A SAMPLING THEOREM FOR MANHATTAN GRIDS

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ABSTRACT

This paper presents a sampling theorem for Manhattan-grid sampling, which is a sampling scheme in which data is taken along evenly spaced rows and columns. Given the spacing between the rows, the columns, and the samples along the rows and columns, the theorem shows that an image can be perfectly reconstructed from Manhattan-grid sampling if its spectrum is bandlimited to a cross-shaped region whose arm lengths and widths are determined by the aforementioned sample spacings. The nature of such *cross-bandlimited* images is demonstrated by filtering an image with a cross-bandlimiting filter for several choices of sampling parameters.

***Index Terms*—** Sampling theorem, Manhattan grids, cross-bandlimited, rectangular sampling

1. INTRODUCTION

Manhattan-grid sampling is a new form of image sampling in which data is taken along evenly spaced rows and columns. In particular, as illustrated in Fig. 1, samples are taken at intervals of λ_x along horizontal rows spaced $\Delta_y = k_y\lambda_y$ apart, and also at intervals of λ_y along vertical columns spaced $\Delta_x = k_x\lambda_x$, where k_x and k_y are positive integers. This is a special case of *cutset sampling* that has been used to good effect in both lossy and lossless image compression, especially, for bilevel images [1–3]. The first stage of such methods losslessly compresses Manhattan-grid samples, for example with arithmetic coding (AC), which is efficient since the grid samples are closely spaced and, hence, highly correlated. For lossless compression, the other pixels are then AC encoded, conditioned on those in the grid, while for lossy compression no further encoding is used, and the decoder estimates the remaining pixels from those on the grid. Markov random field models guide the arithmetic coding and the estimation. Recently, Manhattan-grid sampling has also been proposed [5] as a general approach to sampling grayscale images, with the motivations that (a) there are physical scenarios for which it is far more natural than traditional lattice sampling, e.g. a ship sampling water temperature, and (b) dense sampling along lines might capture edge transitions more completely than conventional lattice sampling with the same density. With the latter in mind, image reconstruction methods have been developed for Manhattan grids with the goal of preserving edges [5].

This paper presents both a sampling theorem for Manhattan-grid sampling and a concrete reconstruction method. In particular, given the sampling parameters $\Delta_x, \Delta_y, \lambda_x, \lambda_y$, the theorem shows that an image can be perfectly reconstructed from Manhattan-grid sampling if its spectrum is bandlimited to a cross-shaped region whose arm lengths and widths are determined by the sampling parameters, as illustrated in Fig. 3. It also presents a reconstruction method that involves first reconstructing a highpass portion of the image spectrum from just the samples on the horizontal rows, (alternatively, another highpass portion could be reconstructed from samples on the vertical columns), and then reconstructing the remainder of the spectrum after subtracting the aliasing due to the highpass portion.

Sampling theorems are, of course, well known for lattices [6, 7], as well as for unions of lattice cosets [7]. In most of these, the required bandlimitation is such that image spectrum replicas induced by sampling do not overlap. However, there are also cases where the replica spectra do overlap and, nevertheless, image reconstruction is still possible [8]. The present sampling theorem is a special case of the latter. However, instead of using the characterization in [8], which starts with a spectral constraint and determines a sampling pattern, in this paper, we restrict ourselves to the Manhattan-grid sampling pattern and find the natural spectral constraint that enables reconstruction from these samples. In other words, the theorem and the reconstruction method are tailored specifically to Manhattan-grid sampling. As a result, it is much easier to gain insight into this kind of sampling, and the reconstruction is more straightforward.

The layout of our paper is as follows: In Section 2, we review conventional rectangular sampling and relate it to Manhattan-grid sampling. Section 3 defines a class of functions we call *cross-bandlimited functions*. In Section 4, we define a set of recovery filters to be used in reconstructing our original function from the Manhattan grid samples. Section 5 contains the main result of this paper – a sampling theorem that shows if an image is appropriately cross-bandlimited, then it can be reconstructed from Manhattan-grid samples with a specific method. To gain a sense for the nature of the *cross bandlimited* constraint, Section 6 shows the effect of cross-bandlimiting an image for several Manhattan grid sizes. Additionally, these images have the property that they can be perfectly reconstructed from their Manhattan grids.

