CORRELOGRAM FOR UNDERSAMPLED DATA: BIAS AND VARIANCE ANALYSIS

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ABSTRACT

This paper studies the correlogram spectrum estimation method for the case that only a subset of the Nyquist samples is available. The method is able to estimate the spectrum from undersampled data. The bias and variance of the estimator are derived. We also show that there is a tradeoff between the accuracy of the estimation and the frequency resolution. The asymptotic behavior of the estimator is also investigated, and it is proved that this method is a consistent estimator.

***Index Terms*—**Correlogram, undersampling, consistency.

1. INTRODUCTION

Spectrum estimation from a finite set of noisy measurements is a classical problem with wide applications in communications, astronomy, seismology, radar, sonar signal processing, etc. In practice, the rate at which the measurements are collected can be restricted. However, it is viable to make spectrum estimation from measurements obtained at a rate lower than the Nyquist rate. In [1] and [2], authors have shown that for signals with sparse Fourier representations, it is possible to estimate the Fourier coefficients using a subset of the *Nyquist samples* (samples obtained at the Nyquist rate). In [3] and [4], the possibility of recovering signals sparse in the discrete-time Fourier transform (DTFT) domain from compressive samples obtained at a rate lower than the Nyquist rate has been demonstrated.

In [5], authors have considered power spectral density (PSD) estimation based on the autocorrelation matrices of the data. We refer to this method as the correlogram for undersampled data. This method is able to reconstruct the spectrum from a subset of the Nyquist samples while it does not require the signal to be sparse.

In this paper, the behavior of the correlogram for undersampled data is investigated. We show that there is a tradeoff between the estimation accuracy and the frequency resolution. The method is reviewed in Section 2 with some modifications. In Section 3, the bias of the estimator is computed and it is shown that the estimation is unbiased for any signal length. In Section 4, the covariance matrix of the estimator is derived, and it is proved that the estimation variance tends to zero asymptotically. Therefore, the correlogram for undersampled data is a consistent estimator. Finally, the tradeoff between the estimation variance and the frequency resolution is illustrated in Section 5.

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2. CORRELOGRAM FOR UNDERSAMPLED DATA

Consider a wide-sense stationary (WSS) stochastic process $x(t)$ bandlimited to $W/2$ with PSD $P_x(\omega)$. Let $x(t)$ be sampled using the multi-coset (MC) sampler as described in [5]. Samples are collected by a multi-channel system. The i-th channel ($1 \leq i \leq q$) samples $x(t)$ at the time instants $t = (nL + c_i)T$ for $n = 0,\ 1, \dots,\ N-1$, where N is the number of samples obtained from each channel, T is the Nyquist period ($T = 1/W$), L is a suitable integer, and $q < L$ is the number of sampling channels. The time offsets c_i are distinct, random positive integer numbers less than L. Let the output of the i-th channel be denoted by $y_i(n) = x\left((nL + c_i)T\right)$. The i-th channel can be easily implemented by a system that shifts $x(t)$ by $c_i T$ seconds and then samples uniformly at a rate of $1/LT$ Hz. The samples obtained in this manner form a subset of the Nyquist samples. The average sampling rate is q/LT Hz, and it is less than the Nyquist rate since $q < L$.

