OPTIMALLY WEIGHTED MUSIC ALGORITHM FOR FREQUENCY ESTIMATION OF REAL HARMONIC SINUSOIDS

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ABSTRACT

In this paper, the problem of fundamental frequency estimation for real harmonic sinusoids is addressed. By making use of the subspace technique and Markov-based eigenanalysis, an optimally weighted harmonic multiple signal classification (OW-HMUSIC) estimator is devised. The fundamental frequency estimates are computed in an iterative manner. The performance of the proposed method is derived. Computer simulations are performed to compare the proposed approach with nonlinear least squares and HMUSIC methods as well as Cramér-Rao lower bound.

Index Terms— Fundamental frequency estimation, subspace method, harmonic signal, Markov optimum weighting, multi-pitch

1. INTRODUCTION

Fundamental frequency estimation of harmonic sinusoids is a classical but active problem in spectral analysis research, finding applications in a wide range of areas such as speech and audio signal processing, automotive control systems and angular speed determination of rotating targets in radar. Due to the importance of this problem, kinds of optimum fundamental frequency estimators have been devised [1] – [3]. In most cases, these methods are based on the weighted least squares refinement, which is susceptible to accuracy of the respective frequency estimates and their covariance estimate. The nonlinear least squares (NLS) estimator is statistically efficient. However, due to the decoupling difficulty [4], it cannot deal with the multi-pitch scenario. In [4] – [5], the harmonic multiple signal classification (HMUSIC) algorithm is proposed for single-pitch and multi-pitch estimation. However, the HMUSIC estimation accuracy cannot attain Cramér-Rao lower bound (CRLB) [6].

To improve the HMUSIC performance, we propose the optimally weighted HMUSIC (OW-HMUSIC) estimator with the use of the Markov-based eigenanalysis. First, we tackle the single-pitch estimation. There have been a number of weighted subspace-based estimators [7] – [8] using the covariance matrix in the literature. Nevertheless, it is pointed out in [9] that the subspace methods based on data matrix perform better than those utilize covariance matrix, especially when the number of data points is relatively small. In addition, it is difficult for such estimators to deal with harmonic frequency estimation due to their subspace-fitting schemes. Thus, different from them, our estimator is based on the singular value decomposition (SVD) of data matrix and the HMUSIC estimation. To facilitate the statistical error analysis, we modify the HMUSIC formulation. Then we analyze the perturbation of the orthogonality error, and derive the optimum weighting matrix. The fundamental frequency estimates are computed in an iterative manner. Furthermore, we extend the OW-HMUSIC algorithm to the multi-pitch case. Simulation results show that there is obvious improvement of the proposed scheme over [4] – [5]. Especially in the single-pitch scenario, the OW-HMUSIC performance can attain the CRLB. In this work, we perform estimation for the real harmonic sinusoids instead of the complex ones.

The rest of this paper is organized as follows. The proposed estimators for single-pitch and multi-pitch sinusoids are developed in Section 2. The statistical properties of the orthogonality error is analyzed, and the optimum weighting matrix is derived. Its theoretical estimation performance is also analyzed. In Section 3, simulation results are included to show the performance of the proposed approach by comparing with the NLS and HMUSIC methods as well as CRLB. Finally, conclusions are drawn in Section 4.

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2. ALGORITHM DEVELOPMENT

2.1. Single-Pitch Estimation

Consider the real harmonic sinusoidal signal:

$$x(n) = s(n) + q(n), \quad n = 0, 1, \cdots, N-1, \quad (1)$$

$$s(n) = \sum_{l=1}^{L} \alpha_l \cos(l\omega_1 n + \phi_l), \quad (2)$$

where ω_1, A_l and ϕ_l are unknown parameters representing the fundamental frequency, the amplitude and initial phase of the $(l-1)$-th harmonic, respectively, while $q(n)$ is an uncorrelated Gaussian random process with mean zero and variance

σ^2. The objective is to estimate ω_1 from the N samples of $x(n)$.