2. RECTANGULAR AND MANHATTAN GRID SAMPLING

Conventional sampling on a rectangular lattice refers to sampling a function $f(x, y)$ at all points (x, y) in the lattice $\{(i\Delta_x, j\Delta_y); i, j \in \mathbb{Z}\}$. It is convenient to represent such sampling with the multiplication of $f(x, y)$ with a comb function. Let $\delta_2(x, y)$ be an ideal 2D Dirac delta function. Define a 2D comb function as

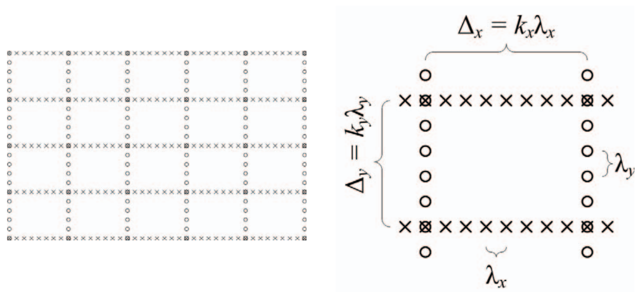
$$\mathcal{C}(x, y; \Delta_x, \Delta_y) = \Delta_x\Delta_y \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \delta_2(x - n\Delta_x, y - m\Delta_y)\,. \quad (1)$$

The Dirac impulses are spaced in the x direction by Δ_x and in the y direction by Δ_y. Rectangular lattice sampling of $f(x, y)$ is then

$$f_{sr}(x, y) = f(x, y)\,\mathcal{C}(x, y; \Delta_x, \Delta_y)\,. \quad (2)$$

We will refer to the Fourier transform of $f_{sr}(x, y)$ as $F_{sr}(u, v)$.

*This work was supported by NSF Grant CCF 0830438.



(a) Entire grid. (b) Close-up with parameters

Fig. 1. Manhattan-grid sampling. The $\times$'s mark the locations of the horizontal samples $f_{sx}(x, y)$, and the $\circ$'s mark the locations of the vertical samples $f_{sy}(x, y)$.

Suppose we want to sample a function on a horizontally dense rectangular lattice. Define *horizontal sampling* as

$$f_{sx}(x, y) \;=\; f(x, y)\, \mathcal{C}\,(x, y; \lambda_x, \Delta_y)\ , \tag{3}$$

where $0 < \lambda_x < \Delta_y < \infty$. We will refer to the Fourier transform of $f_{sx}(x, y)$ as $F_{sx}(u, v)$. Note that in the Fourier domain, $F_{sx}(u, v)$ will contain a set of replicas of $F(u, v)$ that are horizontally spaced by $\frac{1}{\lambda_x}$ and vertically spaced by $\frac{1}{\Delta_y}$. In a similar fashion, we can sample on a rectangular lattice where the vertical samples are taken more densely. We define *vertical sampling* as

$$f_{sy}(x, y) \;=\; f(x, y)\, \mathcal{C}\,(x, y; \Delta_x, \lambda_y)\ , \tag{4}$$

where $0 < \lambda_y < \Delta_x < \infty$. Again, $F_{sy}(u, v)$ will refer to the Fourier transform of $f_{sy}(x, y)$, and it will contain replicas of $F(u, v)$ horizontally spaced by $\frac{1}{\Delta_x}$ and vertically spaced by $\frac{1}{\lambda_y}$.

As illustrated in Fig. 1, Manhattan-grid sampling refers to sampling a function $f(x, y)$ at all points (x, y) on the set $\{(i\lambda_x, j\Delta_y)\} : i, j \in \mathbb{Z}\} \cup \{(i\Delta_x, j\lambda_y)\} : i, j \in \mathbb{Z}\}$, where $\Delta_x = k_x\lambda_x$, and $\Delta_y = k_y\lambda_y$ for some positive integers k_x, k_y. It will be useful to visualize a Manhattan grid as the union of a horizontally dense rectangular lattice with a vertically dense rectangular lattice. Using comb functions, we model Manhattan-grid sampling by multiplying a function $f(x, y)$ with

$$\begin{aligned}\mathcal{G}\,(x, y; \Delta_x, \Delta_y, \lambda_x, \lambda_y) \;&=\; \mathcal{C}\,(x, y; \lambda_x, \Delta_y) \\ &+ \mathcal{C}\,(x, y; \Delta_x, \lambda_y) - \mathcal{C}\,(x, y; \Delta_x, \Delta_y)\ .\end{aligned} \tag{5}$$

The third term on the RHS of the above cancels the overlap of the other two comb functions. Thus, $f_{sm}(x, y) = f(x, y)\,\mathcal{G}(x, y; \Delta_x, \Delta_y, \lambda_x, \lambda_y)$ represents sampling on a Manhattan grid.

