FAST CONVERGENCE RECONSTRUCTION FORMULAS FOR PERIODIC NONUNIFORM SAMPLING OF ORDER 2

Marie Chabert

University of Toulouse, IRIT-ENSEEIHT,
Toulouse, France
marie.chabert@enseeiht.fr

Bernard Lacaze

TéSA
14/16 Port St-Etienne 31000 Toulouse France
bernard.lacaze@tesa.prd.fr

ABSTRACT

This paper studies the reconstruction of a random band-limited process from a nonuniform periodic sampling of order 2. This sampling scheme uses two interleaved sample sequences at the same rate chosen according to the Landau criterion. The associated exact reconstruction formula is composed of two series with cardinal sine functions as general terms. In practical applications, the signal is observed on a finite window. The exact reconstruction formula is then approached by truncation of the series. The series rate of convergence has thus a strong influence on the reconstruction performance in practice. Unfortunately, the cardinal sine functions lead to a slow convergence. In the case of an over-sampled process, we propose interpolation formulas derived from raised cosine filters. We show, through theoretical analysis and numerical simulations, that the regularity of these functions leads to a higher series convergence rate and thus improves the reconstruction performance even for a low over-sampling rate.

***Index Terms*—** Nonuniform sampling of order 2, reconstruction formula.

1. INTRODUCTION

Due to the general increase of data rate and bandwidth, high speed analog to digital converters (ADC) are required in modern digital communication systems [1]. Time interleaved converters offer an efficient solution while maintaining low production costs and have been developed by many electronics companies [1]. The signal is processed in parallel by L ADCs operating at the same rate with different phases. They operate a so-called Periodic Nonuniform Sampling of order L (PNSL) [2], [3], [4], [5]. The rate is chosen according to the Landau criterion related to the signal bandwidth [6]. This sampling scheme reduces by L the mean sampling rate and allows to eliminate interleaving in case of bandpass signals [3], [4], [7]. For simplicity, this paper focusses on PNS2. The exact reconstruction of the original process at a given time is possible and only requires the *a priori* knowledge of the signal spectral band [8]. Unfortunately, the classical reconstruction formula suffers from a very bad convergence rate [1], [9]. This is a problem in practical applications where the signal must be reconstructed from a finite number of samples. However, when the signal is over-sampled, an exact reconstruction formula with a very faster convergence can be obtained [1],[9]. Indeed, the reconstruction formula convergence speed is directly related to the continuity of the interpolation function. In this paper, the exact reconstruction formula for PNS2 using a raised cosine interpolation filter is provided and the effect of the truncation on the reconstruction performance is studied through theoretical analysis and numerical simulations. The choice of a raised cosine filter is ubiquitous in telecommunication systems in case of over-sampling. However even if many papers deal with exact reconstruction formulas for non-uniform sampling, few of them study the problem of the truncation error [1]. This paper propose a method for truncation error analysis that can be easily generalized to smoother interpolation functions for even faster reconstruction of random processes.

1.1. Signal model

The original signal is a stationary random process $\mathbf{Z} = \{Z(t), t \in \mathbb{R}\}$ with power spectral density $s_Z(\omega)$ defined by:

$$\mathrm{E}\left[Z(t)\, Z^*(t-\tau)\right] = \int_\Delta e^{i\omega\tau} s_Z(\omega)\, d\omega$$

where E[..] denotes mathematical expectation and * the complex conjugate. Its spectral band has a finite support Δ included in Δ_l defined by:

$$\Delta_l = ((-2l-1)\pi, (-2l+1)\pi) \cup ((2l-1)\pi, (2l+1)\pi)$$

where $l \in \mathbb{N}$, $l = 0$ (respectively $l \neq 0$) corresponds to a low-pass (respectively band-pass) signal [10], [11].

1.2. Sampling scheme and reconstruction formulas

For band-pass signals ($l \neq 0$), let consider the PNS2 sampling scheme. The sampling instants belong to two interleaved periodic sequences:

$$\mathbf{t}^a = \{n+a, n \in \mathbb{Z}\} \text{ and } \mathbf{t}^b = \{n+b, n \in \mathbb{Z}\}.$$

