SET-MEMBERSHIP RECURSIVE LEAST-SQUARES ADAPTIVE FILTERING ALGORITHM

Reza Arablouei[*] *and Kutluyıl Doğançay*[†*]
[*]Institute for Telecommunications Research
[†]School of Electrical and Information Engineering
University of South Australia, Mawson Lakes SA 5095, Australia

ABSTRACT

A new set-membership adaptive filtering algorithm is developed based on the exponentially-weighted RLS algorithm with a time-varying forgetting factor that is optimized at each iteration by imposing a bounded-magnitude constraint on the *a posteriori* filter output error. The new algorithm is designed to improve the numerical behavior of the previously proposed BEACON algorithm while delivering the same convergence and tracking performance as the BEACON algorithm. Simulation results for a flat-fading MIMO channel estimation application demonstrate the superiority of the new algorithm over the BEACON algorithm in terms of numerical stability.

***Index Terms*—**set-membership adaptive filtering, RLS algorithm, BEACON algorithm, numerical stability, MIMO channel estimation

1. INTRODUCTION

Set-membership (SM) filtering algorithms are set-theoretic estimation methods that unlike the traditional methods, e.g. minimum mean square error (MMSE) or least-squares error (LSE) filters, estimate sets of feasible solutions rather than single-point solutions. The SM approaches are of particular interest in signal processing applications because they feature two major advantages over their traditional counterparts. First, they exhibit superior adaptation and tracking properties. Second, they can effectively make use of innovation in the data and improve computational efficiency by establishing a data-discerning update strategy for the parameter estimates. More specifically, unlike the traditional estimation schemes that implement a continuous update process regardless of the usefulness of the data, the SM algorithms assess the potential of the new data to improve the quality of the estimate and weigh the data accordingly. This intelligent update strategy results in discarding the data with unhelpful information content and obviating the expense of updating when the data are redundant. A more detailed and in-depth background on the SM filtering paradigm can be found in [1]-[3] and the references therein.

An SM filtering algorithm is typically formulated as a set estimation problem and seeks solutions for a case that a certain constraining assumption is made about the filter output error. A usual assumption is a bounded magnitude for the filter output error. Several techniques have been proposed to estimate the target set of solutions, called membership set, under the bounded error constraint. The most prominent ones are the optimal bounding ellipsoid (OBE) algorithms that approximate the membership set by tightly outer-bounding it with ellipsoids in the parameter space and optimize the size of the ellipsoid in some meaningful sense. Different optimality criteria have led to different OBE algorithms. The first OBE algorithm was introduced in [4]. A thorough review of numerous further works developing the other members of the OBE family can be found in [1].

Among all the OBE algorithms, the Bounding Ellipsoidal Adaptive CONstraind least-squares (BEACON) algorithm [5] is particularly attractive since it shares many of the desirable features of the various OBE algorithms. Furthermore, it incorporates simple but efficient innovation check and optimal weight calculation processes, which make it computationally more efficient than other OBE algorithms.

In this paper, we develop a new SM adaptive filtering algorithm based on the exponentially-weighted recursive least-squares (EWRLS) algorithm with a time-varying forgetting factor that is optimized within the framework of the SM filtering. In this sense, the proposed algorithm differs from the OBE algorithms, which are based on the weighted recursive least-squares (WRLS) algorithm with a sequence of weights that does not have the functionality of a sequence of forgetting factors. The proposed algorithm enjoys an appreciably improved numerical behavior compared to the BEACON algorithm while having the same complexity and convergence performance as BEACON.

2. SET-MEMBERSHIP ADAPTIVE FILTERING

Let us consider the affine-in-parameter model

$$d(n) = \boldsymbol{\omega}^* \mathbf{x}(n) + v(n) \tag{1}$$

This work was supported in part by a *Commonwealth Scientific and Industrial Research Organisation (CSIRO)* scholarship.

where $d(n) \in \mathbb{C}$ is the reference signal at time index n, $\boldsymbol{\omega} \in \mathbb{C}^L$ is the column vector of the unknown system parameters, $\mathbf{x}(n) \in \mathbb{C}^L$ is the input vector, $v(n) \in \mathbb{C}$ accounts for measurement noise, $\mathbb{C}$ denotes the set of complex numbers and superscript $*$ stands for complex-conjugate transposition.