3. CROSS-BANDLIMITED FUNCTIONS

Cross-bandlimited functions are well suited to Manhattan grid sampling. Let $f(x, y)$ be a square-integrable function with 2D Fourier transform $F(u, v)$. For $0 < u_1 < u_2 < \infty$ and $0 < v_1 < v_2 < \infty$, define the cross-shaped region $B(u_1, u_2, v_1, v_2)$ in 2D frequency space, illustrated in Fig. 2, to be

$$B \;=\; \{|u| < u_2, |v| < v_1\} \cup \{|u| < u_1, |v| < v_2\}\,. \tag{6}$$

We say $f(x, y)$ or $F(u, v)$ is *cross-bandlimited* to $B(u_1, u_2, v_1, v_2)$ if $F(u, v)$ satisfies

$$F(u, v) \;=\; 0, \quad (u, v) \in B^c. \tag{7}$$

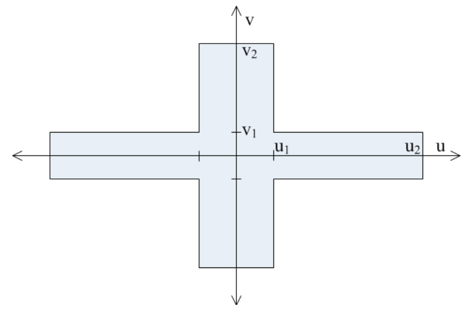


Fig. 2. The cross-bandlimited region $B(u_1, u_2, v_1, v_2)$.

4. IDEAL RECOVERY FILTERS

In order to reconstruct a cross-bandlimited signal, we will need to define a set of recovery filters. First consider the cross-shaped region $S = B\left(\frac{1}{2\Delta_x}, \frac{1}{2\lambda_x}, \frac{1}{2\Delta_y}, \frac{1}{2\lambda_y}\right)$ in 2D frequency space. We partition S using the following regions:

$$S \;=\; S_l \cup S_{hx} \cup S_{hy}\,, \tag{8}$$

where, as illustrated in Fig. 3,

$$S_l \;=\; \left\{(u, v) : |u| < \frac{1}{2\Delta_x}, |v| < \frac{1}{2\Delta_y}\right\} \tag{9a}$$

$$S_{hx} \;=\; \left\{(u, v) : \frac{1}{2\Delta_x} < |u| < \frac{1}{2\lambda_x}, |v| < \frac{1}{2\Delta_y}\right\} \tag{9b}$$

$$S_{hy} \;=\; \left\{(u, v) : |u| < \frac{1}{2\Delta_x}, \frac{1}{2\Delta_y} < |v| < \frac{1}{2\lambda_y}\right\}. \tag{9c}$$

We now define a collection of ideal filters on the support of these regions. This collection will include a lowpass filter, a horizontal highpass filter, and a vertical highpass filter. With I_A denoting the indicator function of the set A, the filters are

$$H_l(u, v) \;=\; \mathrm{I}_{S_l}(u, v) \tag{10a}$$

$$H_{hx}(u, v) \;=\; \mathrm{I}_{S_{hx}}(u, v) \tag{10b}$$

$$H_{hy}(u, v) \;=\; \mathrm{I}_{S_{hy}}(u, v)\,. \tag{10c}$$

If a function $F(u, v)$ is cross-bandlimited to region S, then we can use the partition property of these filters to decompose $F(u, v)$ into its lowpass, horizontal highpass, and vertical highpass bands.

$$F(u, v) \;=\; F_l(u, v) + F_{hx}(u, v) + F_{hy}(u, v) \tag{11}$$

where

$$F_l(u, v) \;=\; H_l(u, v)F(u, v) \tag{12a}$$

$$F_{hx}(u, v) \;=\; H_{hx}(u, v)F(u, v) \tag{12b}$$

$$F_{hy}(u, v) \;=\; H_{hy}(u, v)F(u, v)\,. \tag{12c}$$

5. SAMPLING THEOREM FOR MANHATTAN GRIDS

Before presenting our theorem, we begin with two useful facts.