Given the MC samples, the PSD of the signal can be estimated by transforming the output sequences $y_i(n) = x\left((nL + c_i)T\right)$ into a system of frequency domain equations. Let $Y_i(e^{j\omega \frac{L}{W}})$ and $X(\omega)$ denote the Fourier transform of $y_i(n)$ and $x(t)$, respectively. Then, the system $\boldsymbol{z}(\omega) = \boldsymbol{\Gamma}\boldsymbol{s}(\omega)$ holds [5], where $\boldsymbol{\Gamma} \in \mathbb{C}^{q\times L}$, $\boldsymbol{z}(\omega) = [z_1(\omega),\ z_2(\omega),\ \dots,\ z_q(\omega)]^T \in \mathbb{C}^{q\times 1}$, $\boldsymbol{s}(\omega)=[s_1(\omega),\ s_2(\omega),\ \dots,\ s_L(\omega)]^T \in \mathbb{C}^{L\times 1}$, and $(\cdot)^T$ is the transposition. The elements of $\boldsymbol{z}(\omega)$, $\boldsymbol{\Gamma}$, and $\boldsymbol{s}(\omega)$ are given by $z_i(\omega) = e^{-j\frac{c_i}{W}\omega} Y_i(e^{j\omega\frac{L}{W}}) I_{[-\frac{\pi W}{L}, \frac{\pi W}{L})}$, $[\boldsymbol{\Gamma}]_{i,l} = \frac{W}{L} e^{-j\frac{2\pi}{L} c_i m_l}$, $s_l(\omega) = X(\omega - 2\pi\frac{W}{L} m_l) I_{[-\frac{\pi W}{L}, \frac{\pi W}{L})}$ for $1 \leq i \leq q$, $1 \leq l \leq L$, and $m_l = -\frac{1}{2}(L+1) + l$, where L is an odd number and $I_{[\cdot)}$ represents the indicator function. Let $\boldsymbol{R_z} \in \mathbb{C}^{q\times q}$ and $\boldsymbol{R_s} \in \mathbb{C}^{L\times L}$ be the autocorrelation matrices of $\boldsymbol{z}(\omega)$ and $\boldsymbol{s}(\omega)$. Then, it can be found that

$$\boldsymbol{R_z} \triangleq \lim_{N\to\infty} \frac{1}{N} \int_{-\pi\frac{W}{L}}^{\pi\frac{W}{L}} E\{\boldsymbol{z}(\omega)\boldsymbol{z}^H(\omega)\} d\omega = \boldsymbol{\Gamma}\boldsymbol{R_s}\boldsymbol{\Gamma}^H \quad (1)$$

where $(\cdot)^H$ and $E\{\cdot\}$ stand for the Hermitian transposition and the expectation, respectively. Consider partitioning the

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bandwidth of $x(t)$ into L equal segments. It is shown in [5] that the diagonal elements of $\boldsymbol{R_s}$ represent the average power within these spectral segments, and the off-diagonal elements are zeros. Thus, (1) can be rewritten as

$$u_k \triangleq [\boldsymbol{R_z}]_{a,b} = \left(\frac{W}{L}\right)^2 \sum_{l=1}^{L} e^{-j\frac{2\pi}{L}(c_a-c_b)m_l}[\boldsymbol{R_s}]_{l,l}. \quad (2)$$

The l-th diagonal element of $\boldsymbol{R_s}$, i.e., $[\boldsymbol{R_s}]_{l,l}$, corresponds to the average power within the spectral segment $\left[\pi W - 2\pi\frac{W}{L}l,\ \pi W - 2\pi\frac{W}{L}(l-1)\right)$. Since $\boldsymbol{R_z}$ is a Hermitian matrix with equal diagonal elements, it is sufficient to let the indices a and b just refer to the elements of the upper triangle and the first diagonal element of $\boldsymbol{R_z}$. Therefore, there are $Q = \frac{q(q-1)}{2} + 1$ equations in (2) ($1 \leq k \leq Q$). The matrix-vector form of (2) can be written as $\boldsymbol{u} = \boldsymbol{\Psi v}$, where $\boldsymbol{v} = [v_1,\ v_2,\ \ldots,\ v_L]^T \in \mathbb{R}^{L\times 1}$ consists of the diagonal elements of $\boldsymbol{R_s}$, $\boldsymbol{\Psi} \in \mathbb{C}^{Q\times L}$ and $\boldsymbol{u} = [u_1,\ u_2,\ \ldots,\ u_Q]^T \in \mathbb{C}^{Q\times 1}$. Let $u_1 = [\boldsymbol{R_z}]_{1,1}$ and $u_2, \ldots, u_Q$ correspond to the elements of the upper triangle of $\boldsymbol{R_z}$. The elements of $\boldsymbol{\Psi}$ are given by $[\boldsymbol{\Psi}]_{k,l} = (W/L)^2 e^{-j\frac{2\pi}{L}(c_a-c_b)m_l}$. Since the elements of $\boldsymbol{v}$ are real-valued, the number of equations in $\boldsymbol{u} = \boldsymbol{\Psi v}$ can be doubled by solving $\breve{\boldsymbol{u}} = \breve{\boldsymbol{\Psi}}\boldsymbol{v}$ where $\breve{\boldsymbol{u}} \in \mathbb{R}^{2Q\times 1}$ and $\breve{\boldsymbol{\Psi}} \in \mathbb{R}^{2Q\times L}$ are defined as $\breve{\boldsymbol{u}} \triangleq [Re(\boldsymbol{u}),\ Im(\boldsymbol{u})]^T$ and $\breve{\boldsymbol{\Psi}} \triangleq [Re(\boldsymbol{\Psi}),\ Im(\boldsymbol{\Psi})]^T$. Suppose $\breve{\boldsymbol{\Psi}}$ is full rank and $2Q \geq L$. Then, $\boldsymbol{v}$ can be determined using the pseudoinverse of $\breve{\boldsymbol{\Psi}}$ as