We first define [8]

$$x_c(m_1, m_2) = x(m_1 + m_2 + M), \tag{3}$$
$$x_b(m_1, m_2) = x(m_1 + M - 1 - m_2), \tag{4}$$
$$y(m_1, m_2) = \frac{1}{2}[x_c(m_1, m_2) + x_b(m_1, m_2)], \tag{5}$$

for $m_1 = 0, 1, \cdots, N - 2M$, $m_2 = 0, 1, \cdots, M - 1$, and express (5) in matrix form as

$$\mathbf{Y} = \mathbf{S} + \mathbf{Q}, \tag{6}$$

where $\mathbf{Y}$ is the $M \times (N - 2M + 1)$ data matrix with elements $[\mathbf{Y}]_{m,n} = y(n - 1, m - 1)$, $L < M < (N + 1)/2$. The matrices $\mathbf{S}$ and $\mathbf{Q}$ are the noise-free and noise components of $\mathbf{Y}$, respectively. Following [8], $\mathbf{S}$ can be factorized as:

$$\mathbf{S} = \mathbf{A}\mathbf{\Gamma}\mathbf{H}^T, \tag{7}$$

where

$$\mathbf{\Gamma} = \text{diag}(\alpha_1, \alpha_2, \cdots, \alpha_L), \tag{8}$$
$$\mathbf{A} = [\mathbf{a}_1,\ \mathbf{a}_2,\ \cdots,\ \mathbf{a}_L], \tag{9}$$
$$\mathbf{a}_l = \left[\cos(\frac{l\omega_1}{2}),\ \cdots,\ \cos((M - \frac{1}{2})l\omega_1)\right]^T, \tag{10}$$

and $\mathbf{H}$ is an $(N - 2M + 1) \times L$ matrix. On the other hand, $\mathbf{Y}$ can be decomposed using SVD as

$$\mathbf{Y} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^T = [\mathbf{U}_s\ \mathbf{U}_n] \begin{bmatrix} \mathbf{\Lambda}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_n \end{bmatrix} [\mathbf{V}_s\ \mathbf{V}_n]^T, \tag{11}$$

where $\mathbf{U}_s \in \mathbb{R}^{M \times L}$, $\mathbf{\Lambda}_s \in \mathbb{R}^{L \times L}$, $\mathbf{V}_s \in \mathbb{R}^{(N-2M+1) \times L}$ and $\mathbf{U}_n \in \mathbb{R}^{M \times (M-L)}$, $\mathbf{\Lambda}_n \in \mathbb{R}^{(M-L) \times (M-L)}$, $\mathbf{V}_n \in \mathbb{R}^{(N-2M+1) \times (M-L)}$ are the components of signal and noise subspaces, respectively. Then the real-valued HMUSIC estimate of ω_1, denoted by $\hat{\omega}_1$, is obtained by minimizing the orthogonality error between $\mathbf{A}$ and $\mathbf{U}_n$:

$$\hat{\omega}_1 = \arg\min_{\tilde{\omega}} P(\tilde{\omega}) = \arg\min_{\tilde{\omega}} ||\mathbf{A}^T\mathbf{U}_n||_{\text{F}}^2, \tag{12}$$

with $\tilde{\omega}$ being the variable for ω_1, and $||\cdot||_{\text{F}}$ denotes the Frobenius norm. Like the complex-valued HMUSIC algorithm in [5], this kind of estimator is not statistically efficient. To overcome this problem, we introduce Markov optimum weighting. Due to the non-uniqueness of the columns of $\mathbf{U}_n$ (in fact, any linear combination of these vectors still spans the noise subspace), we cannot analyze their statistical properties. Thus, it is necessary to reformulate the HMUSIC cost function with a different but equivalent form as follows:

$$\begin{aligned} \hat{\omega}_1 &= \arg\min_{\tilde{\omega}} ||\mathbf{A}^T\mathbf{U}_n||_{\text{F}}^2 \\ &= \arg\min_{\tilde{\omega}} ||\mathbf{A}^T\mathbf{U}_n\mathbf{U}_n^T||_{\text{F}}^2 \\ &= \arg\min_{\tilde{\omega}} \mathbf{e}^T\mathbf{e}, \end{aligned} \tag{13}$$