Fact 1. Rectangular sampling theorem. *Suppose we sample a 2D square-integrable function $f(x, y)$ with a comb function $f_{sr}(x, y) = f(x, y)\,\mathcal{C}\,(x, y; \tau_x, \tau_y)$. If $F(u, v) = \mathcal{F}\{f(x, y)\}$*

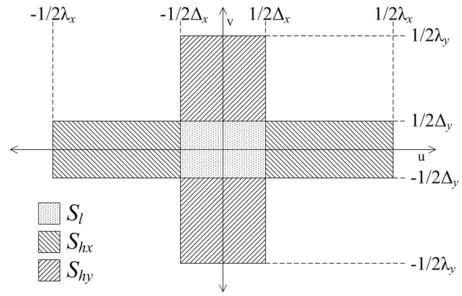


Fig. 3. Support of the recovery filters

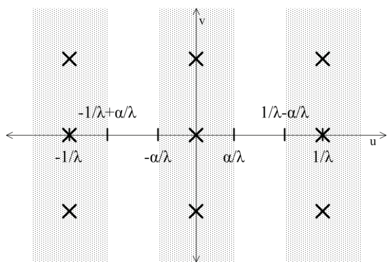


Fig. 4. For horizontal sampling, the shaded region shows the support of the frequency replicas after sampling. The $\times$'s mark their centers.

is bandlimited in the frequency domain to the rectangular region $[-\frac{1}{2\tau_x}, \frac{1}{2\tau_x}] \times [-\frac{1}{2\tau_y}, \frac{1}{2\tau_y}]$*, then* $f(x,y)$ *can be perfectly recovered from a rectangular lowpass filter* $H(u,v)$ *whose support is* $[-\frac{1}{2\tau_x}, \frac{1}{2\tau_x}] \times [-\frac{1}{2\tau_y}, \frac{1}{2\tau_y}]$ *in the frequency domain via*

$$f(x,y) = \mathcal{F}^{-1}\left\{H(u,v)\mathcal{F}\left\{f_{sr}(x,y)\right\}\right\} . \tag{13}$$

An alternative statement of the recoverability is that $F(u,v)$ *and the sampled spectrum* $F_{sr}(u,v)$ *are identical on the region* $[-\frac{1}{2\tau_x}, \frac{1}{2\tau_x}] \times [-\frac{1}{2\tau_y}, \frac{1}{2\tau_y}]$.

Proof. Well known. □

Fact 2. Sampling a function bandlimited to a strip. *Let* $f(x,y)$ *be a 2D square-integrable function bandlimited in the horizontal direction to the vertical strip* $\left[-\frac{\alpha}{\lambda}, \frac{\alpha}{\lambda}\right] \times (-\infty, \infty)$ *for some* $0 < \alpha < \frac{1}{2}$*. If* $f(x,y)$ *is horizontally sampled with rate* $\frac{1}{\lambda}$*, then the spectrum of the sampled signal will be zero on the vertical strips* $(\frac{\alpha}{\lambda}, \frac{1}{\lambda} - \frac{\alpha}{\lambda}) \times (-\infty, \infty)$ *and* $(-\frac{1}{\lambda} + \frac{\alpha}{\lambda}, -\frac{\alpha}{\lambda}) \times (-\infty, \infty)$*. A similar result holds for functions bandlimited in the vertical direction.*

