QUANTIZATION REFERENCE VOLTAGE OF THE MODULATED WIDEBAND CONVERTER

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ABSTRACT

The Modulated Wideband Converter (MWC) is a recently proposed analog-to-digital converter (ADC) based on Compressive Sensing (CS) theory. Unlike conventional ADCs, its quantization reference voltage, which is important to the system performance, does not equal the maximum amplitude of original analog signal. In this paper, the quantization reference voltage of the MWC is theoretically analyzed and the conclusion demonstrates that the reference voltage is proportional to the square root of q, which is a trade-off parameter between sampling rate and number of channels. Further discussions and simulation results show that the reference voltage is proportional to the square root of Nq when the signal consists of N narrowband signals.

Index Terms— Reference voltage, Modulated Wideband Converter (MWC), Compressive Sensing (CS).

1. INTRODUCTION

In telecommunication, radar and other fields of electronic engineering, it is usually of great concern about multiband signals, which consist of several narrowband signals modulated by different carrier frequencies. To acquire such a signal, a common practical method [1] is to demodulate the narrowband signals by their carrier frequencies to baseband, respectively. After low-pass filtered to reject frequencies from other bands, the narrowband signals are converted to digital samples at rate corresponding to their actual bandwidths. Thus each band is acquired separately and the total sampling rate is the sum of the bandwidths. This method can reach the minimal sampling rate while the carrier frequencies must be known a priori.

When the carrier frequencies are unknown, efficient sampling of the multiband signal is a challenging task because the Nyquist frequency of the signal may exceed the capability of the up-to-date analog-to-digital converters (ADCs). In this scenario, several sub-Nyquist sampling strategies have been proposed to handle this challenge. In [2] and [3], a scheme called multicoset sampling is proposed to perform analog-to-digital conversion at low rate, but knowledge of the frequency support must be known for recovery. Periodic nonuniform sampling is another sub-Nyquist sampling strategy where several ADCs are applied to form a high sampling rate for the multiband signal [4].

In recent years, Compressive Sensing (CS) has been raised and developed to recover sparse signals from far fewer samples [5]. Due to the fact that multiband signal is sparse in frequency domain, several novel analog-to-digital architectures have been designed based on CS theory. In Random Demodulator [6, 7], the original signal is multiplied by a pseudorandom sequence, integrated and sampled at sub-Nyquist rate. In the recovery stage, CS algorithms are performed to recover the original signal from these samples.

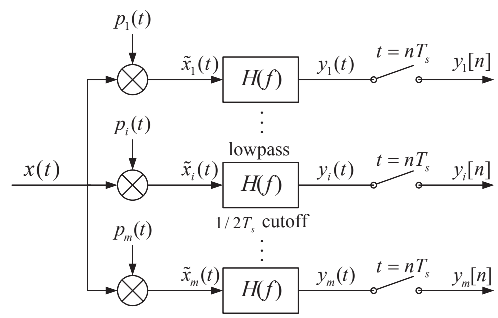

Fig. 1. A block diagram of the sampling stage of the Modulated Wideband Converter

The Modulated Wideband Converter (MWC) [8–10] is another CS-based sub-Nyquist sampling system. It also consists of two stages: sampling, as depicted in Fig. 1, and reconstruction. In the sampling stage, modulated sampling is conducted by mixers and lowpass filters in multiple channels. In the reconstruction stage, sparsity constraint algorithms are applied to recover the original signal from the multiple observations. The sampling rate f_s and the frequency f_p of the periodic mixing function $p_i(t)$ satisfies $f_s = qf_p$ with odd q, which is a strategy collapsing q channels to a single one. The strategy allows designers to trade off between sampling rate and the number of hardware devices. Recent researches focused on this system include a calibration system with simple structure and low computational complexity to obtain actual measurement matrix [11].