The mean sampling rate, which equals 2 in this case, is adapted to the signal bandwidth, 4π, according to the Landau criterion [6]: whereas the Nyquist criterion considers the maximum frequency of the power spectral density, the Landau criterion considers the spectral bandwidth. Under the condition $2l(b-a) \notin \mathbb{Z}$, the signal $\mathbf{Z}$ can be exactly reconstructed, at a given time t, from two sampling sequences using the following formula [3], [4]:

$$Z(t) = \frac{A_a(t)\sin 2\pi l(t-b) + A_b(t)\sin 2\pi l(a-t)}{\sin 2\pi l(a-b)} \quad (1)$$

with:

$$A_x(t) = \sum_{n\in\mathbb{Z}} \frac{\sin\pi(t-n-x)}{\pi(t-n-x)} Z(n+x). \quad (2)$$

This formula is the sum of two series with cardinal sine functions as general terms. The reconstruction formula convergence rate is thus given by n^{-1}, equivalent to the rate of the classical Shannon reconstruction formula for uniform sampling. This low convergence rate is a drawback in practical applications where the observation window length and thus the number of available samples is limited. In this case, the signal reconstruction is an approximation or estimation $\widetilde{Z}(t)$ of $Z(t)$. Let N denote the number of available samples of one of the two sequences in the observation window. The truncated PNS2 reconstruction formula expresses as:

$$\widetilde{A}_x(t) = \sum_{n=-N/2}^{n=+N/2} \frac{\sin\pi(t-n-x)}{\pi(t-n-x)} Z(n+x). \qquad (3)$$

For simplicity, N is assumed even in the following. Since the slow decay of this series is unacceptable in many practical applications, over-sampling is introduced to allow the choice of fast decaying interpolation functions.

2. FAST CONVERGENCE RECONSTRUCTION FORMULAS

2.1. Principle: the fundamental isometry

The reconstruction formula (1) is obtained through the fundamental isometry relating the random process $Z(t)$ and the complex exponential $e^{i\omega t}$ [7]. Let $\mathbf{H}$ denote the Hilbert space generated by the random process $Z(t)$ i.e. the set of its linear combinations. The scalar product in H is defined by $\langle Z(t), Z(t')\rangle_{\mathbf{H}} = E[Z(t)Z^*(t')]$. Let H denote the Hilbert space generated by the random process $e^{i\omega t}$. The scalar product in H is defined by $\langle g,h\rangle_{\mathbf{H}} = \int g(\omega)h^*(\omega)s_Z(\omega)d\omega$. The isometry results from the following equality $E[Z(t)Z^*(t')] = \int_\Delta e^{i\omega\tau}s_Z(\omega)d\omega$. Then, from a particular decomposition of $e^{i\omega t}$ function of sampling instants $\{t_n, n\in\mathbb{N}\}$:

$$e^{i\omega t} = \sum_{n=-\infty}^{n=+\infty} \alpha_n(t)e^{i\omega t_n} \qquad (4)$$

we can obtain, through the isometry, the signal reconstruction formula:

$$Z(t) = \sum_{n=-\infty}^{n=+\infty} \alpha_n(t)Z(t_n) \qquad (5)$$

This formula is obtained by developing in Fourier series the function $e^{i\omega t}$ defined for $\omega \in (-\pi,\pi)$, extended to the real axis through 2π-periodization with respect to ω. However, this function is not continuous except for $t\in\mathbb{Z}$ and its Fourier series convergence rate is thus given by n^{-1} [12]. The reconstruction formula convergence rate can be improved if it is obtained by Fourier series development of a continuous function with possibly continuous derivatives. This is possible in the over-sampling case. The improved reconstruction formulas proposed in this paper have been derived from the transfer function of the raised cosine filter which is continuous and with continuous derivative [13]. Note that the fastest reconstruction would be obtained with infinitely differentiable compactly supported filters. However, unlike classical finite regularity filters, such as the raised cosine for instance, their impulse response has no closed form expression [1].

2.2. Raised cosine filter

Let $\mathcal{F}$ a linear time-invariant filter with impulse response $f(t)$ and transfer function $F(\omega) = \int_{-\infty}^{\infty} f(u)e^{-i\omega u}du$. The filter output $\mathbf{Z}$ is the process $\mathbf{U} = \mathcal{F}[\mathbf{Z}]$ defined by:

$$U(t) = \int_{-\infty}^{\infty} f(u)Z(t-u)du$$

The raised cosine filter $\mathcal{F}_\alpha$ with roll-off parameter $\alpha \neq \pi$ (this condition excludes the ideal low-pass filter) is defined by the transfer function [10]:

