# ROBUST SOURCE NUMBER ENUMERATION FOR R-DIMENSIONAL ARRAYS IN CASE OF BRIEF SENSOR FAILURES

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## ABSTRACT

There has been much activity on model selection for multi-dimensional data in recent years under the assumption of a Gaussian noise distribution. However, methods which are optimal for Gaussian noise are very sensitive against brief sensor failures. We suggest two robust model order selection schemes for multi-dimensional data based on the MM-estimator of the covariance of the $r$-mode unfoldings of the complex valued data tensor. Simulation results are given for 2-D and 3-D uniform rectangular arrays based source enumeration, both for Gaussian noise and a brief sensor failure.

***Index Terms***— robust model selection, tensors, source number enumeration, robust covariance matrix estimation

## 1. INTRODUCTION

Estimating the number of signal components impinging on a sensor array is a crucial step in various signal processing tasks, such as direction of arrival estimation, source separation and Doppler frequency estimation. There has been much research on matrix based array signal processing techniques during the last decades which consider various signal and noise constellations, depending on the application at hand. Various model order selection criteria have been developed and shown to be optimal in some sense, e.g., efficient or consistent under given assumptions, such as the Gaussian distribution of the noise.

Recently, there has been an increased interest in multi-dimensional array signal processing, which is advantageous, since problems are seen from multiple perspectives. Multiple dimensionality can refer to spatial dimensions, e.g., 2-dimensional or 3-dimensional arrays, but also to combinations of several dimensions like space, time, frequency, and polarization. A further important advantage of using multi-dimensional data lies in the identifiability due to the higher rank of the multi-dimensional data.

Recent publications have extended classic model order selection criteria [1] to the multi-dimensional case by using tensor notation [2]. It has been shown that taking into account the multi-dimensional structure of the data improves the estimation of the model order. Not much attention has been payed, however, how these criteria perform for small departures from the assumptions. A practical case which we investigate is what happens when a sensor fails for a short period of time. One could assume that the above techniques can deal with this problem, especially when the number of dimensions and correspondingly sensors, increases.

In this work, we show that even a single sensor failure during one snapshot can cause classical matrix and tensor based model order selection criteria to break down, i.e., the ability to estimate the correct number of sources impinging onto the array decreases drastically. We also give robust extensions of classical criteria, for $R$-dimensional arrays. Robustness here refers to statistical robustness [5], i.e., the methods we suggest are nearly optimal for the Gaussian case and are not much affected when a fraction of the data is corrupted, e.g., by a faulty sensor. Simulation results are given to illustrate the performance of our method for 2-D and 3-D arrays.

The paper is organized as follows: Section 1 gives an introduction and provides a motivation for robust source number enumeration for $R$-dimensional arrays. Section 2 describes source number enumeration for $R$-D arrays. Section 3 introduces robust source number enumeration for $R$-D arrays and two examples are given. Section 4 provides simulation results for the two examples for Gaussian noise and brief sensor failure cases. Section 5 concludes the paper with some remarks on future work.

## 2. SOURCE NUMBER ENUMERATION FOR R-D ARRAYS

Recently, multi-dimensional versions of many classical model order selection schemes have been derived. For a comprehensive overview, the reader is referred to [2]. In this paper we treat only the $R$-dimensional extensions of Akaike's Information Criterion and the Minimum Description Length, which are denoted as $R$-D AIC and $R$-D MDL, respectively. Both

AIC and MDL are based on the structure of the signal and noise subspace, i.e., the eigenvalues. The $R$-D extension consists in replacing the eigenvalues in the classical criteria by the global eigenvalues, i.e., the $R$-D subspace and adjusting the free number of parameters in the penalty terms.

Following the notation of [2, 3], in the remainder of the paper, scalars are denoted as italic letters, column vectors as lower-case bold-face letters, matrices as bold-face capital letters and tensors are written as bold-face calligraphic letters. The $(i,j)$-th element of the matrix $\mathbf{A}$ is denoted as $a_{i,j}$ and the $(i,j,k)$-th element of a third order tensor $\boldsymbol{\mathcal{A}}$ as $a_{i,j,k}$. The superscripts ${}^T$, ${}^H$, ${}^{-1}$ and ${}^*$ denote transposition, Hermitian transposition, matrix inversion, and complex conjugation, respectively. Furthermore, $r$-mode vectors of a tensor are obtained by keeping all indices fixed except for the $r$-th index which is varied within its range. The $r$-mode unfolding of a tensor $\boldsymbol{\mathcal{A}} \in \mathbb{C}^{I_1 \times I_2 \times \ldots \times I_R}$ is denoted by $[\boldsymbol{\mathcal{A}}]_{(r)} \in \mathbb{C}^{I_r \times (I_1 \ldots I_{r-1} I_{r+1} \ldots I_R)}$. The $r$-mode unfolding is therefore nothing else than a matrix containing the $r$-mode vectors of the tensor. The $r$-mode product of a tensor $\boldsymbol{\mathcal{A}}$ and matrix $\mathbf{U} \in \mathbb{C}^{J_r \times I_r}$ is denoted as $\boldsymbol{\mathcal{A}} \times_r \mathbf{U} \in \mathbb{C}^{I_1 \times I_2 \ldots \times J_r \ldots \times I_R}$. It is obtained by multiplying all $r$-mode vectors of $\boldsymbol{\mathcal{A}}$ from the left-hand side by the matrix $\mathbf{U}$.