Constraining the magnitude of the output estimation error, $e(n) = d(n) - \mathbf{w}^*\mathbf{x}(n)$, to be smaller than a pre-determined threshold γ yields a specification on $\mathbf{w}$, which is an estimate of $\boldsymbol{\omega}$. Consequently, there will be a set of feasible solutions for $\mathbf{w}$ rather than a single estimate. The set of all filter vectors $\mathbf{w}$ satisfying the error constraint for all possible input-desired output pairs in the model space $\mathcal{S}$ is called the *feasibility set* and is defines as

$$\Theta = \bigcap_{(\mathbf{x},d)\in\mathcal{S}} \{\mathbf{w} \in \mathbb{C}^L : |d - \mathbf{w}^*\mathbf{x}| \leq \gamma\}. \tag{2}$$

Direct calculation of Θ is formidable and computationally prohibitive. Hence, the adaptive SM filtering algorithms seek solutions that belong to a *membership set* $\Psi(n)$, which is a superset of Θ and is devised to be the minimal set estimate for Θ at time instant n. The membership set is defined by

$$\Psi(n) = \bigcap_{i=1}^{n} \mathcal{H}(i) \tag{3}$$

where $\mathcal{H}(n)$ is the *constraint set* that contains all vectors $\mathbf{w}$ satisfying the error bound at time instant n:

$$\mathcal{H}(n) = \{\mathbf{w} \in \mathbb{C}^L : |d(n) - \mathbf{w}^*\mathbf{x}(n)| \leq \gamma\}. \tag{4}$$

The membership set $\Psi(n)$ is an L-dimensional convex polytope and still not easy to compute. Therefore, the OBE algorithms estimate a sequence of ellipsoids instead that tightly outer-bound $\Psi(n)$.

3. THE BEACON ALGORITHM

Although BEACON is built upon the OBE concept, it can be regarded as a WRLS algorithm with a time-varying weighting factor, $\ell(n)$, where the input autocorrelation matrix, $\boldsymbol{\mathcal{R}}(n)$, and the filter coefficients are updated via

$$\boldsymbol{\mathcal{R}}(n) = \boldsymbol{\mathcal{R}}(n-1) + \ell(n)\mathbf{x}(n)\mathbf{x}^*(n), \tag{5}$$

$$\mathbf{w}(n) = \mathbf{w}(n-1) + \ell(n)\boldsymbol{\mathcal{P}}(n)\mathbf{x}(n)e^*(n) \tag{6}$$

with the *a priori* estimation error being defined by

$$e(n) = d(n) - \mathbf{w}^*(n-1)\mathbf{x}(n). \tag{7}$$

In practice, $\boldsymbol{\mathcal{P}}(n) = \boldsymbol{\mathcal{R}}^{-1}(n)$ is updated rather than $\boldsymbol{\mathcal{R}}(n)$:

$$\boldsymbol{\mathcal{P}}(n) = \boldsymbol{\mathcal{P}}(n-1) - \frac{\ell(n)\boldsymbol{\mathcal{P}}(n-1)\mathbf{x}(n)\mathbf{x}^*(n)\boldsymbol{\mathcal{P}}(n-1)}{1 + \ell(n)\mathbf{x}^*(n)\boldsymbol{\mathcal{P}}(n-1)\mathbf{x}(n)}. \tag{8}$$

At time instant n, if $|e(n)| \leq \gamma$, it is interpreted that $\mathbf{w}(n-1)$ is inside the constraint set $\mathcal{H}(n)$ so there is no need to update it, i.e. $\mathbf{w}(n) = \mathbf{w}(n-1)$. Conversely, $|e(n)| > \gamma$ means that $\mathbf{w}(n-1)$ is outside $\mathcal{H}(n)$ and needs to be updated to a new vector $\mathbf{w}(n)$ that lies inside $\mathcal{H}(n)$. In this case, an update is carried out via (8) and (6) while the optimum value for the weighting factor $\ell(n)$ is found by satisfying the bounded-error-magnitude constraint

$$|d(n) - \mathbf{w}^*(n)\mathbf{x}(n)| = \gamma \tag{9}$$

which ensures that $\mathbf{w}(n)$ is a member of $\mathcal{H}(n)$ and consequently $\Psi(n)$. Multiplying both sides of (6) by $\mathbf{x}^*(n)$ and subtracting from $d^*(n)$ yields

$$d(n) - \mathbf{w}^*(n)\mathbf{x}(n) = \frac{1}{1 + \ell(n)\mathcal{G}(n)} e(n) \tag{10}$$

where

$$\mathcal{G}(n) = \mathbf{x}^*(n)\boldsymbol{\mathcal{P}}(n-1)\mathbf{x}(n). \tag{11}$$