where $\mathbf{e}$ denotes the orthogonality error vector of the form $\mathbf{e} = \text{vec}(\mathbf{A}^T\mathbf{U}_n\mathbf{U}_n^T)$ with vec being the vectorization operator. Using the results of [10] and additional manipulation, we derive the first-order perturbation of $\mathbf{e}$ originating from $\mathbf{U}_n\mathbf{U}_n^T$ as:

$$\begin{aligned} \mathbf{e} &\approx -\text{vec}\left(\mathbf{A}^T\mathbf{U}_s\mathbf{\Lambda}_s^{-1}\mathbf{V}_s^T\Delta\mathbf{Y}^T\mathbf{U}_n\mathbf{U}_n^T\right) \\ &= -(\mathbf{U}_n\mathbf{U}_n^T) \otimes (\mathbf{A}^T\mathbf{U}_s\mathbf{\Lambda}_s^{-1}\mathbf{V}_s^T)\text{vec}(\Delta\mathbf{Y}^T) \\ &= -\frac{1}{2}(\mathbf{U}_n\mathbf{U}_n^T) \otimes (\mathbf{A}^T\mathbf{U}_s\mathbf{\Lambda}_s^{-1}\mathbf{V}_s^T)(\mathbf{T}_1 + \mathbf{T}_2)\mathbf{q} \\ &= \mathbf{D}\mathbf{q}, \end{aligned} \tag{14}$$

where $\otimes$ stands for the Kronecker product, $\Delta\mathbf{Y}$ is the perturbation of $\mathbf{Y}$, while $\mathbf{q}$, $\mathbf{T}_1$ and $\mathbf{T}_2$ are defined as

$$\mathbf{q} = [q(0),\ q(1),\ \cdots,\ q(N-1)]^T, \tag{15}$$
$$\mathbf{T}_1 = \left[\mathbf{0}_{(MN') \times M} \quad \mathbf{T}_1'\right], \tag{16}$$
$$\mathbf{T}_2 = \left[\mathbf{T}_2' \quad \mathbf{0}_{(MN') \times M}\right], \tag{17}$$

with $N' = N - 2M + 1$. The $\mathbf{T}_1'$ and $\mathbf{T}_2'$ are defined as

$$\mathbf{T}_1' = \left[\mathbf{E}_1^T\ \cdots\ \mathbf{E}_M^T\right]^T, \quad \mathbf{T}_2' = \left[\mathbf{E}_M^T\ \cdots\ \mathbf{E}_1^T\right]^T, \tag{18}$$

with $\mathbf{E}_i = [\mathbf{0}_{N' \times (i-1)}\ \mathbf{I}_{N' \times N'}\ \mathbf{0}_{N' \times (M-i)}]$, $i = 1, \cdots, M$. Based on the above perturbation eigenanalysis, the Markov optimum weighting matrix is determined as [11]

$$\mathbf{W} = \left[E\left\{\mathbf{e}\mathbf{e}^T\right\}\big|_{\tilde{\omega}=\omega_1}\right]^\dagger \sigma^2 \approx \left(\mathbf{D}\mathbf{D}^T\right)^\dagger, \tag{19}$$

where † stands for pseudoinverse. Then ω_1 is estimated as

$$\hat{\omega}_1 = \arg\min_{\tilde{\omega}} J(\tilde{\omega}), \tag{20}$$

where $J(\tilde{\omega}) = \mathbf{e}^T\mathbf{W}\mathbf{e}$. Noting that $\mathbf{W}$ is a function of the unknown ω_1, the following relaxing procedure is employed:

- Step 1. Set $\mathbf{W}$ as the (ML)-order identity matrix.
- Step 2. Find $\hat{\omega}_1$ by searching for the minimum of (20).
- Step 3. Compute $\mathbf{W}$ using (19).
- Step 4. Repeat Steps 2 and 3 until a stopping criterion is reached.

The above estimator is termed as the OW-HMUSIC method.