Proof. Suppose $f(x,y)$ is bandlimited in the horizontal frequency u by $\left[-\frac{\alpha}{\lambda}, \frac{\alpha}{\lambda}\right]$. After sampling by $\frac{1}{\lambda}$ in the horizontal direction, the horizontal coordinates of the replicas of $F(u,v)$ will be centered at integer multiples of $\frac{1}{\lambda}$. As illustrated in Fig. 4, the spectral replicas centered at $u = 0$ will have support on the horizontal interval $[-\frac{\alpha}{\lambda}, \frac{\alpha}{\lambda}]$. The closest replicas that may have their spectral support on $[-\frac{1}{\lambda}, \frac{1}{\lambda}]$ are centered at $u = \pm\frac{1}{\lambda}$. These bands have a half-width of $\pm\frac{\alpha}{\lambda}$, and their support lies on the intervals $[-\frac{1}{\lambda}, -\frac{1}{\lambda} + \frac{\alpha}{\lambda}]$ and $[\frac{1}{\lambda} - \frac{\alpha}{\lambda}, \frac{1}{\lambda}]$. It is easy to see that the only frequency intervals that are guaranteed to be zero on $[-\frac{1}{\lambda}, \frac{1}{\lambda}]$ are $(-\frac{1}{\lambda} + \frac{\alpha}{\lambda}, -\frac{\alpha}{\lambda})$ and $(\frac{\alpha}{\lambda}, \frac{1}{\lambda} - \frac{\alpha}{\lambda})$. Note that $\alpha < \frac{1}{2}$ prevents overlapping frequencies; at $\alpha = \frac{1}{2}$, we cannot guarantee any frequency regions to be zero. □

The following is the main result of the paper.

Theorem 1. Manhattan-grid sampling theorem. *Suppose we sample a square-integrable function* $f(x,y)$ *using* $(\Delta_x, \Delta_y, \lambda_x, \lambda_y)$ *Manhattan-grid sampling. If the 2D Fourier transform* $F(u,v) = \mathcal{F}\left\{f(x,y)\right\}$ *is cross-bandlimited to* $B\left(\frac{1}{2\Delta_x}, \frac{1}{2\lambda_x}, \frac{1}{2\Delta_y}, \frac{1}{2\lambda_y}\right)$*, then* $f(x,y)$ *can be recovered exactly. Furthermore, if* $F_{sx}(u,v)$ *is the spectrum of the horizontal samples and* $F_{sy}(u,v)$ *is the spectrum of the vertical samples, then* $f(x,y)$ *can be recovered via*

$$f(x,y) \;=\; \mathcal{F}^{-1}\left\{F_l(u,v) + F_{hx}(u,v) + F_{hy}(u,v)\right\} , \tag{14}$$

where

$$F_{hx}(u,v) \;=\; H_{hx}(u,v)F_{sx}(u,v) \tag{15a}$$

$$F_{hy}(u,v) \;=\; H_{hy}(u,v)F_{sy}(u,v) , \tag{15b}$$

and with $\star$ *denoting 2D convolution,*

$$\begin{aligned} F_l(u,v) \;&=\; H_l(u,v)F_{sy}(u,v) \\ &- H_l(u,v)\left[F_{hx}(u,v) \star \Delta_x\lambda_y\,\mathcal{C}\left(u,v;\frac{1}{\Delta_x},\frac{1}{\lambda_y}\right)\right], \end{aligned} \tag{15c}$$

or, equivalently,

$$\begin{aligned} F_l(u,v) \;&=\; H_l(u,v)F_{sx}(u,v) \\ &- H_l(u,v)\left[F_{hy}(u,v) \star \lambda_x\Delta_y\,\mathcal{C}\left(u,v;\frac{1}{\lambda_x},\frac{1}{\Delta_y}\right)\right]. \end{aligned} \tag{15d}$$

Proof. It suffices to prove (14). Begin by letting $f(x,y)$ be a square-integrable function cross-bandlimited to $B\left(\frac{1}{2\Delta_x}, \frac{1}{2\lambda_x}, \frac{1}{2\Delta_y}, \frac{1}{2\lambda_y}\right)$. We sample $f(x,y)$ using $(\Delta_x, \Delta_y, \lambda_x, \lambda_y)$ grid sampling. We will use the conventions in Section 2 to denote the horizontally sampled spectrum $F_{sx}(u,v)$ and the vertically sampled spectrum $F_{sy}(u,v)$.