Thus, by equating the RHSs of (9) and (10), $\ell(n)$ is found as

$$\ell(n) = \frac{1}{\mathcal{G}(n)}\left(\frac{|e(n)|}{\gamma} - 1\right). \tag{12}$$

The BEACON algorithm is summarized in Table I. In this algorithm, the norm of $\boldsymbol{\mathcal{R}}(n)$ grows in time constantly because of the accumulative term on the RHS of (5). Growth of $\boldsymbol{\mathcal{R}}(n-1)$ decreases $\mathcal{G}(n)$ and so increases $\ell(n)$ while a larger $\ell(n)$ can in turn accelerate the growth of $\boldsymbol{\mathcal{R}}(n)$. This positive feedback mechanism can typically push the internal parameters, $\boldsymbol{\mathcal{P}}(n)$ and $\ell(n)$, out of the realizable ranges in finite-precision implementations. The consequent overflow/underflow of the parameter values can eventually result in cessation of the adaptation since $\lim_{n\to\infty} \boldsymbol{\mathcal{P}}(n) = \mathbf{0}$.

4. THE SET-MEMBERSHIP RLS ALGORITHM

Let us define

$$\mathbf{R}(n) = \ell^{-1}(n)\boldsymbol{\mathcal{R}}(n), \tag{13}$$

$$\mathbf{P}(n) = \mathbf{R}^{-1}(n), \tag{14}$$

and

$$\lambda(n) = \frac{\ell(n-1)}{\ell(n)}. \tag{15}$$

Multiplying both sides of (5) with $\ell^{-1}(n)$ and using the above definitions, we have

$$\mathbf{R}(n) = \lambda(n)\mathbf{R}(n-1) + \mathbf{x}(n)\mathbf{x}^*(n), \tag{16}$$

$$\mathbf{w}(n) = \mathbf{w}(n-1) + \mathbf{P}(n)\mathbf{x}(n)e^*(n). \tag{17}$$

Applying matrix inversion lemma to (16) yields

$$\mathbf{P}(n) = \lambda^{-1}(n)\left(\mathbf{P}(n-1) - \frac{\mathbf{P}(n-1)\mathbf{x}(n)\mathbf{x}^*(n)\mathbf{P}(n-1)}{\lambda(n) + \mathbf{x}^*(n)\mathbf{P}(n-1)\mathbf{x}(n)}\right). \tag{18}$$

Table I, The BEACON algorithm.

- Initialization:
  - $\boldsymbol{\mathcal{P}}(0) = \delta\mathbf{I}$ where δ is a small positive number and $\mathbf{I}$ is the identity matrix
  - $\mathbf{w}(0) = \mathbf{0}$
- At iteration n:
  - $e(n) = d(n) - \mathbf{w}^*(n-1)\mathbf{x}(n)$
  - if $|e(n)| > \gamma$
    - $\mathcal{G}(n) = \mathbf{x}^*(n)\boldsymbol{\mathcal{P}}(n-1)\mathbf{x}(n)$
    - $\ell(n) = \frac{1}{\mathcal{G}(n)}\left(\frac{|e(n)|}{\gamma} - 1\right)$
    - $\boldsymbol{\mathcal{P}}(n) = \boldsymbol{\mathcal{P}}(n-1) - \frac{\ell(n)\boldsymbol{\mathcal{P}}(n-1)\mathbf{x}(n)\mathbf{x}^*(n)\boldsymbol{\mathcal{P}}(n-1)}{1+\ell(n)\mathbf{x}^*(n)\boldsymbol{\mathcal{P}}(n-1)\mathbf{x}(n)}$
    - $\mathbf{w}(n) = \mathbf{w}(n-1) + \ell(n)\boldsymbol{\mathcal{P}}(n)\mathbf{x}(n)e^*(n)$
  - otherwise
    - $\boldsymbol{\mathcal{P}}(n) = \boldsymbol{\mathcal{P}}(n-1)$
    - $\mathbf{w}(n) = \mathbf{w}(n-1)$

Table II, The set-membership RLS algorithm.