In order to fully use the accuracy of an ADC, the quantization reference voltage is an essential parameter to consider about. The reference voltage of an conventional ADC simply equals the maximum value of the input analog signal. In the MWC, however, the quantization takes place at the end of the sampling stage and the maximum of the samples is sensitive to several parameters, especially the randomness of the mixing functions. Thus the reference voltage can not be derived from the input signal intuitively. Moreover, simulation results exploit that the maximum of the samples varies widely even when the input signal is fixed. Therefore, an improper reference voltage may have a severe impact on the performance of the whole system. Though important in practice, few research-

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es have been focused on this problem and the reference voltage in practical implementation is obtained by experiments.

This paper aims to theoretically study the reference voltage of quantizer in the MWC. A reasonable method to acquire reference voltage is proposed. By analyzing the input signal under several assumptions, we draw the conclusion that the reference voltage is proportional to the square root of the collapse parameter $q = f_s/f_p$. Further discussions show that the assumptions made in the analysis are acceptable and that the conclusion is valid for arbitrary multiband signals.

This paper is organized as follows. Section 2 briefly introduces the structure of the MWC and related mathematical expressions. Section 3 studies the reference voltage and come to the conclusion under rational assumptions. The theoretical results in Section 3 are numerically validated in Section 4.

2. THE MODULATED WIDEBAND CONVERTER

As can be seen from Fig. 1, the input signal $x(t)$ enters m channels simultaneously. In the i-th channel, it is multiplied by a T_p-periodic pseudorandom sequence $p_i(t)$, whose frequency is $f_p = 1/T_p$. Specifically, $p_i(t)$ is chosen as a piecewise constant function that randomly alternates between the levels ± 1 for each of M equal time intervals. The mixed signal is then truncated by an ideal low-pass filter with cutoff $1/(2T_s)$. Finally, the filtered signal $y_i(t)$ is sampled at rate $f_s = 1/T_s$. According to [9], the parameters f_s and f_p satisfy $f_s = qf_p$ with odd q. Further assume that $x(t)$ is bandlimited to $[-f_{\mathrm{NYQ}}/2, f_{\mathrm{NYQ}}/2]$, where f_{NYQ} is much larger than the total sampling rate of the system.

Formally, the mathematical description of $p_i(t)$ is

$$p_i(t) = \alpha_{ik}, \quad k\frac{T_p}{M} \le t < (k+1)\frac{T_p}{M}, \quad 0 \le k \le M-1,$$

with $\alpha_{ik} \in \{+1, -1\}$ and $p_i(t + nT_p) = p_i(t)$ for every $n \in \mathbb{Z}$. Since $p_i(t)$ is T_p-periodic, it has the Fourier expansion

$$p_i(t) = \sum_{l=-\infty}^{+\infty} c_{il} \mathrm{e}^{\mathrm{j}\frac{2\pi}{T_p}lt},$$

where $\{c_{il}\}$ are Fourier coefficients. Define $\phi = \mathrm{e}^{-\mathrm{j}2\pi/M}$, thus

$$c_{il} = d_l \sum_{k=0}^{M-1} \alpha_{ik}\phi^{lk}, \tag{1}$$

where

$$d_l = \frac{1}{T_p}\int_0^{\frac{T_p}{M}} \mathrm{e}^{-\mathrm{j}\frac{2\pi}{T_p}lt}\,\mathrm{d}t.$$

In [9], it has been proved that the Fourier transform of $y_i(t)$ is expanded as

$$Y_i(f) = \sum_{l=-L_0}^{L_0} c_{il}X(f - lf_p), \quad f \in [-f_s/2, f_s/2], \tag{2}$$