First, we will prove that the highpass band $F_{hx}(u,v)$ defined in (12b) can be recovered from the spectrum of the horizontal samples $F_{sx}(u,v)$ using (15a). Similarly, the other highpass band $F_{hy}(u,v)$ can be recovered via (15b) by a symmetry argument. Consider the spectrum of horizontal samples

$$F_{sx}(u,v) \;=\; F(u,v) \star \lambda_x\Delta_y\,\mathcal{C}\left(u,v;\frac{1}{\lambda_x},\frac{1}{\Delta_y}\right) \tag{16}$$

where the $\lambda_x\Delta_y$ scaling is due to taking the Fourier transform of $\mathcal{C}\left(x,y;\lambda_x,\Delta_y\right)$. As before, this spectrum consists of a series of replicas of $F(u,v)$ in the frequency domain. Substituting (11) into (16) and filtering with $H_{hx}(u,v)$, we obtain

$$\begin{aligned} &H_{hx}(u,v)F_{sx}(u,v) \;=\; \\ &\quad H_{hx}(u,v)\left[F_{hx}(u,v) \star \lambda_x\Delta_y\mathcal{C}\left(u,v;\frac{1}{\lambda_x},\frac{1}{\Delta_y}\right)\right] \\ &\quad + H_{hx}(u,v)\left[F_l(u,v) \star \lambda_x\Delta_y\mathcal{C}\left(u,v;\frac{1}{\lambda_x},\frac{1}{\Delta_y}\right)\right] \\ &\quad + H_{hx}(u,v)\left[F_{hy}(u,v) \star \lambda_x\Delta_y\mathcal{C}\left(u,v;\frac{1}{\lambda_x},\frac{1}{\Delta_y}\right)\right]. \end{aligned} \tag{17}$$

In the first term on the RHS of the above, the signal $F_{hx}(u,v)$ is bandlimited to $[-\frac{1}{2\lambda_x}, \frac{1}{2\lambda_x}] \times [-\frac{1}{\Delta_y}, \frac{1}{\Delta_y}]$. As noted in Section 2, rectangular sampling in the space domain causes the spectrum $F_{hx}(u,v) \star \lambda_x\Delta_y\,\mathcal{C}\left(u,v;\frac{1}{\lambda_x},\frac{1}{\Delta_y}\right)$ to contain shifted replicas of $F(u,v)$, By Fact 1, this spectrum is identical to $F_{hx}(u,v)$ on the interval $[-\frac{1}{2\lambda_x}, \frac{1}{2\lambda_x}] \times [-\frac{1}{\Delta_y}, \frac{1}{\Delta_y}]$. Therefore, $F_{hx}(u,v)$ can be extracted by filtering with $H_{hx}(u,v)$, and thus the first term of (17) equals $F_{hx}(u,v)$.

(a) Original image "Bank" (b) Cross-bandlimited, 4×4 (c) Cross-bandlimited, 8×8 (d) Cross-bandlimited, 16×16

Fig. 5. Examples of cross-bandlimited images. $n \times n$ Manhattan grid sampling corresponds to $\Delta_x = \Delta_y = ns$, $\lambda_x = \lambda_y = 1s$, where s is the original pixel spacing. These images can be reconstructed from every nth row and nth column of pixels.

Now we show that the second and third terms on the RHS of (17) are zero. Before sampling, the spectra $F_l(u,v)$ and $F_{hy}(u,v)$ are bandlimited in the horizontal direction on the interval $[-\frac{1}{2\Delta_x}, \frac{1}{2\Delta_x}]$. Thus, the spectrum $[F_l(u,v)+F_{hy}(u,v)]\star\lambda_x\Delta_y\,\mathcal{C}\left(u,v;\frac{1}{\lambda_x},\frac{1}{\Delta_y}\right)$ contains replicas of $F_l(u,v)$ and $F_{hy}(u,v)$. By Fact 2, the replicas of $F_l(u,v)$ and $F_{hy}(u,v)$ do not overlap with the support of our horizontal highpass filter $H_{hx}(u,v)$, which has horizontal support $u \in [-\frac{1}{2\lambda_x}, -\frac{1}{2\Delta_x}] \cup [\frac{1}{2\Delta_x}, \frac{1}{2\lambda_x}]$. Thus, filtering the last two terms of (17) with $H_{hx}(u,v)$ will force them to zero. As a result, the RHS of (17) equal the RHS of (15a). (15b) can be derived similarly.