- Initialization:
  - $\mathbf{P}(0) = \delta\mathbf{I}$
  - $\mathbf{w}(0) = \mathbf{0}$
- At iteration n:
  - $e(n) = d(n) - \mathbf{w}^*(n-1)\mathbf{x}(n)$
  - if $|e(n)| > \gamma$
    - $G(n) = \mathbf{x}^*(n)\mathbf{P}(n-1)\mathbf{x}(n)$
    - $\lambda(n) = \frac{G(n)}{\left(\frac{|e(n)|}{\gamma} - 1\right)}$
    - $\mathbf{P}(n) = \lambda^{-1}(n)\left(\mathbf{P}(n-1) - \frac{\mathbf{P}(n-1)\mathbf{x}(n)\mathbf{x}^*(n)\mathbf{P}(n-1)}{\lambda(n)+\mathbf{x}^*(n)\mathbf{P}(n-1)\mathbf{x}(n)}\right)$
    - $\mathbf{w}(n) = \mathbf{w}(n-1) + \mathbf{P}(n)\mathbf{x}(n)e^*(n)$
  - otherwise
    - $\mathbf{P}(n) = \mathbf{P}(n-1)$
    - $\mathbf{w}(n) = \mathbf{w}(n-1)$

Substituting (12) into (15) gives

$$\lambda(n) = \ell(n-1)\frac{\mathcal{G}(n)}{\left(\frac{|e(n)|}{\gamma} - 1\right)}. \quad (19)$$

By rewriting (11) as

$$\mathcal{G}(n) = \mathbf{x}^*(n)\ell^{-1}(n-1)\mathbf{P}(n-1)\mathbf{x}(n), \quad (20)$$

we can express (19) as

$$\lambda(n) = \frac{G(n)}{\left(\frac{|e(n)|}{\gamma} - 1\right)}. \quad (21)$$

where $G(n) = \mathbf{x}^*(n)\mathbf{P}(n-1)\mathbf{x}(n)$.

The resultant new algorithm is summarized in Table II. We call this algorithm *set-membership recursive least-squares* (SM-RLS) algorithm. The reason for choosing this name is that the new algorithm is in fact an EWRLS algorithm with a time-varying forgetting factor where the forgetting factor is optimized to satisfy the set-membership-induced error bound, (9). Multiplying both sides of (17) by $\mathbf{x}^*(n)$ and subtracting from $d^*(n)$ gives

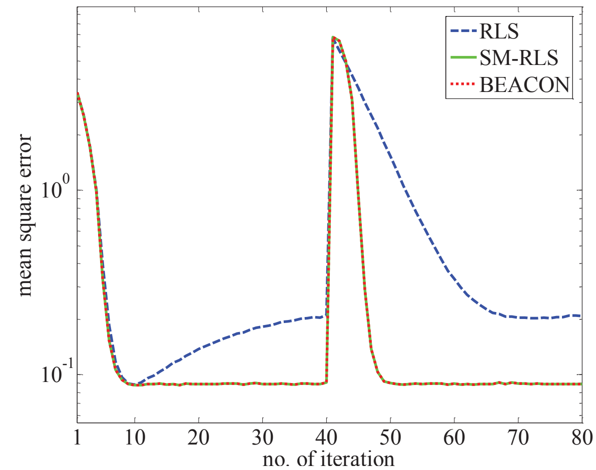

Fig. 1. Mean square error performance of different algorithms.

$$d(n) - \mathbf{w}^*(n)\mathbf{x}(n) = \frac{1}{1+\lambda^{-1}(n)G(n)}e(n). \quad (22)$$

Therefore, we can alternatively obtain (21) by equating the RHSs of (22) and (9) and solving it with respect to $\lambda(n)$.

The BEACON and SM-RLS algorithms calculate the same filter coefficients though with a major difference in the way that they carry out the coefficient update process. In BEACON, $\ell(n)$ is a weighting factor, whereas in SM-RLS, $\lambda(n)$ acts as a forgetting factor. As a result, in SM-RLS, increase of $\mathbf{R}(n-1)$ decreases $G(n)$ and hence decreases $\lambda(n)$, whereas a smaller $\lambda(n)$ leads to a smaller $\mathbf{R}(n)$. Unlike in BEACON, this negative feedback mechanism helps SM-RLS maintain its numerical stability.

5. SIMULATIONS

In this section, we compare performance of the SM-RLS algorithm with the conventional RLS algorithm and the BEACON algorithm for an application of flat-fading MIMO channel estimation studied in [6]. For this purpose, a MIMO communication system with four transmitter and four receiver antennas is considered. The sub-channels between all the transmitter and receiver pairs are independent Rayleigh fading and vary in time based on Jakes model [7] with a normalized Doppler frequency $f_D T_s = 0.01$ where f_D is the maximum Doppler frequency shift and T_s is the transmission symbol period. A sudden random change in the channel taps is also introduced halfway through the simulations. Four FIR filters each having four taps constitute the MIMO channel estimator. Similar to [6], in the SM-RLS and BEACON algorithms, the norm of the error vector composed by the errors of all the filters is used for the considered MIMO case in place of the absolute of the scalar error in the SISO case. The transmitted signal is uncoded and modulated using QPSK scheme. It is grouped