$$F_\alpha(\omega) = \begin{cases} 1 & \text{pour } \omega\in(-\alpha,\alpha) \\ \cos^2\left[\frac{\pi}{2(\pi-\alpha)}(\omega-\alpha)\right] & \text{pour } \omega\in(\alpha,\pi) \\ \cos^2\left[\frac{\pi}{2(\pi-\alpha)}(\omega+\alpha)\right] & \text{pour } \omega\in(-\pi,-\alpha) \end{cases} \qquad (6)$$

on the interval $(-\pi,\pi)$, extended by periodicity to $\mathbb{R}$ by $F_\alpha(\omega) = F_\alpha(\omega+2\pi)$. The reconstruction formulas proposed in this paper use particular properties of $F_\alpha(\omega)$ that lead to exact reconstruction with a very high convergence rate.
The first property, which guarantees exact reconstruction, is that, if $s_Z(\omega) = 0$ for $\omega\notin\Delta_{l,\alpha}$ with:

$$\Delta_{l,\alpha} = (-2l\pi-\alpha, -2l\pi+\alpha)\cup(2l\pi-\alpha, 2l\pi+\alpha) \qquad (7)$$

then

$$Z(t) = \mathcal{F}_\alpha[Z](t) \qquad (8)$$

since $F_\alpha(\omega) = 1$ on the process spectral band. The process power spectral density is not modified by the filtering. Now consider the Fourier series of $F_\alpha(\omega)e^{i\omega t}$ on the interval $(-\pi,\pi)$:

$$F_\alpha(\omega)e^{i\omega t} = \sum_{n\in\mathbb{Z}} a_n(t)e^{in\omega}, \omega\in(-\pi,\pi). \qquad (9)$$

The functions $a_n(t)$ are defined by:

$$a_n(t) = \frac{-\beta^2}{2\pi}\frac{\sin\alpha(t-n)+\sin\pi(t-n)}{(t-n)\left[(t-n)^2-\beta^2\right]} \text{ with } \beta = \frac{\pi}{\pi-\alpha}. \qquad (10)$$

Note that the equation (9) is valid only for $\omega\in(-\pi,\pi)$, except in the case where t belongs to $\mathbb{Z}$. The equality $F_\alpha(\omega) = 1$ over $\Delta_{0,\alpha}\subset(-\pi,\pi)$ is equivalent to:

$$\int_{-\infty}^{\infty}\left|e^{i\omega t} - \sum_{n\in\mathbb{Z}} a_n(t)e^{in\omega}\right|^2 s_Z(\omega)d\omega = 0.$$

This equality allows to establish the reconstruction formula given in the next paragraph for low-pass and band-pass signals.
The second property which guarantees the high convergence rate, is that the derivative of $F_\alpha(\omega)e^{i\omega t}$ is continuous over the interval $(-\pi,\pi)$. Consequently, this function Fourier series expansion convergence rate is given by n^{-3} [12]. This general property can be easily verified from (10): the general term $a_n(t)$ effectively decreases in n^{-3} when n goes to infinity.

2.3. Exact fast convergence reconstruction formula

For a low-pass signal, after uniform sampling, under the condition that $s_Z(\omega) = 0, \omega\notin\Delta_{0,\alpha}$, the following reconstruction formula applies :

$$Z(t) = \sum_{n\in\mathbb{Z}} a_n(t)Z(n). \qquad (11)$$

where the $a_n(t)$ are defined in (10).
For a band-pass signal, the reconstruction uses a decomposition of $\mathbf{Z}$ into two terms, one corresponding to the positive frequencies and the other to the negative frequencies:

$$\mathbf{Z} = \mathbf{Z}_+ + \mathbf{Z}_-$$

with

$$\begin{cases} s_{Z_+}(\omega) = 0, & \omega \notin (2l\pi - \alpha, 2l\pi + \alpha) \\ s_{Z_-}(\omega) = 0, & \omega \notin (-2l\pi - \alpha, -2l\pi + \alpha) \end{cases}$$

