMD to successfully extract the high frequency component $\cos(2\pi t)$ depends highly on the average number of extrema per unit length (or the extrema rate) $r_e(a, f)$. If this number is close to that of the high frequency component, EMD is more likely to separate the two tones, otherwise it is unlikely to discover the extrema of high frequency component thus might fail in tone separating. Furthermore, theoretical analysis showed that perfect separation ($c^{(n)}(a, f) \approx 0$) is possible for signals that satisfy $\{f < 1/3\} \cap \{a \leq (f \sin(\frac{3\pi f}{2}))^{-1}\} \cup \{af < 1\}$, while tones are badly separated ($c^{(n)}(a, f) \approx 1$) for signals satisfying $af^2 > 1$. No regular pattern of performance was observed in the transition region as shown in Figure 1 (denoted by the light grey region).

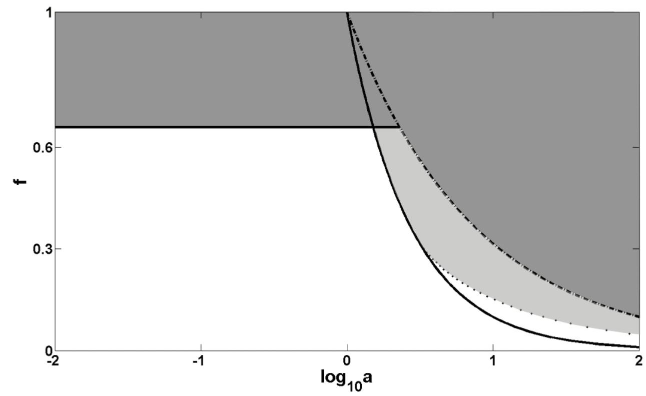


Fig. 1. Main result from [2]: Performance measure of separation for two-tones signals - 2D projection onto the (a, f)-plane. Critical Curves are plotted as dashed($af = 1$), dash-dotted($af^2 = 1$) and dotted($af \sin(\frac{3\pi f}{2}) = 1$) lines. The black thick line stands for the contour $\langle c_1^{(10)}(a, f, \varphi)\rangle_\varphi = 0.5$

3. SIGNAL MODEL AND PERFORMANCE MEASURE

In this paper performance of tone separation in the transition region is studied. A simplified model similar to the one proposed by Rilling [2] is used, with $2\pi t$ replaced by x for the convenience of derivation. Also φ is kept as a parameter in all calculations rather than being averaged out:

$$F(x; a, f, \varphi) = \cos(x) + a\cos(xf + \varphi), x \in \mathbb{R}$$

In the following sections F is used to denote the mixture of two tones, F_1 for the high frequency component $\cos(x)$, and F_2 for the low frequency component $a\cos(xf + \varphi)$ so that:

$$F = F_1 + F_2$$

The same performance measure is used but will be refereed to as *error function* in the following sections. Parameter n is dropped since we don't consider the effect of different number of sifting operations in this work:

$$c(a, f, \varphi) \triangleq \frac{\|d_1(t; a, f) - \cos(2\pi t)\|_{L^2(T)}}{\|a\cos(2\pi ft + \varphi)\|_{L^2(T)}}$$

4. ANALYSIS

In this section we derive the relationship between a, f, φ and the error function $c(a, f, \varphi)$ in the transition region defined by $\{af > 1\} \cap \{af^2 < 1\} \cap (\{f > 1/3\} \cup \{a > (f\sin(\frac{3\pi f}{2}))^{-1}\})$. As pointed out by Rilling [2], there is no regular distribution of extrema if (a, f) falls in the transition region (Figure 2). In the transition region, our derivation is carried out for two separate cases:

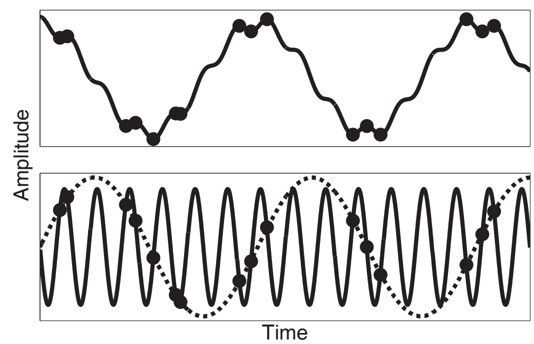


Fig. 2. Distribution of extrema when (a, f, φ) are in transition region. Top plot is $F = F_1 + F_2$, solid line in bottom plot is F_1' and dashed line in bottom plot is $-F_2'$. Each intersection in bottom plot is marked with a dot, corresponding to one extremum point in F.