where

$$L_0 = \left\lceil \frac{f_{\mathrm{NYQ}} + f_s}{2f_p} \right\rceil - 1.$$

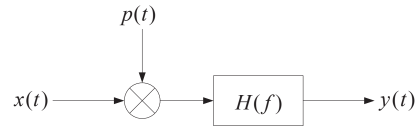

Fig. 2. The simplified sampling stage

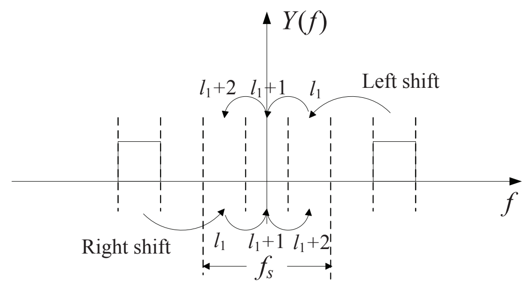

Fig. 3. The simplification from (2) to (4) (Parameters: $q = 3, l_0 = 3, l_1 = 2$).

3. REFERENCE VOLTAGE OF QUANTIZATION

3.1. Preliminary

We first consider about a basic scenario where the signal is only composed of one modulated narrowband signal, and then discuss about arbitrary multiband signals. In the sequel, the Fourier transform of $x(t)$, denoted by $X(f)$, consists of two symmetric bands centered at $\pm l_0 f_p$ respectively, where $(l_0 - 1/2)f_p \ge f_s/2$. The width of both bands satisfies $B \le f_p$. Formally,

$$X(f) = X_0(f - l_0 f_p) + X_0(f + l_0 f_p), \tag{3}$$

where

$$X_0(f) = 0, \quad f \notin [-f_p/2, f_p/2].$$

Notice that the structure of each channel is identical, and that the maximum sample of $y_i[n]$ is no larger than the maximum value of analog signal $y_i(t)$. As a result, the sampling stage can be simplified to Fig. 2 when reference voltage is being analyzed. For convenience, in the sequel, the subscript i which denotes the channel number is omitted.

For the basic scenario (3), the series on the right side of (2) have only $2q$ nonzero terms. Then

$$Y(f) = \sum_{l=l_1}^{l_1+q-1} \left[c_l X(f - lf_p) + c_{-l}X(f + lf_p)\right], \quad f \in [-f_s/2, f_s/2], \tag{4}$$

where $l_1 = l_0 - (q-1)/2$. Fig. 3 exploits the simplification from (2) to (4) intuitively.

Considering (4) in time domain, one has

$$y(t) = \sum_{l=l_1}^{l_1+q-1} c_l \int_{-f_s/2}^{f_s/2} X(f - lf_p)\mathrm{e}^{\mathrm{j}2\pi ft}\,\mathrm{d}t + \sum_{l=l_1}^{l_1+q-1} c_{-l} \int_{-f_s/2}^{f_s/2} X(f + lf_p)\mathrm{e}^{\mathrm{j}2\pi ft}\,\mathrm{d}t. \tag{5}$$

Since $[-f_s/2, f_s/2]$ contains only one sideband of $X(f - lf_p)$, Hilbert transform can be applied to represent such single sideband signal. Setting $\hat{x}(t) = x(t) * \{1/(\pi t)\}$ as Hilbert transform of $x(t)$, $\hat{X}(f)$ denotes the Fourier transform of $\hat{x}(t)$. According to the definition of Hilbert transform, $[X(f) - \mathrm{j}\hat{X}(f)]/2$ represents the negative-frequency band of $X(f)$. Hence

$$\begin{aligned}
&\sum_{l=l_1}^{l_1+q-1} c_l \int_{-f_s/2}^{f_s/2} X(f - lf_p)\mathrm{e}^{\mathrm{j}2\pi ft}\,\mathrm{d}t \\
&= \sum_{l=l_1}^{l_1+q-1} c_l \mathrm{e}^{\mathrm{j}2\pi lf_p t} \int_{-\infty}^{+\infty} \frac{X(f) - \mathrm{j}\hat{X}(f)}{2}\mathrm{e}^{\mathrm{j}2\pi ft}\,\mathrm{d}t \\
&= \sum_{l=l_1}^{l_1+q-1} c_l \mathrm{e}^{\mathrm{j}2\pi lf_p t}\frac{\tilde{x}^*(t)}{2}, \qquad (6)
\end{aligned}$$

where $\tilde{x}(t) = x(t) + \mathrm{j}\hat{x}(t)$ is the analytic signal of $x(t)$.