The two processes $\mathbf{Z}_+$ and $\mathbf{Z}_-$ have spectral bands with length smaller than 2π and can be reconstructed from one sampling sequence $\mathbf{t}_a$ or $\mathbf{t}_b$ using the isometry and the following relation:

$$e^{i\omega t} = e^{2i\pi\varepsilon l(t-a)} \sum_{n\in\mathbb{Z}} a_n(t-a)\, e^{i(n+a)\omega} \tag{12}$$

for $\omega \in (-\alpha + 2\pi\varepsilon l, +\alpha + 2\pi\varepsilon l)$ with $\varepsilon = \pm 1$. For instance, the reconstruction formula of Z_- from the samples measured at times $\mathbf{t}_b$ is given by:

$$Z_-(t) = e^{-2i\pi l(t-b)} \sum_{n\in\mathbb{Z}} a_n(t-b)\, Z_-(n+b)\,.$$

The reconstruction of Z_- and Z_+ using each of the sampling sequence $\mathbf{t}_a$ and $\mathbf{t}_b$ allow to build the following system of equations:

$$\begin{cases} Z_+(t)\, e^{-2i\pi l(t-a)} + Z_-(t)\, e^{2i\pi l(t-a)} = A_a(t) \\ Z_+(t)\, e^{-2i\pi l(t-b)} + Z_-(t)\, e^{2i\pi l(t-b)} = A_b(t) \end{cases}$$

with:

$$A_x(t) = \sum_{n\in\mathbb{Z}} a_n(t-x)\, Z(n+x)$$

This system resolution leads to the following reconstruction formula:

$$Z(t) = \frac{A_a(t)\sin 2\pi l(t-b) + A_b(t)\sin 2\pi l(a-t)}{\sin 2\pi l(a-b)}$$

This formula is valid for the low-pass signals such that $s_Z(\omega) = 0, \omega \notin \Delta_{l,\alpha}, l \neq 0$. For low-pass and band-pass signals, the convergence rate is given by n^{-3}.

3. PERFORMANCE ANALYSIS

In a first step, the proposed formulas have been tested on a random band-pass signal ($l = 3$, $\alpha = 0.251 * \pi$) which is the sum of a sine wave with random phase (uniformly distributed over $[0, 2\pi]$) and a white noise in the considered band. The power spectral density of the original signal is presented on figure 1. This signal is then sampled according to the interleaved sequences $\mathbf{t}_a$ and $\mathbf{t}_b$ with $a = 0.1$ and $b = 0.777$. Figure 2 displays the two sampling sequences and the reconstruction with the classical and improved formula in the case of a finite window with size $N = 10$. Figure 3 displays the reconstruction error for both formulas. According to Figure 2, the signal is exactly reconstructed at the available sample locations. However, the reconstruction is not exact on the entire window. In particular, side effects can be observed as shown by figure 3. The reconstruction is obviously better at the window center: the signal is then reconstructed with an equivalent number of samples on each side. In practice, the reconstruction is performed on a sliding window and the signal reconstruction is performed at the center. In the following, the performance of the classical and improved method are compared at the center of the window corresponding to $t = 0$. The performance are measured through the estimate of the mean square reconstruction error at this particular instant. The figure 4 shows the influence of the window size N: as expected, the convergence is faster with the proposed formula. The mean square reconstruction error displayed on figures 4 and 5 is estimated from $n_{it} = 1000$ signal runs. Figure 5 shows the influence of the over-sampling parameter α. When the signal is over-sampled, the proposed method should be preferred to the classical one.

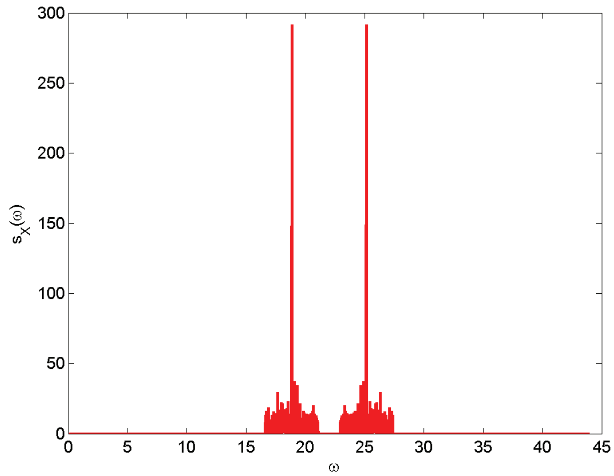

Fig. 1. Power spectral density of the original signal

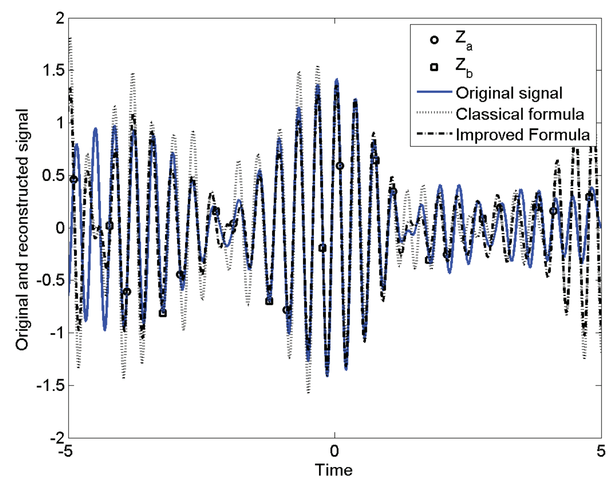

Fig. 2. Signal reconstruction

4. CONCLUSION

This paper studies the reconstruction of random process from periodic non-uniform sampling of order 2. An improved reconstruction formula in terms of convergence rate is proposed in the over-sampled

