# **REPRESENTATION OF PWM SIGNALS THROUGH TIME WARPING**

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# ABSTRACT

In this work a novel approach for representing pulse width modulated (PWM) signals is introduced. PWM signals are usually represented according to the way they are generated, that is, by manipulating the sign of the comparison between the input signal and a reference wave. On the contrary, the new representation consists of a warped Fourier series, that is, a series of properly phase-modulated sinusoids, such that the zero-crossings of each warped harmonic comprehends the zero-crossing of the PWM signal. Yet, in contrast with the original signal, the spectrum of the resulting components decays exponentially, so they can be sampled at reasonable low rate while maintaining aliasing negligible. Being bandlimited and keeping zero-crossings unaltered, this representation is suitable for computationally efficient PWM signal generation.

Index Terms- Pulse Width Modulation, Time Warping

### 1. INTRODUCTION

Pulse Width Modulated (PWM) signals consist of a train of pulses such that the information in the modulating signal is represented by the pulses width or equivalently by the resulting zero-crossing instants [1]. When the PWM signal is properly generated, the modulating signal can be accurately restored by simple low-pass filtering. Historically, this modulation has been mostly employed in power control for switching power converters and related applications, and, more recently, it has been effectively applied in class-D amplifiers in order to overcome the disadvantages of linear amplifiers and to exploit the efficiency of non-linear ones.

Warping techniques are ubiquitous in signal processing: for instance, frequency warping deals with the possibility of generalizing time-frequency transformations while time warping is generally employed in image or voice processing for feature extraction and pattern recognition [2].

We plow here to exploit warping techniques to give an alternative representation of PWM signals. In fact, although there are some different ways to represent and generate PWM signals, the deformation of the time axis, that is the time warping, has not been explored yet. This kind of representation carries some advantages which will be illustrated throughout the paper. The general idea is to consider a periodic square wave with constant 50% cycle and apply a deformation of its time axis so that is becomes identical to what is obtained by encoding a signal by classical PWM methods, i.e. by comparing it with periodic triangle wave. Probably the most straightforward way of finding a suitable time warping is to consider a piecewise linear function where each piece is identified by the zero-crossings of the PWM signal. The resulting time warping function can then be used to describe the instantaneous phase deviation to be applied to a periodic square wave to reproduce the same PWM signal. Regrettably, such an intuitive approach yields a phase modulation with an heavy tailed spectrum that, for example, can be hardly reproduced by discretetime digital means.

More sophisticated warping should be taken into account and this is precisely the task of the following Sections that, after an overview of PWM signals (Section 2) develop the general warping model (3) and propose some feasible warping functions (4).

### 2. PWM IN BRIEF

PWM is commonly presented through the circuit by which the modulated signal is obtained. A continuous-time bandlimited signal s(t), with normalized Nyquist frequency equal to 1 and dynamic range in (-1, 1), is compared to a triangle wave  $\xi(f_c t)$  whose normalized first harmonic is at a frequency  $f_c > 1/2$ , so that it is entirely out of the signal band. The triangle wave is given by:

$$\xi(t) = \begin{cases} -1 + 4(t-k) & k \le t < k + 1/2 \\ -1 - 4(t-k) & k - 1/2 \le t < k \end{cases}, \ k \in \mathbb{Z}.$$

Such a PWM signal is referred to as double-edge, since the modulating signal is sampled twice per reference wave period. The difference  $s(t) - \xi(f_c t)$  is thresholded to symmetrical positive and negative levels equal to +1 and -1 respectively:

$$\phi_{s,f_c}(t) = \operatorname{sign}(s(t) - \xi(f_c t)). \tag{1}$$

The obtained modulated signal contains the input signal plus an infinite number of non-linear contributions whose spectra are centered on the triangle wave harmonics [3].