- *Case A*: $\frac{1}{f} = 2k + 1, k \in \mathbb{Z}$
- *Case B*: $\frac{1}{f} \neq 2k + 1, k \in \mathbb{Z}$

4.1. Case A

If $\frac{1}{f}$ is an odd integer, F_2 has period that is integer multiples of F_1 and extrema rate are the same around every extremum point on F_2. In this case if an extremum point related to F_1 is not visible in F around $x = 0$, all extrema related to F_1 in the entire time span are not visible, so that extrema rate $r_e(a, f)$ is equal to that of the low frequency component F_2. It is obvious that EMD will fail in separating tones since there is no way for EMD to sift out the high frequency component. Consider the simple case where $\varphi = 0$ in Figure 3. In this Figure F_1' is plotted together with $-F_2'$ so that each intersection denotes one extremum in the mixture signal F. If $-F_2'$ intersects with F_1' only at $x = 0$ around 0, extrema related to F_1 are not visible in F. Since $\frac{1}{f}$ is odd integer also implies that $f < \frac{1}{3}$, the worst cases situation scenario is a tangency condition which is already satisfied for the transition region, as stated in [2] and its approximation is defined by:

$$a > (f\sin(\frac{3\pi f}{2}))^{-1}$$

Now consider the case where $\varphi \neq 0$. Since F_1 is periodic with period 2π, intersections are the same (thus performances are the same) if

$$\varphi = 2\pi kf, k \in \mathbb{Z} \tag{1}$$

Now suppose the initial phase deviates slightly from φ to $\varphi + \Delta\varphi$ as shown in Figure 4. In this case $-F_2'$ starts to move to either direction and will eventually become tangent with F_1 so that there are two intersections around $x = 0$, which implies one extremum of the high frequency component is

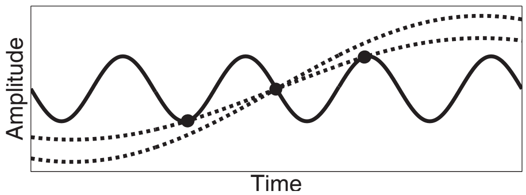


Fig. 3. Tangency condition.

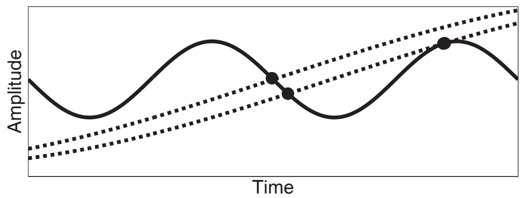


Fig. 4. Small deviation of phase from $\varphi = 0$.

now visible in F. To solve for $\varphi + \Delta\varphi$ in the tangent condition, without loss of generality, we assume $\varphi = 0$. Now $F_2 = a\cos(xf + \Delta\varphi)$ and we have:

$$\begin{cases} F_1' = -F_2' \\ F_1'' = -F_2'' \\ \frac{\pi}{2} \le \frac{\Delta\varphi}{f} \le \frac{3\pi}{2} \end{cases}$$

where $F_1'(x) = -\sin(x)$, $F_2'(x) = -af\sin(xf + \varphi)$, $F_1''(x) = -\cos(x)$ and $F_2''(x) = -af^2\cos(xf + \varphi)$. solve for $\Delta\varphi$ gives

$$\Delta\varphi = \sin^{-1}\left(-\frac{1}{af}\sqrt{\frac{a^2f^4-1}{f^2-1}}\right) + f\left(\pi + \sin^{-1}\left(\sqrt{\frac{a^2f^4-1}{f^2-1}}\right)\right) \quad (2)$$

If the deviation of φ from the one defined in Equation (1) is smaller than $\Delta\varphi$, EMD will fail in separating the two tones. To summarize, in the transition region, when $\frac{1}{f}$ is an odd integer, error function can be calculated as follows:

$$c(a, f, \varphi) = \begin{cases} 1 & x \in \mathbb{S} \\ 0 & x \notin \mathbb{S} \end{cases} \quad (3)$$

where

$$\mathbb{S} = \{(f, \varphi) | 2\pi kf - \Delta\varphi < \varphi < 2\pi kf + \Delta\varphi, k \in \mathbb{Z}\}$$

with $\Delta\varphi$ as defined in Equation (2).