$$\boldsymbol{v} = (\breve{\boldsymbol{\Psi}}^T\breve{\boldsymbol{\Psi}})^{-1}\breve{\boldsymbol{\Psi}}^T\breve{\boldsymbol{u}}. \quad (3)$$

The elements of $\breve{\boldsymbol{u}}$ are comprised of the elements of $\boldsymbol{R_z}$. It is shown in [5] that $\boldsymbol{R_z}$ can be estimated from a finite number of samples as

$$[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{a,b} = 2\pi\frac{W}{NL}\sum_{n=0}^{N-1} y_a\left(n-\frac{c_a}{L}\right)y_b^*\left(n-\frac{c_b}{L}\right) \quad (4)$$

where $(\cdot)^*$ denotes the conjugate of a complex number. The fractional delays c_a/L and c_b/L can be implemented by fractional delay filters such as the Lagrange interpolator that is a finite impulse response (FIR) filter [6]. FIR fractional delay filters perform the best when the total delay is approximately equal to half of the order of the filter [7]. This can be achieved by adding a suitable integer delay D to the fractional delays (D cancels out in (4)). Then, we can rewrite (4) as

$$[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{a,b} = 2\pi\tfrac{W}{NL}\sum_{n=0}^{N-1}\sum_{\substack{r=\max\\(0,n-N_h+1)}}^{n}\sum_{\substack{p=\max\\(0,n-N_h+1)}}^{n} h_a(n-r)y_a(r)h_b(n-p)y_b^*(p) \quad (5)$$

where $h_a(\cdot)$ and $h_b(\cdot)$ are the impulse responses of the fractional delay filters and N_h is the length of these responses. Next, $\hat{\breve{\boldsymbol{u}}}$ and $\hat{\boldsymbol{v}}$ are formed as estimations for $\breve{\boldsymbol{u}}$ and $\boldsymbol{v}$, respectively. The elements of $\hat{\boldsymbol{v}}$ represent an estimation for the average power within each spectral segment.

3. BIAS COMPUTATION

Let $x(t)$ be a zero-mean white Gaussian random process with PSD $P_x(\omega) = \sigma^2$.[1] The estimation bias is found by computing the expected value of $\hat{\boldsymbol{v}}$. From (3) we have

$$E\{\hat{\boldsymbol{v}}\} = (\breve{\boldsymbol{\Psi}}^T\breve{\boldsymbol{\Psi}})^{-1}\breve{\boldsymbol{\Psi}}^T E\{\hat{\breve{\boldsymbol{u}}}\}. \quad (6)$$