Two simple examples of PWM generation are shown in Fig. 1. In the first case  $f_c = 1$ , so that triangle wave is intuitively able to sample enough information from s(t). In the second case  $f_c = 1/8$ , thus within the signal band, the transitions of  $\phi_{s,f_c}$  are not able to catch the variations of s(t). Moreover, around t = 26, s(t) is sampled three times on the same linear piece, causing an increase of average transition frequency. In the rest of the paper we will always assume that  $f_c$  is large enough to guarantee that the average transition frequency is equal to  $f_c$  itself.

## 3. PWM REPRESENTATION THROUGH TIME WARPING

The basic idea behind the representation of PWM signal through a time warping procedure is first to consider an unmodulated train of pulses

$$\phi_{0,f_c}(t) = \operatorname{sign}(-\xi(f_c t))$$
  
=  $\sum_{k \in \mathbb{Z}} \operatorname{2rect}(2(t-k)) - 1$   
=  $\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \cos(2\pi(2m+1)f_c t)$ 

where rect is the rectangular function. To this we add a signal-dependent time shift  $\delta_{s,f_c}(t)$  on the cosines to obtain

$$\begin{aligned} \phi_{s,f_c}(t) &= \phi_{0,f_c}(t+\delta_{s,f_c}(t)) \\ &= \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \cos(2\pi(2m+1)f_c(t+\delta_{s,f_c}(t))). \end{aligned}$$

By doing so, we can assure that the PWM signal shares its zero-crossing instants with all the terms of the Fourier series.

To find the expression of the warping function  $t + \delta_{s,f_c}(t)$ , let us start by recalling that, according to warping theory, a warping function has to defined such that its derivative is always positive. If this condition is satisfied, the zero-crossings of the unmodulated PWM signal  $\phi_{0,f_c}$  are shifted one with respect to the other while their order is kept unaltered. The zero-crossings of the unmodulated signal are given by

$$\frac{1}{f_c}\left(k\pm\frac{1}{4}\right) \quad k\in\mathbb{Z}$$

so we get that in all time instants t for which the modulated PWM signal has a zero-crossing, s(t) has to be equal to  $\xi(t)$ , which turns in the following constraint for  $k \in \mathbb{Z}$ 

$$f_c(t + \delta_{s,f_c}(t)) = k \pm \frac{1}{4} \Rightarrow s(t) = -1 \pm 4(f_c t - k).$$
 (2)

As anticipated in the Introduction, a time shift that satisfies the previous requirement can be obtained from the sets



Fig. 1: A case of correctly generated PWM signal (upper figure) and undersampled PWM signal (lower figure), where also an unwanted oversampling occurs around t = 26.

of trailing zero-crossings  $\alpha_k$  and of leading zero-crossings  $\beta_k$  such that

$$s(\alpha_k) = -1 + 4(f_c \alpha_k - k)$$
  

$$s(\beta_k) = -1 - 4(f_c \beta_k - k)$$

All we have to do is to linearly interconnect the samples  $s(\alpha_k)$ and  $-s(\beta_k)$ 

$$\delta_{s,f_c}(t) = \begin{cases} \frac{1}{4f_c} \left[ \frac{s(\alpha_k) + s(\beta_k)}{\alpha_k - \beta_k} (t - \beta_k) - s(\beta_k) \right] \\ \frac{1}{4f_c} \left[ \frac{s(\alpha_k) + s(\beta_{k+1})}{\alpha_k - \beta_{k+1}} (t - \alpha_k) + s(\alpha_k) \right] \end{cases}$$
(3)

for  $\beta_k \leq t \leq \alpha_k$  and  $\alpha_k \leq t \leq \beta_{k+1}$  respectively. This definition is not a good choice for the following reasons:

- it is desirable that each warped cosine has a fast decaying spectrum, which means that a smooth warping function should be employed;
- the definition of the warping function should not require to know a-priori the zero-crossings of the PWM signal;
- the warping function should continuously depend on s(t) rather than being piecewise defined.

Warping functions obeying the above specifications are *good* ones and, in general, may be thought of as smooth interpolations of the piecewise-linear example given above that do not rely on the a-priori knowledge of zero-crossings.