The $R$-D AIC chooses the model order $\hat{d}$ as the value $k \in \{1, \ldots, K\}$, which minimizes

$$\phi_{\text{AIC}}(k) = -N(\alpha^{(G)} - p) \log\left(\frac{g^{(G)}(p)}{a^{(G)}(p)}\right) + p(2\alpha^{(G)} - p). \quad (1)$$

Here, $\alpha^{(G)}$ is the total number of sequentially defined eigenvalues, $a^{(G)}(p)$ and $g^{(G)}(p)$ are the arithmetic and geometric means of the smallest $p = K - k$ global eigenvalues, which are given by the product of the eigenvalues computed for every $r$-mode of the tensor [2]. Similarly, the $R$-D MDL criterion is given by

$$\phi_{\text{MDL}}(k) = -N(\alpha^{(G)} - p) \log\left(\frac{g^{(G)}(p)}{a^{(G)}(p)}\right) + \frac{1}{2} p(2\alpha^{(G)} - p) \log(N) \quad (2)$$

and only differs in the penalty term. The $r$-mode eigenvalues are estimated by use of the sample covariance matrix of the $r$-mode unfolding of the data tensor $\boldsymbol{\mathcal{X}}$, i.e.,

$$\hat{\mathbf{R}}_{XX}^{(r)} = \frac{M_r}{\prod_{i=1}^{R} M_i} [\boldsymbol{\mathcal{X}}]_{(r)} [\boldsymbol{\mathcal{X}}]_{(r)}^H \in \mathbb{C}^{M_r \times M_r}. \quad (3)$$

## 3. ROBUST SOURCE NUMBER ENUMERATION FOR R-D ARRAYS

It is well known in the array signal processing community that the sample covariance matrix is optimal in the Maximum Likelihood sense under the Gaussian noise assumption. However for slight deviations from the Gaussian assumption it looses drastically in performance [4, 5]. This can be explained by means of the influence function which describes the bias impact of infinitesimal contamination at an arbitrary point on the estimator, standardized by the fraction of contamination [5]. In the case of the sample covariance matrix, the influence of a bad data point, e.g., generated by a sensor failure is unbounded [4]. Hence, none of the classical subspace based source enumeration methods are robust against a sensor failure and the optimality of these methods is quickly lost.

We suggest, therefore, to induce robustness into $R$-D source number enumeration by replacing the sample covariance matrix of the $r$-mode unfolding of the data tensor by a highly efficient and statistically robust estimate of the covariance matrix. One such estimator is the multivariate MM-estimator of the covariance matrix [6, 7]. The principle of the estimator is based on a two step procedure: $(i)$ get an initial estimate of the covariance by using a very robust estimator, e.g., an S-estimator and $(ii)$ compute the final estimate of the covariance matrix using an estimator which is efficient at the assumed model, e.g., a correctly tuned M-estimator. S-estimators are a generalization of LS-estimators, where the standard deviation is replaced by a robust scale estimate, while M-estimators are a generalization of Maximum Likelihood estimators where the log-likelihood function is replaced by a function which leaves the majority of the data unchanged, while bounding the effect of the strongly deviating data points [5].

