ANALYSIS OF THE EDGE-EFFECTS IN FREQUENCY-DOMAIN TDOA ESTIMATION

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ABSTRACT

Passive estimation of the Time-Difference of Arrival (TDOA) of a common signal at two (or more) sensors is a fundamental problem in signal processing, with applications mainly in emitter localization. A common approach to TDOA estimation is the maximization of the sample cross-correlation between the received signals. For various reasons, this correlation is sometimes computed via the frequency-domain, following a Discrete Fourier Transform (DFT) of the signals - in which case the linear correlation is essentially replaced with a cyclic correlation. Although the two computations differ merely by some relatively short "edge-effects", these edge-effects can entail more impact than commonly predicted by their relative (usually negligible) effective durations. In this work we analyze the mean square TDOA estimation error resulting from the use of cyclic instead of linear correlations, showing that for some signals the loss can be more severe than what would be predicted by a simple linear dependence on the delay value.

***Index Terms*—** TDOA, TOA, Time-Delay Estimation, Cyclic Correlation, Edge-Effects, End-Effects.

1. INTRODUCTION

Estimation of the Time-Difference of Arrival (TDOA) of a signal intercepted at two sensors is a fundamental, well-studied problem in signal processing. The estimated time difference if often used in the context of localizing the transmitting source, be it in the context of acoustic [5] (or underwater acoustic) signals or electromagnetic signals [2] [3] - but also in other applications (e.g., synchronization). Classical estimation approaches, such as the Generalized Cross-Correlation (GCC) [1] and performance analysis in terms of bounds [3] or small-errors analysis [2] [4] of specific estimators have been proposed and studied over the past three decades.

A common (perhaps the most simple) approach to TDOA estimation is to search for the peak of the ordinary sample-cross-correlation between the two received signals [1], [4]. The sample-cross-correlation can be conveniently computed in time-domain, however in certain scenarios which might involve frequency-domain pre-processing of the signals, or more complicated signal-models (such as a multipath model [5]), it becomes more convenient to maximize the time-domain sample-cross-correlation via frequency-domain delay-matching, following Discrete Fourier Transformation (DFT) of the observed signals' samples. The two operations are nearly equivalent, with the exception of some "edge-effects" which are due to the fact that frequency-domain multiplication of DFTs is equivalent to *cyclic*, rather than to linear time-domain convolution.

These "edge-effects" are usually dismissed as negligible, and are regarded as being equivalent to some additive "noise" (or "interference"). Such an approach is advocated, e.g., in [6], where the equivalent Signal to Interference Ratio (SIR) is roughly quantified as the ratio between the observation length and the true TDOA between the received signals (which is the length of the "edge"). Thus, when the observation length is sufficiently long with respect to the TDOA, these effects are ignored, especially when "true" additive noise at a lower SNR is present. However, as we shall show in this paper, it turns out that this point of view might be too optimistic in practice, since considerably larger estimation errors can be incurred in frequency-domain processing due to the differences between linear and cyclic cross-correlations.

Our goal in this work is therefore to take a closer look at the implied differences between using a cyclic and a linear cross-correlation for TDOA estimation. Our detailed error analysis shows that the sensitivity of the TDOA estimation error to the associated edge-effects is generally much more involved than predicted by the simplified "equivalent SIR" model, and is heavily dependent on the particular signal's autocorrelation. We derive closed-form expressions, corroborated by simulation results, which demonstrate that for certain signals the mean square error (MSE) of the resulting TDOA estimates can be significantly larger than what would be anticipated by assuming the "equivalent" SIR effect.

2. THE SIGNAL MODEL AND THE SAMPLE CROSS-CORRELATIONS

We assume that the continuous-time source signal $s(t)$ is a stochastic wide-sense stationary (WSS) bandlimited zero-mean Gaussian process, possibly contaminated, at each sensor, by additive, WSS bandlimited zero-mean Gaussian noise processes $v_1(t)$ and $v_2(t)$,

$$\begin{aligned} x_1(t) &= s(t) + v_1(t) \\ x_2(t) &= as(t-d) + v_2(t), \end{aligned} \tag{1}$$

where a is the relative signal gain between the sensors and d is the TDOA. For simplicity of the exposition we assume that all signals are real-valued.