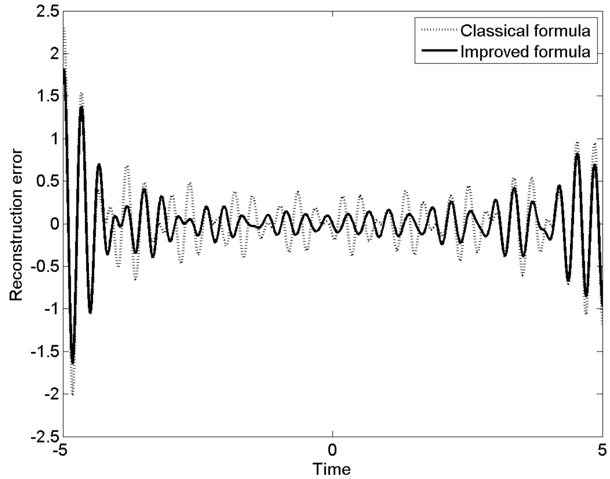

Fig. 3. Reconstruction error

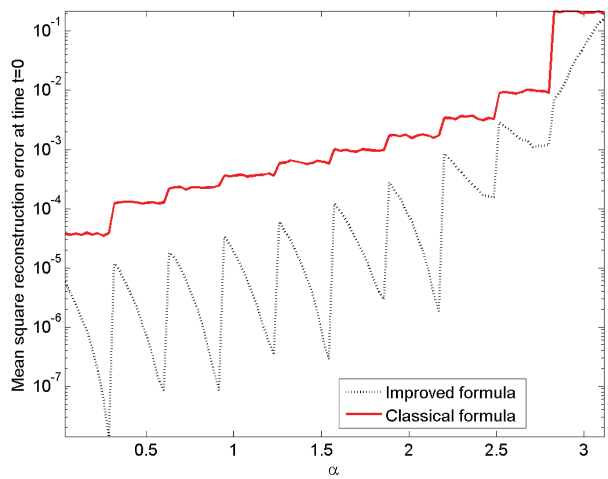

Fig. 5. Influence of the over-sampling factor

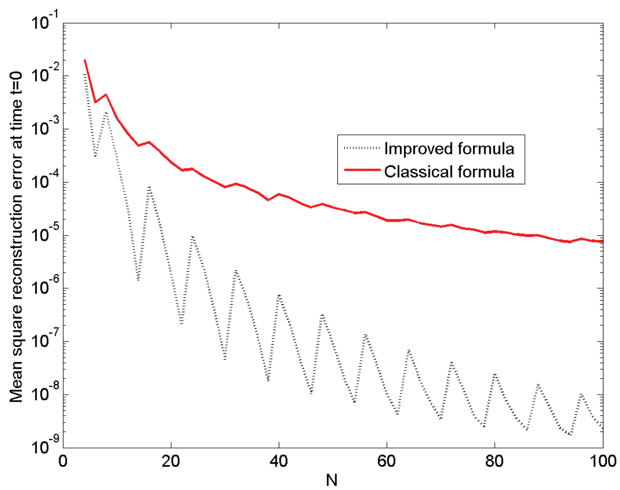

Fig. 4. Influence of the number of samples

case. This formula involves a raised cosine filter as the interpolation function. The reconstruction formula proposed in this paper leads to a faster convergence than the classical one, as shown through theoretical and numerical performance analysis. In particular, the convergence rate is studied as function of the signal spectral bandwidth. Note that using this principle, reconstruction formulas with even larger convergence rate can be designed from interpolators with an increasing number of continuous derivatives. In the case of fast communication systems, the main advantage of these reconstruction formulas is their explicit and relatively simple expression in the time domain. Consequently, the reconstruction is straightforward without transition to the frequency domain through Fourier transform and inverse Fourier transform.

5. REFERENCES

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