For the MM-estimator, the influence function is bounded and smooth, which means that the bias impact of a large contamination is bounded and small changes in the data lead to small changes in the estimate. By ensuring these conditions, we can achieve nearly optimal estimates of the covariance matrix for Gaussian noise and a similar performance in case of a brief sensor failure. By applying robust covariance estimation to all $r$-mode unfoldings of the data tensor, we obtain a robust $R$-D AIC that chooses the model order $\hat{d}$ as the value $k \in \{1, \ldots, K\}$, which minimizes

$$\phi_{\text{AIC,rob}}(k) = -N(\alpha^{(G)} - p) \log\left(\frac{g_{\text{rob}}^{(G)}(p)}{a_{\text{rob}}^{(G)}(p)}\right) + p(2\alpha^{(G)} - p). \quad (4)$$

Here, $a_{\text{rob}}^{(G)}(p)$ and $g_{\text{rob}}^{(G)}(p)$ are computed analogously to $a^{(G)}(p)$ and $g^{(G)}(p)$ with the difference that the eigenvalues are estimated using the robust MM-covariance matrix estimator. Similarly, the robust $R$-D MDL is given by

$$\phi_{\text{MDL,rob}}(k) = -N(\alpha^{(G)} - p) \log\left(\frac{g_{\text{rob}}^{(G)}(p)}{a_{\text{rob}}^{(G)}(p)}\right) + \frac{1}{2} p(2\alpha^{(G)} - p) \log(N). \quad (5)$$

Since the MM-estimator was designed for real-valued data, for complex valued data, stacking of the real- and imaginary-parts of the $r$-mode unfoldings must be performed, i.e.,

$$[\tilde{\boldsymbol{\mathcal{X}}}]_{(r)} = \begin{bmatrix} \text{Real}\left\{[\boldsymbol{\mathcal{X}}]_{(r)}\right\} \\ \text{Imag}\left\{[\boldsymbol{\mathcal{X}}]_{(r)}\right\} \end{bmatrix}. \quad (6)$$

By employing the MM-estimator on (6), we obtained the robust estimate of its covariance matrix $\hat{\mathbf{R}}^{(r)}_{\tilde{X}\tilde{X},\text{robust}}$ that can be expressed as

$$\hat{\mathbf{R}}^{(r)}_{\tilde{X}\tilde{X},\text{robust}} = \begin{bmatrix} \hat{\mathbf{R}}^{(r)}_{\tilde{X}\tilde{X}_A} & \hat{\mathbf{R}}^{(r)}_{\tilde{X}\tilde{X}_B} \\ \hat{\mathbf{R}}^{(r)}_{\tilde{X}\tilde{X}_C} & \hat{\mathbf{R}}^{(r)}_{\tilde{X}\tilde{X}_D} \end{bmatrix}. \quad (7)$$

Therefore, the robust estimate of the covariance matrix of the $r$-mode unfolding of the data tensor $\boldsymbol{\mathcal{X}}$ can be identified as

$$\hat{\mathbf{R}}^{(r)}_{XX,\text{robust}} = \hat{\mathbf{R}}^{(r)}_{\tilde{X}\tilde{X}_A} + \hat{\mathbf{R}}^{(r)}_{\tilde{X}\tilde{X}_D} + j(\hat{\mathbf{R}}^{(r)}_{\tilde{X}\tilde{X}_C} - \hat{\mathbf{R}}^{(r)}_{\tilde{X}\tilde{X}_B}). \quad (8)$$

### 3.1. Robust Source Number Enumeration R-D Uniform Rectangular Arrays

In the following, we give two examples which illustrate the applicability of the proposed methods. We consider the relatively simple setup of 2-D and 3-D rectangular arrays. Extensions into higher dimensions are straight-forward and other applications such as Electro-Encephalography (EEG), where the dimensions would be time, frequency, and channels can be formulated analogously.

#### *3.1.1. Example 1: Robust Source Number Enumeration for a 2-D Uniform Rectangular Array*

Consider a two-dimensional Uniform Rectangular Array (URA) of dimensions $M_1 \times M_2$ with $d$ sources impinging onto the array. The spatial frequencies for the $i$-th source for the two dimensions are represented by $\boldsymbol{\mu}_i = \left[\mu_i^{(1)}, \mu_i^{(2)}\right]^T$, $i = 1, \dots, d$. The vector $\boldsymbol{a}^{(r)}(\mu_i^{(r)})$ denotes the array response in the $r$-th dimension for the $i$-th source, where $r = 1, 2$ in this example. Let $N$ denote the number of available snapshots and $M = M_1 \cdot M_2$ be the total number of sensors. Let $\mathbf{S}$ be the complex valued source symbol matrix of dimensions $d \times N$. In the classical matrix based approach, all the spatial dimensions are stacked into column vectors. Here, we construct a measurement tensor $\boldsymbol{\mathcal{X}} \in \mathbb{C}^{M_1 \times M_2 \times N}$ as

$$\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{A}} \times_3 \mathbf{S}^T + \boldsymbol{\mathcal{W}}, \quad (9)$$

where $\boldsymbol{\mathcal{W}}$ is i.i.d. complex circular stationary noise tensor and $\boldsymbol{\mathcal{A}}$ is the array steering tensor constructed as