Dealing with the second term on the right side of (5) in the same way, it can be derived that

$$\begin{aligned}
|y(t)| &= \left|\sum_{l=l_1}^{l_1+q-1} \Re\left\{c_l \mathrm{e}^{\mathrm{j}2\pi lf_p t}\tilde{x}^*(t)\right\}\right| \\
&= |\tilde{x}(t)| \left|\sum_{l=l_1}^{l_1+q-1} |c_l| \cos\left(\theta_l + 2\pi lf_p t - \theta_{\tilde{x}}(t)\right)\right|, \qquad (7)
\end{aligned}$$

where $c_l = |c_l|\,\mathrm{e}^{\mathrm{j}\theta_l}$, $\tilde{x}(t) = |\tilde{x}(t)|\,\mathrm{e}^{\mathrm{j}\theta_{\tilde{x}}(t)}$ and $\Re(\cdot)$ denotes the real part of its argument. The reference voltage is determined by the maximum value of $|y(t)|$.

Define $|y|_{\max}$ as the maximum value of $|y(t)|$. $|y|_{\max}$ is a random variable, due to the fact that the mixing function $p(t)$ randomly alternates between ± 1. Mathematically, (7) indicates that the randomness of $|y|_{\max}$ results from the randomness of $\{c_l\}$, which are Fourier coefficients of $p(t)$. The threshold of $|y|_{\max}$ is chosen such that the probability

$$\mathrm{P}\left\{|y|_{\max} \le |y|_{\mathrm{th}}\right\} = P_0, \qquad (8)$$

where P_0 is a constant. It is reasonable to use $|y|_{\mathrm{th}}$ as reference voltage with a proper P_0, say 99%.

3.2. Main Contribution

The following proposition reveals the property of the reference voltage defined as (8).

Proposition 1. *In the MWC mentioned above, consider fixed input signal (3). Further assume*
(a) $|y(t)|$ and $|\tilde{x}(t)|$ reach their maximum at the same time;
(b) $\{\theta_l\}$ are i.i.d. random variables with uniform distribution on $[-\pi, \pi]$, and $\{|c_l|\}$ are i.i.d. random variables independent from $\{\theta_l\}$.

Then the reference voltage $|y|_{\mathrm{th}}$ defined by (8) is proportional to $\sqrt{q} = \sqrt{f_s/f_p}$. Specifically,

$$|y|_{\mathrm{th}} = \sqrt{q}\,|\tilde{x}(t_0)|\,\sigma Y_{\mathrm{th}}, \qquad (9)$$

where $t_0 = \operatorname{argmax}_t \{\tilde{x}(t)\}$, $\sigma^2 = \operatorname{var}\{|c_l|\cos\theta_l\} = \operatorname{var}\{\Re\{c_l\}\}$, and Y_{th} satisfies

$$2\Phi(Y_{\mathrm{th}}) - 1 = \frac{1}{\sqrt{2\pi}}\int_{-Y_{\mathrm{th}}}^{Y_{\mathrm{th}}} \mathrm{e}^{-t^2/2}\,\mathrm{d}t = P_0.$$

where $\Phi(x)$ is the cumulative distribution function of the standard normal distribution.