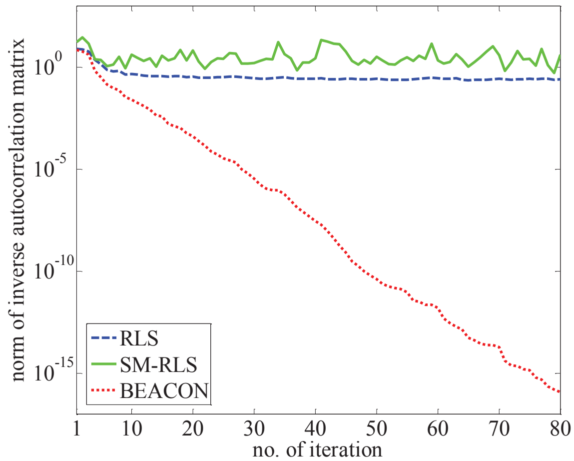

Fig. 2. The Frobenius norm of the inverse autocorrelation matrix versus time for different algorithms.

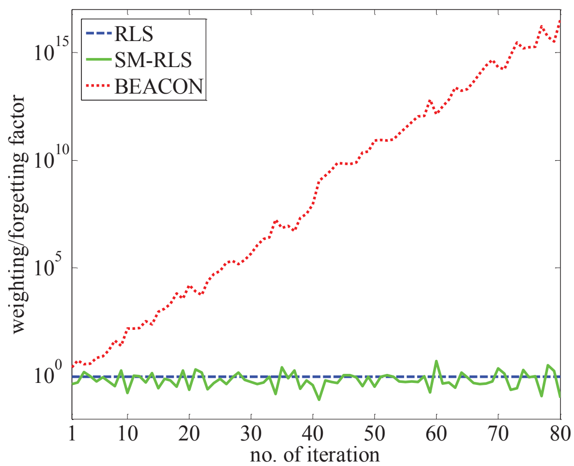

Fig. 3. Time evolution of the optimal weight of the BEACON algorithm and the optimal forgetting factor of the SM-RLS algorithm together with the fixed forgetting factor of the RLS algorithm.

in packets of data each containing 80 vectors of transmitted symbols and these vectors are the common input to the filters of the MIMO channel estimator. For the RLS algorithm, a fixed forgetting factor of 0.9 is used regarding the assumed normalized Doppler frequency. For the SM-RLS and BEACON algorithms, the error threshold is set to $\gamma = 0.1$. The energy per bit to noise power spectral density ratio is also $E_b/N_0 = 12$ dB.

Fig. 1 compares mean square error (MSE) performance of different algorithms when ensemble-averaged over 10^4 independent runs. As expected, Fig. 1 clearly shows that the SM-RLS and BEACON algorithms perform similarly. It should be noted that both the SM algorithms updated in average at 80 percent of the iterations in this experiment.

Fig. 2 shows the Frobenius norm of the inverse autocorrelation matrix versus time for different algorithms. Fig. 3 shows time evolution of the optimal weighting factor of the BEACON algorithm and the optimal forgetting factor of the SM-RLS algorithm together with the fixed forgetting factor of the RLS algorithm. The important observation from Figs. 2 and 3 is that, as predicted, the BEACON algorithm is prone to numerical problems, in particular, when its update frequency is high. It is evident from Figs. 2 and 3 that BEACON's internal parameters, $\boldsymbol{\mathcal{P}}(n)$ and $\ell(n)$, feature monotonic exponential increase/decrease in time. This makes their dynamic range extremely wide and consequently BEACON's practicable run-time very limited. The experiment presented here shows that BEACON's optimal weight can grow from 1 to 10^{16} in only 80 iterations. On the other hand, Figs. 2 and 3 show that the internal parameters of SM-RLS fluctuate around their steady-state values and are of much smaller dynamic range rendering SM-RLS more suitable than BEACON for practical applications.

6. CONCLUSION

A new set-membership adaptive filtering algorithm, called SM-RLS, was proposed. It was developed to improve numerical behavior of the previously proposed BEACON algorithm. Unlike the OBE algorithms that are known as set-membership weighted RLS algorithms, the proposed algorithm is a set-membership exponentially-weighted RLS algorithm with a time-varying forgetting factor that is optimized within the set-membership adaptive filtering context. Similar to BEACON, the proposed algorithm exhibits a remarkable convergence and tracking performance; however, it provides a dramatically improved numerical behavior in comparison with BEACON.

7. REFERENCES

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