2.2. Multi-Pitch Estimation

In the multi-pitch scenario, the signal model becomes

$$x(n) = \sum_{k=1}^{K}\sum_{l=1}^{L} \alpha_{k,l}\cos\left(l\omega_k n + \phi_{k,l}\right) + q(n), \tag{21}$$

with ω_k, $\{\alpha_{k,l}\}$ and $\{\phi_{k,l}\}$ being unknown fundamental frequency, amplitudes and initial phases of the k-th pitch. Here we assume that the number of harmonics, L, is the same for all pitches for simplicity. However, the following approach, when combining with the joint order estimation technique [5], can also deal with the case of different harmonic numbers.

To estimate ω_k, first we construct the same data matrix as (6), whose left singular vectors associated with its $(M - KL)$ smallest singular values span the noise subspace $\mathbf{U}_n$. Then the HMUSIC estimates are got from the K main minima of:

$$P(\tilde{\omega}) = ||\mathbf{A}^T\mathbf{U}_n||_{\mathrm{F}}^2. \quad (22)$$

Taking the HMUSIC estimates as initial values, constructing the weighting matrix of (19) for each of them, and searching for its closest minimum of (20), we can solve the OW-HMUSIC estimates of the multi-pitch fundamental frequencies in an iterative way similar to the single-pitch estimation.

2.3. Performance Analysis

The variance of the OW-HMUSIC approach is derived as follows. From the above development, we can find that the single-pitch and multi-pitch estimation are both unconstrained optimization problems with the cost function $J(\tilde{\omega})$ of (20). Defining $\xi_k = \frac{\partial \mathbf{e}}{\partial \tilde{\omega}}\big|_{\tilde{\omega}=\omega_k}$, $k = 1, \cdots, K$, and applying the variance formula for unconstrained optimization problems at sufficiently small noise conditions [12], we compute the variance of the OW-HMUSIC frequency estimates as

$$\mathrm{var}(\hat{\omega}_k) = \xi_k^T \mathbf{W}\mathbf{\Phi}\mathbf{W}\xi_k / (\xi_k^T\mathbf{W}\xi_k)^2, k = 1, \cdots, K, \quad (23)$$

where $\mathbf{\Phi}$ is the covariance matrix of $\mathbf{e}$:

$$\begin{aligned} \mathbf{\Phi} &= \mathrm{cov}(\mathbf{e})|_{\tilde{\omega}=\omega_k} = E\{\mathbf{e}\mathbf{e}^T\}|_{\tilde{\omega}=\omega_k} \\ &\approx \sigma^2 (\mathbf{D}\mathbf{D}^T)|_{\tilde{\omega}=\omega_k}. \end{aligned} \quad (24)$$

3. SIMULATION RESULTS

In this section, we perform Monte Carlo simulations to evaluate the fundamental frequency estimation performance of the proposed approach. The estimation accuracy is evaluated using the root mean square error (RMSE), defined as $\mathrm{RMSE} = \sqrt{\frac{1}{SK}\sum_{k=1}^{K}\sum_{s=1}^{S}(\hat{\omega}_k^{(s)} - \omega_k)^2}$, with ω_k and $\hat{\omega}_k^{(s)}$ being the true fundamental frequency and its estimate, respectively, and S being the number of trials. We use the number of iterations as the stopping criterion in the OW-HMUSIC algorithm, which is assigned as 3. The row number of the data matrix is set as $M = \lfloor 0.4\ N \rfloor$, which is found empirically to result in good performance. All the results provided are averages of 1000 independent runs.