Proof. According to assumption (a), denoting $\theta_l + 2\pi lf_p t_0 - \theta_{\tilde{x}}(t_0)$ by $\hat{\theta}_l$, one has

$$|y|_{\max} = |\tilde{x}(t_0)| \left|\sum_{l=l_1}^{l_1+q-1} |c_l| \cos\left(\hat{\theta}_l\right)\right|.$$

Notice that $\hat{\theta}_l$ is also uniform distributed on $[-\pi, \pi]$ as θ_l. Setting $z_l = |c_l|\cos\hat{\theta}_l$, the equivalent problem is to find $Z_{\mathrm{th}} = |y|_{\mathrm{th}}/|\tilde{x}(t_0)|$ such that

$$\mathrm{P}\left\{\left|\sum_{l=l_1}^{l_1+q-1} z_l\right| \le Z_{\mathrm{th}}\right\} = P_0. \qquad (10)$$

Since $\mathrm{E}\{\cos\hat{\theta}_l\}$ equals 0 when $\hat{\theta}_l$ is uniform distributed on $[-\pi, \pi]$, $\{z_l\}$ are zero-mean i.i.d. random variables. According to Central Limit Theorem [12], the variable

$$Z = \sum_{l=l_1}^{l_1+q-1} \frac{z_l}{\sigma\sqrt{q}}$$

satisfies standard normal distribution when q approaches infinity. Hence

$$\begin{aligned}
\mathrm{P}\left\{\left|\sum_{l=l_1}^{l_1+q-1} z_l\right| \le Z_{\mathrm{th}}\right\} &= \mathrm{P}\left\{-\frac{Z_{\mathrm{th}}}{\sigma\sqrt{q}} \le Z \le \frac{Z_{\mathrm{th}}}{\sigma\sqrt{q}}\right\} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-Y_{\mathrm{th}}}^{Y_{\mathrm{th}}} \mathrm{e}^{-t^2/2}\,\mathrm{d}t, \qquad (11)
\end{aligned}$$

where $Y_{\mathrm{th}} = Z_{\mathrm{th}}/\left(\sigma\sqrt{q}\right)$. Combining (10) and (11), it is obvious that Y_{th} is a constant satisfying

$$\frac{1}{\sqrt{2\pi}}\int_{-Y_{\mathrm{th}}}^{Y_{\mathrm{th}}} \mathrm{e}^{-t^2/2}\,\mathrm{d}t = P_0.$$

Thus

$$|y|_{\mathrm{th}} = |\tilde{x}(t_0)|\,Z_{\mathrm{th}} = \sqrt{q}\,|\tilde{x}(t_0)|\,\sigma Y_{\mathrm{th}},$$

and this completes the proof. □

3.3. Discussion

Assumption (a) is drawn from the mechanism of the system. Mapping (3) to time domain, one has $x(t) = x_0(t)\cos(2\pi l_0 f_p t)$. Thus

$$\begin{aligned}
|\tilde{x}(t)| &= |x(t) + \mathrm{j}\hat{x}(t)| \\
&= |x_0(t)\cos(2\pi l_0 f_p t) + \mathrm{j}x_0(t)\sin(2\pi l_0 f_p t)| \\
&= |x_0(t)|\,. \qquad (12)
\end{aligned}$$

Equation (12) shows that $|\tilde{x}(t)|$ and $|x_0(t)|$ reach the maximum at the same time. On the other hand, $y(t)$ is obtained through three steps from $x_0(t)$: modulated by $\cos(2\pi l_0 f_p t)$ with large l_0 typically, multiplied by $p(t)$ and lowpass filtered. In the first step, modulation with the carrier at high frequency can hardly change the instant when the signal reaches the maximum. The second step does not make any change because $|p(t)| = 1$. For the last step, since $f_s \ge f_p$, the lowpass filter does not affect the "envelope" of the signal. Based on the analysis above, it is convincing that $|y(t)|$ and $|x_0(t)|$ also reach the maximum at the same time, which leads to assumption (a).