In order to determine $E\{\hat{\breve{\boldsymbol{u}}}\}$, it is required to find the expected value of the real and imaginary parts of $\hat{\boldsymbol{R}}_{\boldsymbol{z}}$. The expectation can be performed before taking the real or imaginary parts of $\hat{\boldsymbol{R}}_{\boldsymbol{z}}$, as these operators are linear. We can find $E\{[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{a,b}\}$ by taking expectation from both sides of (5). The problem is now reduced to finding $E\{y_a(r)y_b^*(p)\}$, which is obtained as

$$E\{y_a(r)y_b^*(p)\} = E\{x((rL+c_a)T)x^*((pL+c_b)T)\} = \sigma^2 \quad (7)$$

for $rL + c_a = pL + c_b$ (or $a = b$, $r = p$), and it equals zero otherwise. This results from the fact that $x(t)$ is a white process with PSD $P_x(\omega) = \sigma^2$. Applying (7) to (5), we find that $E\{[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{a,b}\} = 0$ for $a \neq b$ and

$$E\{[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{a,b}\} = \frac{2\pi W}{NL}\sum_{n=0}^{N-1}\sum_{\substack{r=\max\\(0,n-N_h+1)}}^{n} h_a^2(n-r)\sigma^2 = \frac{2\pi W}{L}H_a\sigma^2 \quad (8)$$

for $a = b$, where $H_a \triangleq \frac{1}{N}\sum_{m=0}^{N_h-1}(N-m)h_a^2(m)$. Recalling that the first diagonal element of $\hat{\boldsymbol{R}}_{\boldsymbol{z}}$ is used in $\hat{\breve{\boldsymbol{u}}}$ and taking the real and imaginary parts of (8), we have $E\{\hat{\breve{\boldsymbol{u}}}\} = 2\pi\frac{W}{L}H_1\sigma^2\boldsymbol{e}_1$, where $\boldsymbol{e}_1$ is a column vector of length $q(q-1)+2$ with all its elements equal to zero except for the first element which is 1. Then, the expected value of $\hat{\boldsymbol{v}}$ can be found using (6) as

$$E\{\hat{\boldsymbol{v}}\} = 2\pi\frac{W}{L}H_1\sigma^2(\breve{\boldsymbol{\Psi}}^T\breve{\boldsymbol{\Psi}})^{-1}\breve{\boldsymbol{\Psi}}^T\boldsymbol{e}_1. \quad (9)$$

We analyze next the asymptotic behavior of the estimator. First, it can be easily shown that $\hat{\boldsymbol{R}}_{\boldsymbol{z}}$ is an asymptotically unbiased estimator of $\boldsymbol{R_z}$. Since $\hat{\breve{\boldsymbol{u}}}$ consists of the elements of $\hat{\boldsymbol{R}}_{\boldsymbol{z}}$ and the operations of taking the real and imaginary parts are linear, it follows that $\hat{\breve{\boldsymbol{u}}}$ is also an asymptotically unbiased estimator of $\breve{\boldsymbol{u}}$. Furthermore, letting the number of samples tend to infinity in (6) and using (3), we find that

$$\lim_{N\to\infty}E\{\hat{\boldsymbol{v}}\} = (\breve{\boldsymbol{\Psi}}^T\breve{\boldsymbol{\Psi}})^{-1}\breve{\boldsymbol{\Psi}}^T\lim_{N\to\infty}E\{\hat{\breve{\boldsymbol{u}}}\} = (\breve{\boldsymbol{\Psi}}^T\breve{\boldsymbol{\Psi}})^{-1}\breve{\boldsymbol{\Psi}}^T\breve{\boldsymbol{u}} = \boldsymbol{v}.$$

In other words, $\hat{\boldsymbol{v}}$ is also an asymptotically unbiased estimator of $\boldsymbol{v}$. Consider the fact that $x(t)$ has equal power in all spectral segments (the elements of $\boldsymbol{v}$ are all the same). Since $\hat{\boldsymbol{v}}$ is asymptotically unbiased, it follows that the estimator makes the same estimation for all spectral segments (the elements of $\lim_{N\to\infty}E\{\hat{\boldsymbol{v}}\}$ are equal). Using estimated values for $a=b=1$ in (2), taking expectation from both sides, and letting the number of samples tend to infinity, we obtain that

$$\lim_{N\to\infty}E\{[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{1,1}\} = \left(\frac{W}{L}\right)^2\boldsymbol{1}_L^T\lim_{N\to\infty}E\{\hat{\boldsymbol{v}}\} \quad (10)$$