We now prove that the lowpass band $F_l(u,v)$ can be recovered via (15c). A similar argument can be used to derive (15d). Define $G_x(u,v)$ to be the components of $F_{hx}(u,v)$ that alias in the lowpass domain when vertically sampled at (Δ_x, λ_y), i.e.

$$G_x(u,v) = H_l(u,v)\left[F_{hx}(u,v)\star\Delta_x\lambda_y\,\mathcal{C}\left(u,v;\frac{1}{\Delta_x},\frac{1}{\lambda_y}\right)\right] \tag{18}$$

Now consider the spectrum of $F_{sy}(u,v)$ filtered with $H_l(u,v)$:

$$\begin{aligned} H_l(u,v)F_{sy}(u,v) \;=\;& H_l(u,v)\left[F_l(u,v)\star\Delta_x\lambda_y\,\mathcal{C}\left(u,v;\frac{1}{\Delta_x},\frac{1}{\lambda_y}\right)\right] \\ &+H_l(u,v)\left[F_{hx}(u,v)\star\Delta_x\lambda_y\,\mathcal{C}\left(u,v;\frac{1}{\Delta_x},\frac{1}{\lambda_y}\right)\right] \\ &+H_l(u,v)\left[F_{hy}(u,v)\star\Delta_x\lambda_y\,\mathcal{C}\left(u,v;\frac{1}{\Delta_x},\frac{1}{\lambda_y}\right)\right] \end{aligned} \tag{19}$$

By Fact 1, the first term of (19) is simply $F_l(u,v)$. The second term of (19) is $G_x(u,v)$ by definition. The third term of (19) is zero. To see why, note that $F_{hy}(u,v)$ is bandlimited to $[-\frac{1}{2\Delta_x}, \frac{1}{2\Delta_x}] \times [-\frac{1}{2\lambda_y}, \frac{1}{2\lambda_y}]$ and is being sampled by $\mathcal{C}(x,y;\Delta_x,\lambda_y)$. By Fact 1, the spectrum of $F_{hy}(u,v)\star\Delta_x\lambda_y\mathcal{C}\left(u,v;\frac{1}{\Delta_x},\frac{1}{\lambda_y}\right)$ is identical to $F_{hy}(u,v)$ on the region $[-\frac{1}{2\Delta_x}, \frac{1}{2\Delta_x}] \times [-\frac{1}{2\lambda_y}, \frac{1}{2\lambda_y}]$. Since the highpass spectrum $F_{hy}(u,v)$ is zero on the support of the lowpass filter $H_l(u,v)$, the third term is zero. Thus, we are left with:

$$H_l(u,v)F_{sy}(u,v) \;=\; F_l(u,v) + G_x(u,v) \tag{20}$$

All that is needed is to subtract (18) from (20) to obtain the desired result (15c). Likewise, (15d) follows by a symmetry argument.

Summing (15a), (15b), and either (15c) or (15d), and taking the inverse Fourier transform yields (14). □

6. THE EFFECTS OF CROSS BANDLIMITING

We filtered a 512×512 image with a cross-bandlimiting filter corresponding to sampling parameters $\Delta_x = \Delta_y = ns$ and $\lambda_x = \lambda_y = 1s$, where s refers to the original pixel spacing and n is some positive integer. Figure 5(a) shows the original image. Figures 5(b), 5(c) and 5(d) show the image after it has been cross-bandlimited using 4×4, 8×8, and 16×16 Manhattan grids, respectively.

The cross-bandlimiting filter has caused the images in Figures 5(b), 5(c) and 5(d) to have several properties and artifacts. First, the horizontal and vertical edges are very sharp. This is related to the fact that the cross-bandlimiting filter retains all purely horizontal and purely vertical frequencies. However, many of the diagonal edges are blurry, and several ringing artifacts can be seen. Overall, the images have a quilt-like texture, which seems to mimic the cross-shaped geometry of our sampling and frequency support.

As verification of the sampling theorem, we also sampled the image on an $ns \times ns$ Manhattan grid, which corresponds to sampling every nth row and every nth column of the original image. This is approximately $\frac{2n-1}{n^2}$ of the total number of pixels. We were able to perfectly recover the filtered images using the method in Section 5.

7. REFERENCES

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