We further denote the respective (true) continuous-time correlations as $R_s(\tau) = E[s(t+\tau)s(t)]$ and $R_i(\tau) = E[v_i(t+\tau)v_i(t)]$ (for $i = 1, 2$), and assume that all three signals are mutually uncorrelated.

The received signals are usually sampled at their Nyquist rate, which we shall assume for simplicity to be 1 - implying that the continuous-time signals are bandlimited between $-\frac{1}{2}$ and $\frac{1}{2}$. Note, however, that the use of an interpolated version of the sample-correlation is equivalent to the use of the sample-correlation of an interpolated version of the signals. Obviously, computing the former (namely, correlating the signals and then interpolating the resulting correlation sequence) is considerably preferable in terms of computational load over computing the latter (namely, over interpolating the signals and then correlating). However, for analyzing the difference between the cyclic and linear correlations, it is more convenient to assume that the signals are initially over-sampled by a factor of L.

We therefore assume that the signals are sampled at sample rate L, in sampling intervals of $\Delta = \frac{1}{L}$, over an observation period T - yielding $N = L \cdot T$ samples for each signal. We further assume that the true delay d is an integer multiple of the sampling interval, such that $d = m \cdot \Delta$, where m is a positive integer. We shall assume that $m << N$, namely that $d << T$, the true TDOA is much smaller than the observation interval, an assumption which is commonly justified. The continuous-time model (1) thereby assumes its discrete-time version (denoting $x_1[n] \triangleq x_1(n\Delta)$, etc.):

$$\begin{aligned} x_1[n] &= s[n] + v_1[n], \qquad n = 0, 1, ...N-1 \\ x_2[n] &= as[n-m] + v_2[n], \end{aligned} \tag{2}$$

and we shall further denote $R_s[\ell] \triangleq R_s(\ell\Delta)$ and $R_i[\ell] \triangleq R_i(\ell\Delta)$ $(i = 1, 2)$ as the sampled versions of the true correlations.

The linear sample-cross-correlation at lag ℓ is given by

$$\widehat{R}[\ell] = \frac{1}{N} \sum_{n=0}^{N-1} x_2[n+\ell] x_1[n], \tag{3}$$

where we assume that $x_2[n]$ is observed also in the necessary small vicinity outside the $[0, N-1]$ observation interval, such that $x_2[n+\ell]$ is available for the relevant values of ℓ.

Conversely, the cyclic sample-cross-correlation at lag ℓ is given by

$$\widehat{\widehat{R}}[\ell] = \frac{1}{N} \sum_{n=0}^{N-1} x_2[n \oplus \ell] x_1[n], \tag{4}$$

where the operator '$\oplus$' denotes addition modulu N, such that if $n+\ell$ is in the range $[0, N-1]$, we have $n \oplus \ell = n + \ell$, but if $n + \ell \geq N$ or $n + \ell < 0$, we have $n \oplus \ell = n + \ell - N$ or $n \oplus \ell = n + \ell + N$, respectively. It is well-known that $\widehat{\widehat{R}}[\ell]$ is also the extent of "delay-matched" correlation in the frequency-domain, namely

$$\widehat{\widehat{R}}[\ell] = \frac{1}{N^2} \sum_{k=0}^{N-1} \tilde{x}_2[k] \cdot e^{j2\pi k\ell/N} \cdot \tilde{x}_1^*[k] \tag{5}$$

where $\tilde{x}_i[k] = \sum_{n=0}^{N-1} x_i[n] \cdot e^{-j2\pi kn/N}$, $k = 0, ..., N-1$, $i = 1, 2$ are the DFTs of the received signals.

3. TDOA ESTIMATION ERROR ANALYSIS

The TDOA estimate is obtained by finding the maximum of the implied continuous-time versions of either $\widehat{R}[\ell]$ or $\widehat{\widehat{R}}[\ell]$. Using a small-errors assumption and assuming that Δ is sufficiently small, the maximizer can be obtained from a parabolic interpolation in the vicinity of $\widehat{R}[m]$ (or $\widehat{\widehat{R}}[m]$), namely

$$\widehat{d} = \left(m + \frac{1}{2} \cdot \frac{\widehat{R}[m+1] - \widehat{R}[m-1]}{2\widehat{R}[m] - \widehat{R}[m+1] - \widehat{R}[m-1]} \right) \cdot \Delta \tag{6}$$