where $\boldsymbol{1}_L$ is the column vector of length L with all its elements equal to 1. Considering normalized fractional delay filters ($\sum_{m=0}^{N_h-1}h_a^2(m) = 1$) and referring to the definition of H_a, we also find that $\lim_{N\to\infty}H_a = 1$. Therefore, using (8), we find that $\lim_{N\to\infty}E\{[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{1,1}\} = 2\pi\frac{W}{L}\sigma^2$. Combining this result with (10) results in $\lim_{N\to\infty}E\{\hat{\boldsymbol{v}}\} = \frac{2\pi}{W}\sigma^2\boldsymbol{1}_L$. Letting the number of samples tend to infinity in (9) yields

$$\lim_{N\to\infty}E\{\hat{\boldsymbol{v}}\} = 2\pi\frac{W}{L}\sigma^2(\breve{\boldsymbol{\Psi}}^T\breve{\boldsymbol{\Psi}})^{-1}\breve{\boldsymbol{\Psi}}^T\boldsymbol{e}_1 = \frac{2\pi}{W}\sigma^2\boldsymbol{1}_L. \quad (11)$$

It follows from (11) that all the elements of the first column of $(\breve{\boldsymbol{\Psi}}^T\breve{\boldsymbol{\Psi}})^{-1}\breve{\boldsymbol{\Psi}}^T$ are equal to L/W^2. Therefore, (9) can be simplified as $E\{\hat{\boldsymbol{v}}\} = \frac{2\pi}{W}H_1\sigma^2\boldsymbol{1}_L$.

[1] A general signal can be written as a filtered Gaussian process.

Finally, we define $\hat{P}_x(\omega)$ as $\hat{P}_x(\omega) \triangleq \frac{W}{2\pi H_1}\hat{v}_l$ for ω in the l-th spectral segment ($1 \le l \le L$), where $\hat{v}_l$ are the elements of $\hat{\boldsymbol{v}}$. This gives an unbiased estimator of the average power in each spectral segment. These estimations can also be arranged in the following vector

$$\hat{\boldsymbol{p}} \triangleq \frac{W}{2\pi H_1}\hat{\boldsymbol{v}}. \tag{12}$$

4. VARIANCE COMPUTATION

Theorem 1: The correlogram estimation based on undersampled data is a consistent estimator of the average power in each spectral segment.

Proof. Using (3) and (12), the covariance matrix of the estimator for the Gaussian signal case is given by

$$\mathcal{C}_{\hat{\boldsymbol{p}}}=\left(\frac{W}{2\pi H_1}\right)^2(\breve{\boldsymbol{\Psi}}^T\breve{\boldsymbol{\Psi}})^{-1}\breve{\boldsymbol{\Psi}}^T\boldsymbol{U}\breve{\boldsymbol{\Psi}}(\breve{\boldsymbol{\Psi}}^T\breve{\boldsymbol{\Psi}})^{-1}-\sigma^4\mathbf{1}_{LL} \tag{13}$$

where $\boldsymbol{U} \triangleq E\{\hat{\breve{\boldsymbol{u}}}\hat{\breve{\boldsymbol{u}}}^T\} \in \mathbb{R}^{2Q\times 2Q}$ and $\mathbf{1}_{LL}$ is the $L\times L$ matrix with all its elements equal to 1.

Let x and y be two arbitrary complex numbers. The following equations hold [8]

$$Re(x)Re(y) = 0.5\,(Re(xy) + Re(xy^*)) \tag{14}$$

$$Im(x)Im(y) = -0.5\,(Re(xy) - Re(xy^*))\,. \tag{15}$$

The elements of $\boldsymbol{U}$ can be obtained using (14), (15), $E\{[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{a,b}[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{c,d}\}$, and $E\{[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{a,b}[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]^*_{c,d}\}$ where (a,b) and (c,d) correspond to the elements of $\hat{\boldsymbol{R}}_{\boldsymbol{z}}$ used in $\hat{\breve{\boldsymbol{u}}}$. Using (5), we obtain