Assumption (b) is based on the observation of $\{c_l, l_l \leq l \leq l_1 + q - 1\}$. Considering (1), since l_1 is far larger than q, one has $l_1+q-1 \approx l_1$, which results in the approximation that $\{c_l, l_1 \leq l \leq l_1+q-1\}$ are i.i.d. Furthermore, the distribution of θ_l is determined mainly by the sum of $\alpha_k \phi^{lk}, 0 \leq k \leq M-1$. In the complex plane, $\{\phi^{lk}, 0 \leq k \leq M-1\}$ represents M points located uniformly on the unit circle and M is typically very large. Therefore assume that the argument of their linear combination satisfies uniform distribution is an acceptable way to simplify the analysis.

Now we discuss about arbitrary multiband signals. Consider a multiband signal $y(t)$ which consists of N basic signals denoted by (3) with the same maximum value. If there is no overlap in time domain among these N signals, the problem can be reduced to study one input signal. When N signals reach their maximum at the same time, say t_0, the following analysis intuitively shows that the reference voltage is proportional to $\sqrt{Nq}$. In frequency domain, according to (4), $[-f_s/2, f_s/2]$ has q shift copy of each band. Thus, observing the multiband signal within a short time interval around t_0, the energy of $Y(f)$ is Nq times the energy of each basic signal. Hence the amplitude $|y(t_0)|$ is $\sqrt{Nq}$ times the maximum of each basic signal.

4. SIMULATION

The following simulation calculates the reference voltage $|y|_{\mathrm{th}}$ numerically with $P_0 = 99\%$. For specific parameter settings, the experiment is conducted 5000 trials and the value which is larger than 99% of the results is chosen as $|y|_{\mathrm{th}}$. The number of the channels is $m = 30$ and the frequency of the mixing function is $f_p \approx 51.3$MHz. The input $x(t)$ is a multiband signal consisting of N pairs of bands, defined as

$$x(t) = \sum_{i=1}^{N} \sqrt{EB}\mathrm{sinc}\left(B\left(t-\tau\right)\right)\cos\left(2\pi f_i \left(t-\tau\right)\right) \qquad (13)$$

where $E = 10$, $B = 50$MHz, $\tau = 0.4\mu$s, $f_i = \{25 i f_p\}$. The reference voltage is calculated under the parameter choice $N = \{1, 2, 3\}$ and $\{q = 2q' + 1, \quad q' = 0, 1, ..., 9\}$.

The results are plotted in Fig. 4. It is shown that $|y|_{\mathrm{th}} / \sqrt{q}$ is almost constant when $q \leq 3$. Though (11) is obtained under the condition that q approaches infinity, the results indicate that q need not be very large. Furthermore, for each fixed N respectively, calculate arithmetic mean of $|y|_{\mathrm{th}} / \sqrt{q}$. The ratio of the results is $1 : 1.37 : 1.63$, which is close to $1 : \sqrt{2} : \sqrt{3}$. The results above correspond to the theoretical analysis and the discussions.

5. CONCLUSION

In this paper, we propose a method to acquire the reference voltage of the MWC. Then we theoretically analyze the reference voltage under rational assumptions of the input signal and the mixing functions. The conclusion is drawn that the quantization reference voltage is proportional to the square root of collapse parameter $q = f_s/f_p$. Furthermore, discussions and simulation results show that the conclusion is valid for arbitrary multiband signals. Specifically, when the multiband signal consists of N pairs of bands, the reference voltage is proportional to $\sqrt{Nq}$.

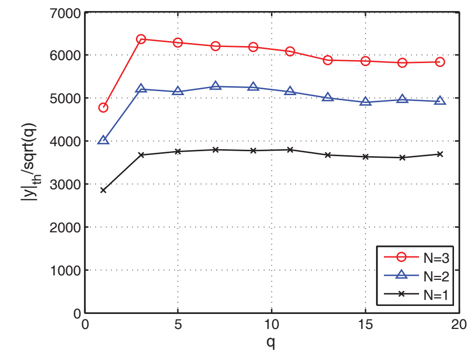

Fig. 4. Reference voltage with respect to collapse parameter q and the number of signals N

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