(or with $\widehat{\widehat{R}}[\ell]$ substituting $\widehat{R}[\ell]$). We are therefore interested in analyzing the estimation error

$$\epsilon = \widehat{d} - d = \frac{(\widehat{R}[m+1] - \widehat{R}[m-1])/2\Delta}{(2\widehat{R}[m] - \widehat{R}[m+1] - \widehat{R}[m-1])/\Delta^2}. \tag{7}$$

Note that the numerator is an approximation of the first derivative of the cross-correlation at its peak (which is nominally zero), whereas the denominator is an approximation of the second derivative of the same (which is nominally a negative number, proportional to the mean-square bandwidth of the signal- e.g., [2]). We would therefore assume, under a first-order small-errors framework, that the denominator is a deterministic number taking its mean value, such that the only randomness is in the numerator. As we shall see in our subsequent simulation demonstration, this is a realistic assumption under the studied conditions.

In order to isolate the effect of errors induced by the use of cyclic correlations from errors induced by additive noise, we will first address a noiseless scenario, where the model (2) is reduced into $x_1[n] = s[n]$ and $x_2[n] = as[n-m]$. In this case we have, for the cyclic autocorrelation (for $\ell \geq 0$)

$$\begin{aligned} \widehat{\widehat{R}}[\ell] &= \frac{1}{N} \sum_{n=0}^{N-1} x_2[n \oplus \ell] x_1[n] = \frac{a}{N} \sum_{n=0}^{N-1} s[(n \oplus \ell) - m] s[n] \\ &= \frac{a}{N} \sum_{n=0}^{N-1-\ell} s[n+\ell-m] s[n] + \frac{a}{N} \sum_{n=N-\ell}^{N-1} s[n+\ell-N-m] s[n]. \end{aligned} \tag{8}$$

Substituting (8) into (7), the numerator takes the form

$$\frac{1}{2\Delta}\left(\widehat{\widetilde{R}}[m+1]-\widehat{\widetilde{R}}[m-1]\right)=\frac{a}{2T}\left(\sum_{n=0}^{N-m-2}s[n+1]s[n]\right.$$
$$-\sum_{n=0}^{N-m}s[n-1]s[n]+\sum_{n=N-m-1}^{N-1}s[n+1-N]s[n]$$
$$\left.-\sum_{n=N-m+1}^{N-1}s[n-1-N]s[n]\right)=\frac{a}{2T}\left(-\underbrace{s[0]s[-1]}_{Q_1}\right.$$
$$-\underbrace{s[N-m]s[N-m-1]}_{Q_2}+\underbrace{\sum_{n=0}^{m}s[n-m]s[n-m-1+N]}_{Q_3}$$
$$\left.-\underbrace{\sum_{n=0}^{m-2}s[n-m]s[n-m+1+N]}_{Q_4}\right). \quad (9)$$

We are interested in the second moment of this expression, which would lead us to the MSE. To this end, we first need second moments and joint second moments of the terms marked Q_1, Q_2, Q_3 and Q_4 in (9). We shall exploit the Gaussianity of the signal (using the well-known expression for the mean of the product of four jointly-Gaussian zero-mean random variables) and would also assume that T is sufficiently large with respect to the effective correlation length of the signal, such that samples which are roughly N samples away are uncorrelated (we shall refer to this assumption as the "distant decorrelation" assumption). We thus obtain the following second moments:

$$E[Q_1^2]=E[Q_2^2]=R_s^2[0]+2R_s^2[1] \quad (10)$$

and (under the "distant decorrelation" assumption):

$$E[Q_3^2]=\sum_{\ell=-m}^{m}(m+1-|\ell|)R_s^2[\ell]\ ,\ E[Q_4^2]=\sum_{\ell=-m+2}^{m-2}(m-1-|\ell|)R_s^2[\ell], \quad (11)$$

as well as the joint moments:

$$E[Q_1Q_2]=R_s^2[1],$$
$$E[Q_1Q_3]=E[Q_1Q_4]=E[Q_2Q_3]=E[Q_2Q_4]=0, \quad (12)$$

and

$$E[Q_3Q_4]=\sum_{n_1=0}^{m}\sum_{n_2=0}^{m-2}R_s[n_1-n_2]R_s[n_1-n_2-2]$$
$$=\sum_{\ell=-m+2}^{m}q[\ell]R_s[\ell]R_s[\ell-2], \quad (13)$$

with

$$q[\ell]=\begin{cases}m-1+\ell & \ell\le 0\\ m-1 & 0\le\ell\le 2\\ m+1-\ell & 2\le\ell\end{cases}. \quad (14)$$