## 4. WARPING FUNCTIONS FOR PWM

In this Section we build some good warping functions satisfying the requirements which have been previously listed starting from some general observations. According to the linear



**Fig. 2**: Time shift  $\delta_{s,f_c}$ , as defined in eq. (4), for  $f_c = 1$ , plotted for each pair (s,t) in the domain  $(-1,1) \times (-1/2,1/2)$ . For  $|s| > 2/\pi$  some regions do not satisfy the specifications since the derivative along t is smaller than -1.

piecewise function (3), the time shift has to oscillate between s(t) and -s(t) with a proper scaling and pass by  $s(\alpha_k)$  and  $-s(\beta_k)$ . As a starting point, we consider a generic odd oscillating function such that it is equal to 0 for  $k \in \mathbb{Z}$ . In fact, the instants  $t = k/f_c$  cannot represent zero-crossings, since leading edges  $\alpha_k$  range in  $(k/f_c, (k+1/2)/f_c)$  while trailing edges  $\beta_k$  range in  $((k-1/2)/f_c, k/f_c)$ . The bounds of the interval are never reached since  $s \in (-1, 1)$ . Moreover, the oscillating function has to be monotonic in s, so that the larger is s the larger is the time shift. Finally, as the time shift increases, the oscillating function has to warp itself making the transitions instants move in a proportional way with respect to s.

Our first proposal to meet all the above considerations is

$$\delta_{s,f_c}(t) = -\frac{1}{4f_c} s(t) \sin(2\pi f_c(t+\delta_{s,f_c}(t))).$$
(4)

It is quite easy to verify that this function satisfy the specification, in fact, by considering  $f_c(t + \delta_{s,f_c}(t)) = k \pm 1/4$ , for the left side we get

$$\delta_{s,f_c}(t) = \frac{1}{f_c} \left(k \pm 1/4\right) - t$$

while for the right side we have

$$-\frac{1}{4f_c}s(t)\sin(2\pi f_c(t+\delta_{s,f_c}(t))) = \mp \frac{1}{4f_c}s(t)$$

which, combined together, give  $s(t) = \xi(f_c t)$ .

The fact that the time shift has been defined through a recursion poses little or no problem at all since it can be considered as a nonlinear scalar equation yielding separately for every t that can be effectively solved by numerical means since



**Fig. 3**: First warped harmonic according to time shift in eq. (4), for  $f_c = 1$ . In the regions where  $\delta_{s,f_c}$  does not satisfy the specification, the cosine is not correctly warped. The function is 0 on the line  $s(t) = \xi(t)$  (dashed line). For s = 0 (solid line) the function gives an unwarped cosine.

the needed derivative are known. For example, one may employ an iterative Newton algorithm and repeatedly compute

$$\varepsilon_k = -\frac{1}{4f_c} s \sin(2\pi f_c(t+\delta_k)) - \delta_k$$
  

$$\varepsilon'_k = -\frac{\pi}{2} s \cos(2\pi f_c(t+\delta_k)) - 1$$
  

$$\delta_{k+1} = \delta_k - \frac{\varepsilon_k}{\varepsilon'_k}$$

till convergence. What results from this computation needs to be a warping function, i.e., its derivative must be always positive or, equivalently, the minimum of the derivative of  $\delta_{s,f_c}$ must be larger than -1. The derivative  $\delta'_{s,f_c}$  results

$$\delta_{s,f_c}'(t) = \frac{\psi(s(t), s'(t), \delta(t))}{1 + \frac{\pi}{2}s(t)\cos(2\pi f_c(t + \delta_{s,f_c}))}$$

where  $\psi(s(t), s'(t), \delta(t))$  is a properly defined function of s, s' and  $\delta$ . A complete study of this function for every (s, t) pair in the domain  $(-1, 1) \times (-1/2, 1/2)$  is not straightforward, but it is evident that the denominator can be equal to 0 when  $|s| = 2/\pi$ , thus potentially generating values smaller than -1. In fact, as illustrated in Fig. 2 and Fig. 3, the time shift (4) result to be valid only for  $|s| < 2/\pi$ .