$$\boldsymbol{\mathcal{A}} = \left[\mathbf{A}_1 \sqcup_3 \mathbf{A}_2 \dots \sqcup_3 \mathbf{A}_d\right], \quad (10)$$

where $\sqcup_r$ represents the concatenation operation along mode $r$, and matrix $\mathbf{A}_i$ is obtained from the outer product of the array response vectors $\boldsymbol{a}^{(1)}(\mu_i^{(1)})$ and $\boldsymbol{a}^{(2)}(\mu_i^{(2)})$, $i = 1, \dots, d$. For this case, the global (robust) eigenvalue is given by the product of the three (robust) eigenvalues of the $r$-mode unfoldings of the measurement data tensor $\boldsymbol{\mathcal{X}}$ [2] .

#### *3.1.2. Example 2: Robust Source Number Enumeration for a 3-D Uniform Rectangular Array*

Similar to the 2-D case, with $M_3$ denoting the number of sensors on the third dimension of the array, we model the measurement tensor $\boldsymbol{\mathcal{X}} \in \mathbb{C}^{M_1 \times M_2 \times M_3 \times N}$ as

$$\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{A}} \times_4 \mathbf{S}^T + \boldsymbol{\mathcal{W}}, \quad (11)$$

where $\boldsymbol{\mathcal{W}}$ is i.i.d. complex circular stationary noise tensor. Note that in this 3-D example, the spatial frequencies for the $i$-th source for the three dimensions are represented by $\boldsymbol{\mu}_i = \left[\mu_i^{(1)}, \mu_i^{(2)}, \mu_i^{(3)}\right]^T$, $i = 1, \dots, d$. The vector $\boldsymbol{a}^{(r)}(\mu_i^{(r)})$ represents the array response in the $r$-th dimension for the $i$-th source, where $r = 1, 2, 3$. The array steering tensor $\boldsymbol{\mathcal{A}}$ is then constructed as

$$\boldsymbol{\mathcal{A}} = \left[\boldsymbol{\mathcal{A}}_1 \sqcup_4 \boldsymbol{\mathcal{A}}_2 \dots \sqcup_4 \boldsymbol{\mathcal{A}}_d\right], \quad (12)$$

where tensor $\boldsymbol{\mathcal{A}}_i$ is obtained from the outer product of the array response vectors $\boldsymbol{a}^{(1)}(\mu_i^{(1)})$, $\boldsymbol{a}^{(2)}(\mu_i^{(2)})$ and $\boldsymbol{a}^{(3)}(\mu_i^{(3)})$, $i = 1, \dots, d$.

## 4. SIMULATIONS

The simulation setup for Example 1 is a 2-D URA array setup with parameters as follows: $M_1 = 8$, $M_2 = 8$, $d = 3$, $N = 6$, $\boldsymbol{\mu}_1 = [-\pi/2, -\pi/4]^T$, $\boldsymbol{\mu}_2 = [-\pi/4, -\pi/2]^T$, $\boldsymbol{\mu}_3 = [0, \pi]^T$, $\mathbf{S}$ contains complex valued Gaussian source symbols. Our results are given for varying SNR are based on an average over 100 Monte Carlo runs. Figure 1 illustrates the performance of robust and classical methods for source number enumeration. It is clearly visible here that $R$-D based methods perform better than matrix based methods. Furthermore, the robust methods perform similarly to the non-robust ones.

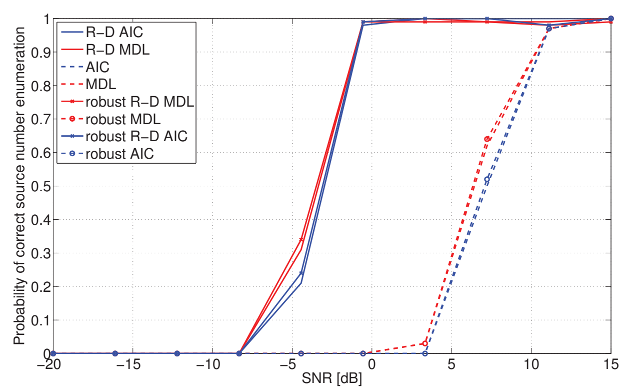


**Fig. 1**. Average probability of detecting the correct number of sources for different SNR using source enumeration with a 2-D ULA in case of Gaussian noise.