$$\begin{aligned}&E\{[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{a,b}[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{c,d}\}\\&=(2\pi\tfrac{W}{NL})^2\textstyle\sum_n\sum_u\sum_r\sum_p\sum_s\sum_m h_a(n-r)h_b(n-p)\times\\&\quad h_c(u-s)h_d(u-m)E\{y_a(r)y_b^*(p)y_c(s)y_d^*(m)\}\end{aligned}$$

where $\sum_n$, $\sum_u$, $\sum_r$, $\sum_p$, $\sum_s$, and $\sum_m$ are notations for $\sum_{n=0}^{N-1}$, $\sum_{u=0}^{N-1}$, $\sum_{r=\max(0,n-N_h+1)}^{n}$, $\sum_{p=\max(0,n-N_h+1)}^{n}$, $\sum_{s=\max(0,u-N_h+1)}^{u}$, and $\sum_{m=\max(0,u-N_h+1)}^{u}$, respectively.

After some computations and using the forth moment of $x(t)$, it can be found that all the off-diagonal elements of $\boldsymbol{U}$ are equal to zero. The first diagonal element of $\boldsymbol{U}$ is obtained as

$$\begin{aligned}E\{[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{1,1}[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{1,1}\}=&\sigma^4\left(2\pi\tfrac{W}{NL}\right)^2\Big(\textstyle\sum_n\sum_u\sum_r\sum_s\\&h_1^2(n-r)h_1^2(u-s)+\textstyle\sum_n S_1(n)\Big)\end{aligned} \tag{16}$$

where $S_1(n)$ is defined as

$$\begin{aligned}S_1(n) \triangleq &\sum_u\sum_r\sum_p\sum_s\sum_m \delta(r-m)\delta(p-s)\times\\&h_1(n-r)h_1(n-p)h_1(u-s)h_1(u-m)\end{aligned}$$

and $\delta(\cdot)$ is the Kronecker delta. It is straightforward to show that for $N_h-1 \le n \le N-N_h$, $S_1(n)$ is given by

$$G_1 \triangleq S_1(n) = \sum_{g=0}^{2N_h-2}\left((h_1(i) * h_1(N_h-1-i))|_g\right)^2$$

where $*$ denotes the convolution operation. Note that G_1 is not a function of n. For $0 \le n < N_h-1$, $S_1(n)$ is given by

$$S_1(n) = \sum_{g=0}^{n+N_h-1}\left(((h_1(i)W_n(i)) * h_1(N_h-1-i))|_g\right)^2$$

where $W_n(i)$ is equal to 1 for $0 \le i \le n$ and 0 elsewhere. For $N-N_h < n \le N-1$, $S_1(n)$ is given by

$$S_1(n) = \sum_{g=0}^{N-n+N_h-2}\left((h_1(i) * h_1(N_h-1-i))|_g\right)^2.$$

Next, (16) can be rewritten as

$$\begin{aligned}E\{[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{1,1}[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{1,1}\}=&\sigma^4\left(\frac{2\pi W}{NL}\right)^2\Big(\sum_n\sum_r h_1^2(n-r)\times\\&\sum_u\sum_s h_1^2(u-s)+(N-2N_h+2)G_1+\sum_{n=0}^{N_h-2}S_1(n)+\sum_{n=N-N_h+1}^{N-1}S_1(n)\Big)\end{aligned}$$

and simplified as

$$\begin{aligned}E\{[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{1,1}[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{1,1}\} = &\sigma^4\left(2\pi\frac{W}{NL}\right)^2\times\\&\Big(N^2H_1^2+(N-2N_h+2)G_1+\Sigma_1\Big)\end{aligned} \tag{17}$$