First, we provide an example of single-pitch estimation. The harmonic signal consists of $L = 4$ sinusoids with fundamental frequency of $\omega_1 = 0.5$. The parameter setting is listed in Table 1. Fig. 1 shows the RMSE of the NLS [4], HMUSIC, OW-HMUSIC methods and CRLB, with $N = 50$ and 100. It is seen that both NLS and OW-HMUSIC estimates attain the optimum accuracy when $\mathrm{SNR} \geq 5$ dB. For the OW-HMUSIC algorithm, the empirical RMSE value also agrees well with its theoretical calculation of (23), which is equal to the CRLB. Furthermore, there is about $3 - 4$ dB gap between the RMSE of the HMUSIC scheme and CRLB.

l	Frequency	Amplitude	Initial Phase
1	0.5	2.0	1
2	1.0	1.5	2
3	1.5	2.5	3
4	2.0	4.0	4

Table 1. Simulation Setting of Single-Pitch Estimation

The next example is about multi-pitch estimation. The harmonic signal consists of $K = 2$ pitches, each with $L_k = 2$ tones. The parameter setting is listed in Table 2, and Fig. 2 shows the RMSE results with $N = 50$ and 100. From these figures, we can see that the RMSE of NLS method keeps nearly constant with SNR, which is similar to [4]. Although the OW-HMUSIC performance cannot reach CRLB in the multi-pitch estimation, it is superior to the HMUSIC scheme by $3 - 4$ dB. In addition, when the data length is larger, the HMUSIC scheme performs better with respect to CRLB and threshold SNR.

k	l_k	Frequency	Amplitude	Initial Phase
1	1	0.3	2.0	1
	2	0.6	1.0	2
2	1	0.5	2.0	3
	2	1.0	1.0	4

Table 2. Simulation Setting of Two-Pitch Estimation

4. CONCLUSION

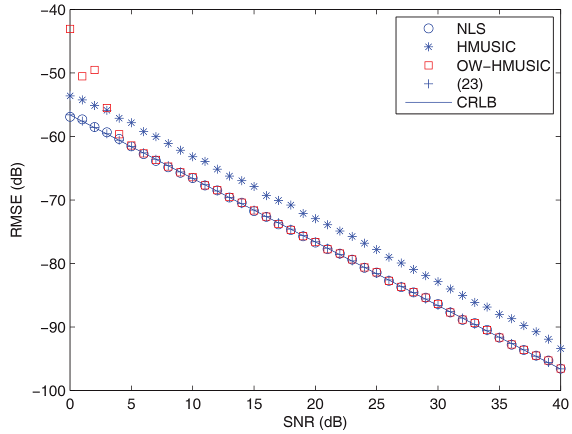
The OW-HMUSIC algorithm for fundamental frequency estimation of real sinusoids is proposed. In our approach, the Markov optimum weighting is utilized in minimizing the orthogonality error, which is derived from the perturbation analysis of orthogonality error. Simulation results show that the proposed method improves the accuracy of the conventional HMUSIC scheme, and can attain CRLB for single-pitch estimation. Further works include harmonic order selection, study of closely-spaced pitches in the multi-pitch scenario, and the application in speech and audio signal processing.

5. REFERENCES

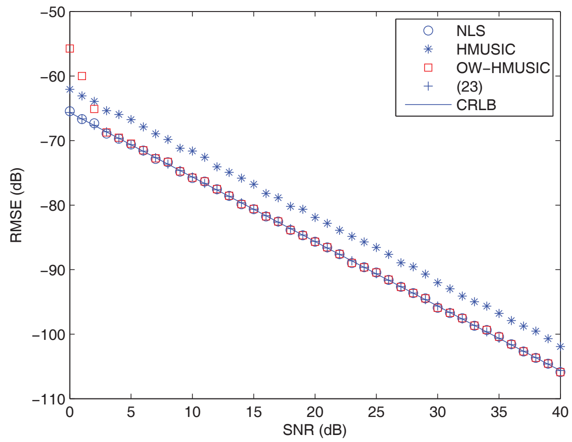
[1] F. K.W. Chan, H.C. So, W.H. Lau and C.F. Chan, "Efficient approach for sinusoidal frequency estimation of gapped data," *IEEE Signal Processing Letters*, vol.17, no.6, pp.611-614, Jun. 2010.

[2] H. Li, P. Stoica and J. Li, "Computationally efficient parameter estimation for harmonic sinusoidal signals," *Signal Processing*, vol.80, no.9, pp.1937-1944, Sep. 2000.