Despite the above mentioned limitations, the time shift (4) has an interesting feature. By properly representing the warped harmonics in Taylor series, their spectrum can be obtained as a series of non-linear function of s shifted in frequency on multiples of  $f_c$ . For this warping function, the only warped harmonic having a baseband component is the first one, i.e.  $4/\pi \cos(2\pi f_c(t + \delta_{s,f_c}(t)))$ , and this component is just s(t).

In order to overcome the limitations of function (4), we define a new time shift by exploiting the warping function



Fig. 4: Time shift  $\delta_{s,f_c}$ , as defined in eq. (5), for  $f_c = 1$ , plotted for each pair (s,t) in the domain  $(-1,1) \times (-1/2,1/2)$ . Unlike the time shift plotted in Fig. 2, this function give rise to a valid warping function on the entire domain.

employed in the Laguerre transform. Basically we start considering that

$$\frac{1}{\pi}\arctan(\nu\tan(\pi t)) \quad \nu \in (0,\infty)$$

represents an antisymmetric warping function on the interval [-1/2, 1/2] depending on the positive parameter  $\nu$ : when  $\nu$  is equal to 1 the warping function is simply a line, which means no warping, when  $\nu$  tends to 0 the warping function tends to the round(t), while when  $\nu$  tends to  $\infty$  the warping function tends to the round(t-1/2) + 1/2. This means that the entire domains  $[0, 1/2] \times [0, 1/2]$  and  $[-1/2, 0] \times [-1/2, 0]$  are covered by this set of functions and any correspondence between s and t can be matched. Hence, the parameter  $\nu$  has to be set as a function of s such that the condition (2) for obtaining a feasible warping function is satisfied. By some calculations we get

$$\nu = \frac{1}{\tan(\pi(s+1)/4)}$$

so that the time shift can be define as

$$\delta_{s,f_c}(t) = \frac{1}{\pi f_c} \arctan\left(\frac{\tan(\pi f_c t)}{\tan(\pi (s(t)+1)/4)}\right) - t$$

which can be rewritten in the following way

$$\delta_{s,f_c}(t) = \frac{1}{\pi f_c} \arctan\left(\frac{\lambda(s(t))\sin(2\pi f_c t)}{1 - \lambda(s(t))\cos(2\pi f_c t)}\right)$$
(5)

where  $\lambda(s)$  is given by

$$\lambda(s) = \frac{1 - \tan(\pi(s+1)/4)}{1 + \tan(\pi(s+1)/4)}.$$

Unlike the previous case, it is not necessary to verify that  $\delta'_{s, f_c} > -1$  since we started from a class of warping function



**Fig. 5**: First warped harmonic according to time shift in eq. (5), for  $f_c = 1$ . Being (5) a valid time shift, the cosine is warped correctly on the entire domain. The curve where the surface is equal to 0 (dashed line) is the line  $s(t) = \xi(t)$ . For s = 0 the function gives an unwarped cosine.

which is widely known and employed. The newly defined  $\delta_{s,f_c}$  is plotted in Fig. 4, while the first warped harmonic is represented in Fig. 5.

As a general remark, note that both time shifts 4 and (5) can be computed without the knowledge of the zero-crossings of the PWM signal to be modeled. Notwithstanding this, the zero-crossings of each of the resulting harmonics is the same as the final signal. Since the spectrum of each of these harmonics decays exponentially, samples taken, for example, from the first of them, can be effectively used as a discrete time representation of the PWM signals. This could be useful for both signal representation and generation.

#### 5. CONCLUSION

An innovative methodology based on time warping for building and representing PWM signals has been explored. We introduced a decomposition in warped harmonics having the same zero-crossings as the PWM signal and fast decaying spectra, such that they can be sampled and used for discretetime representation and building of PWM signals without computing its zero-crossings. Two remarkable examples of feasible warping function, based on a recursive warping and on Laguerre warping respectively, have been presented.

#### 6. REFERENCES

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