Collecting all terms we obtain (and define)

$$Q[m]\triangleq E\left[(-Q_1-Q_2+Q_3-Q_4)^2\right]=2(m+1)R_s^2[0]$$
$$-2(m+4)R_s^2[1]-4(m-1)R_s[0]R_s[2]+4R_s^2[m-1]+2R_s^2[m]$$
$$+4\sum_{\ell=1}^{m-2}\left((m-\ell)R_s[\ell]-(m-\ell-1)R_s[\ell+2]\right)R_s[\ell] \quad (15)$$

The mean of the denominator in the error expression (7) is given, using the "distant decorrelation" assumption and (8), by

$$E\left[\frac{2\widehat{\widetilde{R}}[m]-\widehat{\widetilde{R}}[m-1]-\widehat{\widetilde{R}}[m+1]}{\Delta^2}\right]=\frac{N-m}{N}\cdot\frac{2a(R_s[0]-R_s[1])}{\Delta^2}, \quad (16)$$

and therefore, substituting (9), (15) and (16) for the second moment of ϵ in (7), the mean square TDOA estimation error is given in this case by

$$E[\epsilon^2]=\frac{Q[m]/4T^2}{(1-\frac{m}{N})^2((2R_s[0]-2R_s[1])/\Delta^2)^2}. \quad (17)$$

Using some tedious derivations for limit analysis of $Q[m]$ (omitted from here due to space limitations), it can be shown that as Δ becomes sufficiently small, this expression converges to

$$E[\epsilon^2]\xrightarrow{\Delta\to 0}\frac{3R_s^2(0)+d\int_{-d}^{d}\left(1-\frac{|\tau|}{d}\right)\dot{R}_s^2(\tau)d\tau}{T^2\ddot{R}_s^2(0)}\cdot, \quad (18)$$

where $\dot{R}_s(\tau)$ and $\ddot{R}_s(\tau)$ denote the first and second derivatives (resp.) of $R_s(\tau)$. Note that this expression entails strong dependence on the signal's autocorrelation function, and more so on its (squared) first derivative. Once the true TDOA d is sufficiently large such that the integral becomes insensitive to d, the MSE becomes linear in d (assuming that T is fixed), but for smaller values of d this dependence can be quadratic or stronger - as we demonstrate in simulation later on.

In order to compare to the "equivalent additive noise" model, we now analyze the sensitivity to additive noise, isolating this effect by assuming a linear sample-correlation. We shall assume that the noise exists only in one of the received signals, say $x_1[n]$, just like the edge-effect when using a cyclic correlation. Thus, we assume that the second signal is received with unit gain ($a=1$) and no noise, so $x_1[n]=s[n]+v_1[n]$ and $x_2[n]=s[n-m]$. The linear sample correlation is then given by

$$\widehat{R}[\ell]=\frac{1}{N}\sum_{n=0}^{N-1}x_2[n+\ell]x_1[n]$$
$$=\frac{1}{N}\sum_{n=0}^{N-1}s[n+\ell-m](s[n]+v_1[n]). \quad (19)$$

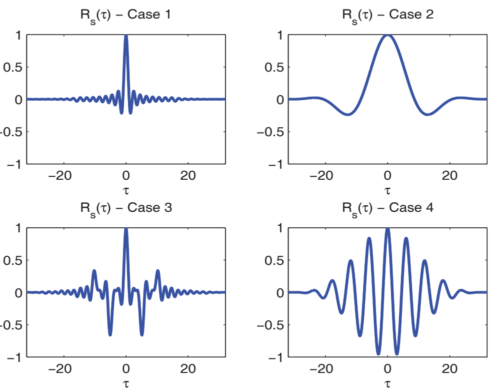

Fig. 1. Signal correlations for the four experiments.