[3] P. Stoica and A. Nehorai, "Statistical analysis of two non-linear least-squares estimators of sine waves parameters in the colored noise," in *Proc. ICASSP*, vol.4,

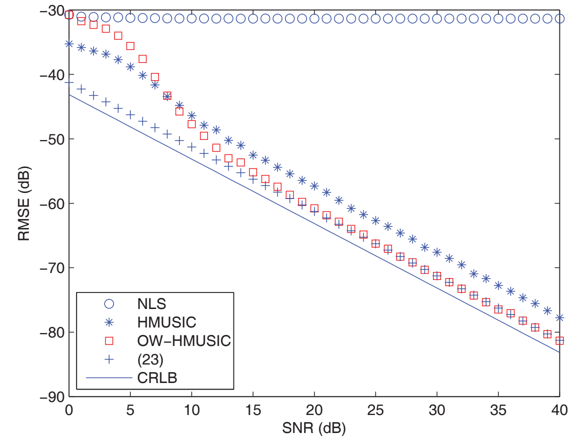


(a)

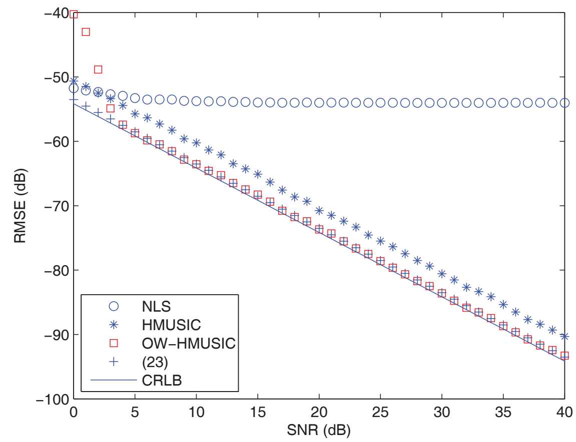


(b)

Fig. 1. RMSEs of single-pitch estimation versus SNR for: (a) $N = 50$ and (b) $N = 100$



(a)



(b)

Fig. 2. RMSEs of two-pitch estimation versus SNR for: (a) $N = 50$ and (b) $N = 100$

pp.2408-2411, Apr. 11-14, 1988, New York, New York, USA.

[4] M.G. Christensen, P. Stoica, A. Jakobsson and S.H. Jensen, “Multi-pitch estimation,” *Signal Processing*, vol.88, no.4, pp.972-983, Apr. 2008.

[5] M.G. Christensen, A. Jakobsson and S.H. Jensen, “Joint high-resolution fundamental frequency and order estimation,” *IEEE Transactions on Audio, Speech, and Language Processing*, vol.15, no.5, pp.1635-1644, Jul. 2007.

[6] S.M. Kay, *Fundamentals of Statistical Signal Processing - Estimation Theory*, Englewood Cliffs, N.J.: PTR Prentice-Hall, 1993.

[7] P. Stoica, A. Eriksson and T. Soderstrom, “Optimally weighted MUSIC for frequency estimation,” *SIAM J. on Matrix Analysis and Applications*, vol.16, no.3, pp.811-827, Jul. 1995.

[8] K. Mahata, “Subspace fitting approaches for frequency estimation using real-valued data,” *IEEE Transactions on Signal Processing*, vol.53, no.8, pp.3099-3110, Aug. 2005.

[9] B. Porat, *Digital Processing of Random Signals: Theory and Methods*, Englewood Cliffs, N.J.: Prentice Hall, 1994.

[10] J. Liu, X. Liu and X. Ma, “First-order perturbation analysis of singular vectors in singular value decomposition,” *IEEE Transactions on Signal Processing*, vol.56, no.7, pp.3044-3049, Jul. 2008.

[11] T. Soderstrom and P. Stoica, *System Identification*, Englewood Cliffs, N.J.: Prentice-Hall International, 1989.

[12] H.C. So, “Source Localization: Algorithms and Analysis,” *Handbook of Position Location: Theory, Practice and Advances*, Chapter 2, S.A. Zekavat and M. Buehrer, Eds., Wiley-IEEE Press, 2011.