Figure 2 illustrates a scenario where we have a very short sensor failure. We simulated this by randomly replacing a single observation at a random sensor position with a complex

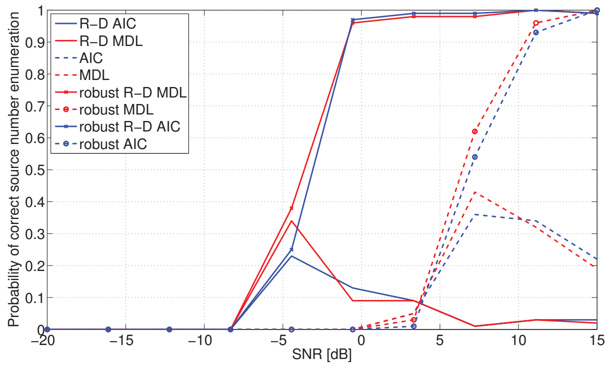


**Fig. 2**. Average probability of detecting the correct number of sources for different SNR using source enumeration with a 2-D ULA in case of brief sensor failure.

i.i.d. impulsive noise, i.e., the contaminating density is complex, zero mean Gaussian with variance equal to $\kappa\sigma^2$. In this example $\kappa = 50$, other impulsive noise types produced similar results. The second simulation setup is a 3-D scenario with the following parameters: $M_1 = 5$, $M_2 = 7$, $M_3 = 9$, $d = 3$, $N = 10$, $\boldsymbol{\mu}_1 = [-\pi/4, 0, \pi/4]^T$, $\boldsymbol{\mu}_2 = [0, \pi/4, \pi/2]^T$, $\boldsymbol{\mu}_3 = [\pi/4, \pi/2, 3\pi/4]^T$. The other parameters were chosen as in the previous example. Figure 3 shows the results

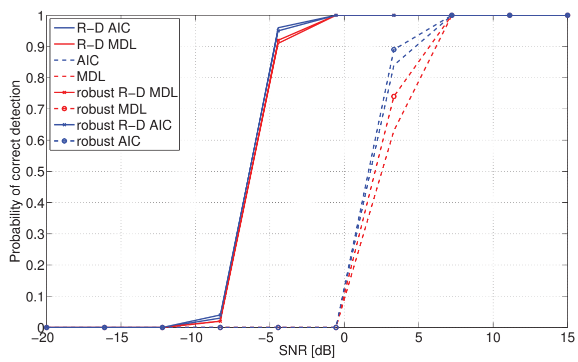


**Fig. 3**. Average probability of detecting the correct number of sources for different SNR using source enumeration with a 3-D ULA in case of Gaussian noise.

for the Gaussian noise case. Again, the $R$-D methods provide a considerable gain in detecting the correct number of sources compared to matrix based methods. Differences between non-robust and robust methods, as well as between AIC and MDL based methods are small. Figure 4 displays the effects of a brief single sensor failure. It becomes apparent that even for the increased number of sensors that are used in the 3-D case, a single failure causes the non-robust methods to break down.

## 5. CONCLUSIONS

We have investigated the problem of source number enumeration for array signal processing in the presence of brief sensor failures. For this, we have introduced robust model selection criteria for multi-dimensional array data based on the robust MM-estimate of the covariance matrix using the $r$-mode unfolding operation of the data tensor. Our method is applicable to complex as well as real-valued data. In this way, we obtain robust estimates of the $r$-mode eigenvalues which are multiplied to get global eigenvalues. These are used to robustify $R$-D model order selection criteria. The proposed criteria have shown nearly optimal performance with 2-D and 3-D uniform rectangular array settings for the Gaussian noise. In case of a brief sensor failure, they provide a similar performance while non-robust methods break down. Also the $R$-D criteria generally outperform the matrix based ones. Not much difference is observed between applying all variants of AIC and MDL. Future work will consider different non-Gaussian noise scenarios, dropping of the independence assumption and applying the proposed criteria to real EEG data.

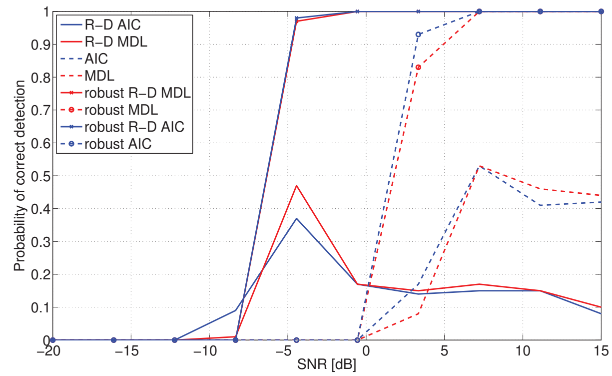


**Fig. 4**. Average probability of detecting the correct number of sources for different SNR using source enumeration with a 3-D ULA in case of a brief sensor failure.

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