Substituting into the numerator of (7) we get

$$\frac{1}{2\Delta}\left(\widehat{R}[m{+}1]-\widehat{R}[m{-}1]\right)=\frac{1}{2T}\left(\sum_{n=0}^{N-1}s[n{+}1]s[n]\right.$$

$$\left.-\sum_{n=0}^{N-1}s[n{-}1]s[n]+\sum_{n=0}^{N-1}s[n{+}1]v_1[n]-\sum_{n=0}^{N-1}s[n{-}1]v_1[n]\right)$$

$$=\frac{1}{2T}\left(\underbrace{s[N]s[N-1]}_{P_1}-\underbrace{s[-1]s[0]}_{P_2}\right.$$

$$\left.+\underbrace{\sum_{n=0}^{N-1}s[n{+}1]v_1[n]}_{P_3}-\underbrace{\sum_{n=0}^{N-1}s[n{-}1]v_1[n]}_{P_4}\right). \quad (20)$$

Once again, we need second moments of the terms marked $P_1,...,P_4$ in (20). These can be derived in a similar fashion to the $Q_1,...,Q_4$ terms above (exploiting the Gaussianity and the "distant decorrelation" assumption, as well as the fact that the signal and noise are mutually uncorrelated), leading to

$$P\triangleq E\left[(P_1-P_2+P_3-P_4)^2\right]=2R_s^2[0]+2R_s^2[1]$$

$$+\sum_{\ell=-N+1}^{N-1}(N-|\ell|)(2R_s[\ell]-2R_s[\ell+2])R_1[\ell]. \quad (21)$$

The mean of the denominator in the error expression (7) is given in this case by

$$E\left[\frac{2\widehat{R}[m]-\widehat{R}[m-1]-\widehat{R}[m+1]}{\Delta^2}\right]=\frac{2(R_s[0]-R_s[1])}{\Delta^2}, \quad (22)$$

and therefore, substituting (20), (21) and (22), the MSE is given by

$$E[\epsilon^2]=\frac{P/4T^2}{((2R_s[0]-2R_s[1])/\Delta^2)^2}, \quad (23)$$

and taking the limit of P for small values of Δ we obtain

$$E[\epsilon^2]\xrightarrow{\Delta\to 0}\frac{R_s^2(0)-T\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)\ddot{R}_s(\tau)R_1(\tau)d\tau}{T^2\ddot{R}_s^2(0)}, \quad (24)$$

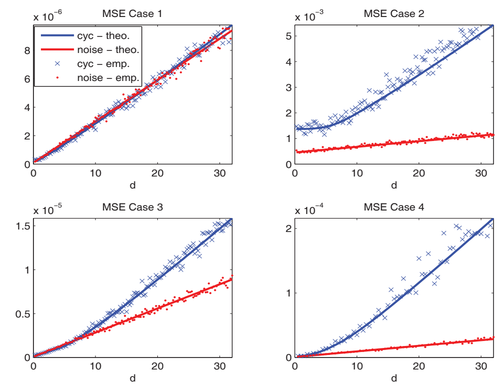

Fig. 2. MSE in the four experiments: empirical vs. theoretically predicted values, cyclic correlation vs. "equivalent" additive noise errors.

4. SIMULATION RESULTS

Our goal in the simulation experiments is two-fold: We demonstrate that the empirical MSE in the TDOA estimation in the two considered scenarios, namely noiseless cyclic correlation and noisy linear correlation, match their theoretically predicted values of (18) and (24) (respectively); And we examine the validity of the common assumption that the error induced by cyclic correlation processing is equivalent to the error induced by noise at an SNR equivalent to the ratio T/d.

In Fig.1 we present the four different correlation functions $R_s(\tau)$ which were used in the four experiments. In each experiment we generated observation signals (with the prescribed correlation) of length $T=1024$, at a sampling rate $L=8$ (so $\Delta=\frac{1}{8}$ and $N=8192$) and averaged the MSE over 400 independent trials for each tested TDOA, ranging from $d=2\Delta=0.25$ to $d=32$. For each value of d we estimated the TDOA using cyclic correlations in noiseless reception, and then using linear correlations with noise present in the first signal at SNR given by N/d. Evidently, the empirical results match the respective theoretical predictions. However, only in Case 1 the MSE due to the cyclic correlation processing is comparable to the anticipated performance with "equivalent" SNR. In all the other cases, the errors induced by the cyclic correlations are significantly higher than anticipated from the simple "equivalent SNR" model.

5. REFERENCES

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