4.2. Case B

In the general case, since there is no regular pattern for the distribution of extrema points, in order to have extrema related to F_1 visible on F, we require that F_1' intersects with $-F_2'$ more than once within one cycle of F_1. The more frequent this happens inside one cycle of F (with period T_F defined by $2\pi\frac{LCM(1,f)}{f}$, where LCM stands for the least common multiplier) the more likely it is for EMD to reveal all the extrema of F_1 within finite number of sifting operations. This type of intersection is depicted in Figure 5.

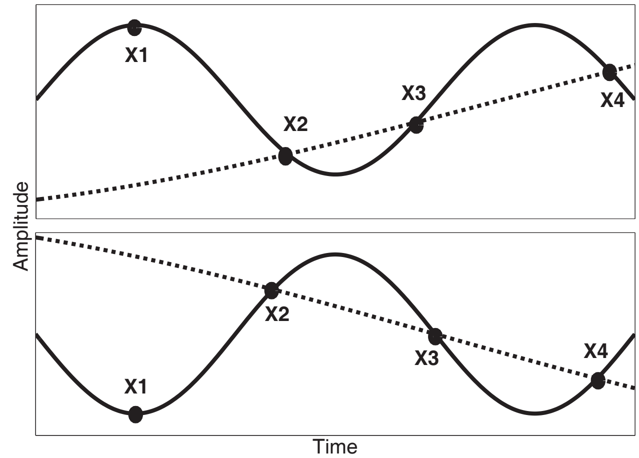
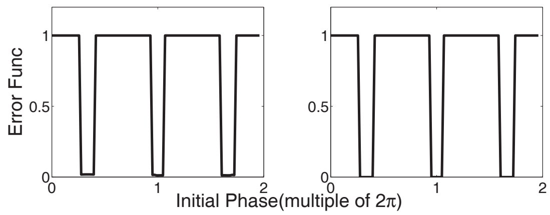


Fig. 5. Necessary condition for extrema to be revealed during EMD sifting process. In both plots solid lines are F_1' and dashed lines are $-F_2'$. F_1' and $-F_2'$ need to have more than one intersections within one cycle of F_1 for extremum point related to F_1 to be visible in F.

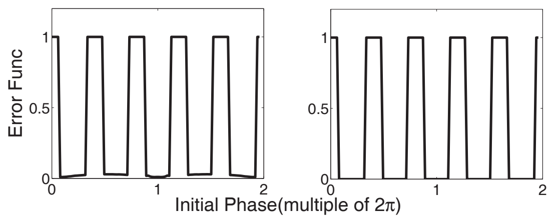
To test the existence of such type of intersection, we solve for $\boldsymbol{X} = [x_1\ x_2\ x_3\ x_4]^T$ in the following equalities and inequalities

$$\begin{cases} |F_1'(x_1)| = 1 \\ F_1'(x_2) = -F_2'(x_2) \\ F_1'(x_3) = -F_2'(x_3) \\ F_1'(x_4) = -F_2'(x_4) \\ x_1 < x_2 < x_3 < x_4 \\ x_1 + \frac{1}{2}\pi < x_2 < x_1 + \frac{3}{2}\pi \\ x_1 + \pi < x_3 < x_1 + 2\pi \\ x_1 + \frac{3}{2}\pi < x_4 < x_1 + \frac{5}{2}\pi \\ sign(F_1'(x_1))(F_1'(x_1) - F_2'(x_1)) > 0 \\ sign(F_1'(x_1))(F_2'(x_1 + 3\pi) - F_1'(x_1 + 3\pi)) > 0 \end{cases} \quad (4)$$

where $F_1'(x) = -\sin(x)$, $F_2'(x) = -af\sin(xf + \varphi)$ and $sign(\bullet)$ is the signum function. The more solutions and the smaller T_F implies a denser distribution of F_1-related extrema in F, under which case it is easier for EMD to extract F_1. On the other hand, if solutions are fewer and T_F is large, in the worst case there is only one set of solution per one cycle of F, which makes it difficult for EMD to sift out all the extrema points of F_1 and it will probably fail to achieve this within finite number of operations. To summarize, in the transition region, for the general case of $\frac{1}{f}$ not being an odd integer, error function can be calculated as follows:



(a) $a = 5, f = \frac{1}{3}$



(b) $a = 5, f = \frac{1}{5}$

Fig. 6. Estimated (left) and Simulation (right) error function for Case A.

$$c(a, f, \varphi) = \begin{cases} 1 & \{\boldsymbol{X_i}\} = \emptyset \\ 0.5 & 0 \leq N \leq 0.2T_F \\ 0 & N \geq 0.2T_F \end{cases} \tag{5}$$

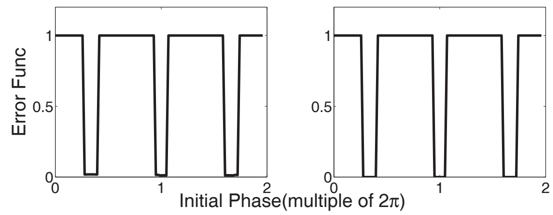
where $\{\boldsymbol{X_i}\}, i = 1, 2 \cdots N$ are the set of solutions to Equation (4).

5. RESULTS

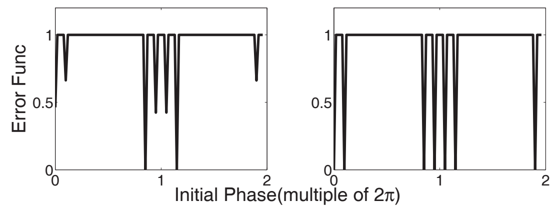
In this section we provide both estimated and simulation results for the two cases defined in section 4. Error function is plotted as function of φ for different values of (a, f).

Results for Case A are depicted in Figure 6. Left columns are results from applying EMD to the tone mixture F and measuring performance using the error function. Right columns are analytical results computed from Equation (3). Analytical solutions are the same with simulation results, with fluctuations happening only around points where error functions are close to 0. This is most likely due to the imperfection of EMD algorithm at signal boundaries.

Results for Case B are depicted in Figure 7. Left columns are results from applying EMD to the tone mixture F and measuring performance using the error function. Right columns are numerical solutions from Equation (5). Estimated performance matches simulation result if the error function takes value in $\{0, 1\}$. Equation (5) fails to estimate performance if it takes values in between 0 and 1. Other conditions have to be taken into account to refine the result.



(a) $a = 26, f = 0.15$



(b) $a = 40, f = 0.15$

Fig. 7. Estimated (left) and Simulation (right) error function for Case B.

6. CONCLUSION

This paper studied tone separation problem using EMD, with emphasize on the effect of phase difference in the transition region of error function. Derivation was carried out for two cases, yield an analytical solution for special case and numerical solution for general case. Comparison between the estimated and simulation error function was presented.

7. REFERENCES

[1] Norden E. Huang, Zheng Shen, Steven R. Long, Manli C. Wu, Hsing H. Shih, Quanan Zheng, Nai-Chyuan Yen, Chi Chao Tung, and Henry H. Liu, "The empirical mode decomposition and the hilbert spectrum for nonlinear and non-stationary time series analysis," *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences*, vol. 454, no. 1971, pp. 903–995, 1998.

[2] G. Rilling and P. Flandrin, "One or two frequencies? the empirical mode decomposition answers," *Signal Processing, IEEE Transactions on*, vol. 56, no. 1, pp. 85 –95, jan. 2008.

[3] E. Delechelle, J. Lemoine, and Oumar Niang, "Empirical mode decomposition: an analytical approach for sifting process," *Signal Processing Letters, IEEE*, vol. 12, no. 11, pp. 764 – 767, nov. 2005.

[4] S. Meignen and V. Perrier, "A new formulation for empirical mode decomposition based on constrained optimization," *Signal Processing Letters, IEEE*, vol. 14, no. 12, pp. 932 –935, dec. 2007.