where $\Sigma_1 \triangleq \sum_{n=0}^{N_h-2}S_1(n) + \sum_{n=N-N_h+1}^{N-1}S_1(n)$. Note that $[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{1,1}$ is real-valued. Therefore, $[\boldsymbol{U}]_{1,1}$ is equal to $E\{[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{1,1}[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{1,1}\}$ as given in (17) and $[\boldsymbol{U}]_{Q+1,Q+1}$ equals zero since the imaginary part of $[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{1,1}$ is zero. For the rest of the diagonal elements of $\boldsymbol{U}$, $E\{[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{a,b}[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{a,b}\}$ is zero. Moreover, $[\boldsymbol{U}]_{k,k}$ ($2 \le k \le 2Q$ and $k \ne Q+1$) is obtained using (14) and (15) as $[\boldsymbol{U}]_{k,k} = \frac{1}{2}Re\left(E\{[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{a,b}[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]^*_{a,b}\}\right)$. Similar to the computations for $E\{[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{a,b}[\hat{\boldsymbol{R}}_{\boldsymbol{z}}]_{a,b}\}$, we have

$$[\boldsymbol{U}]_{k,k} = \frac{1}{2}\sigma^4\left(2\pi\frac{W}{NL}\right)^2\sum_n S_k(n) \tag{18}$$

where $S_k(n) \triangleq \sum_u\sum_r\sum_p\sum_s\sum_m \delta(r-s)\delta(p-m)h_a(n-r)h_b(n-p)h_a(u-s)h_b(u-m)$. It is again straightforward to show that for $N_h-1 \le n \le N-N_h$, $S_k(n)$ is given by

$$\begin{aligned}G_k \triangleq S_k(n) = \sum_{g=0}^{2N_h-2}&\left(h_a(i) * h_a(N_h-1-i)\right)|_g\times\\&\left(h_b(i) * h_b(N_h-1-i)\right)|_g.\end{aligned}$$

For $0 \le n < N_h-1$, $S_k(n)$ is given by

$$\begin{aligned}S_k(n)=\sum_{g=0}^{n+N_h-1}&\left((h_a(i)W_n(i)) * h_a(N_h-1-i)\right)|_g\times\\&\left((h_b(i)W_n(i)) * h_b(N_h-1-i)\right)|_g.\end{aligned}$$

For $N-N_h < n \le N-1$, $S_k(n)$ is given by

$$\begin{aligned}S_k(n) = \sum_{g=0}^{N-n+N_h-2}&\left(h_a(i) * h_a(N_h-1-i)\right)|_g\times\\&\left(h_b(i) * h_b(N_h-1-i)\right)|_g.\end{aligned}$$

Thus, (18) can be rewritten as

$$[\boldsymbol{U}]_{k,k}=\frac{1}{2}\sigma^4\left(2\pi\frac{W}{NL}\right)^2((N-2N_h+2)G_k+\Sigma_k) \tag{19}$$

where $\Sigma_k \triangleq \sum_{n=0}^{N_h-2}S_k(n) + \sum_{n=N-N_h+1}^{N-1}S_k(n)$. All the elements of the matrix $\boldsymbol{U}$ are determined, and thus, the covariance matrix of the estimator can be obtained from (13).

We analyze next the asymptotic behavior of the estimator. Letting the number of samples tend to infinity in (13) and noting that $\lim_{N\to\infty} H_a = 1$, we find that

$$\lim_{N\to\infty}\mathcal{C}_{\hat{\boldsymbol{p}}}=\left(\frac{W}{2\pi}\right)^2(\breve{\boldsymbol{\Psi}}^T\breve{\boldsymbol{\Psi}})^{-1}\breve{\boldsymbol{\Psi}}^T\left(\lim_{N\to\infty}\boldsymbol{U}\right)\breve{\boldsymbol{\Psi}}(\breve{\boldsymbol{\Psi}}^T\breve{\boldsymbol{\Psi}})^{-1}-\sigma^4\mathbf{1}_{LL}. \quad (20)$$

Recall that all the off-diagonal elements of $\boldsymbol{U}$ are zeros and the first diagonal element of $\boldsymbol{U}$ is given by (17). Letting the number of samples tend to infinity in (17), we obtain

$$\lim_{N\to\infty} E\{[\boldsymbol{U}]_{1,1}\} = \sigma^4\left(2\pi\frac{W}{L}\right)^2. \quad (21)$$

The $(Q+1)$-th element of $\boldsymbol{U}$ is zero, and if the number of samples tend to infinity in (19), $\lim_{N\to\infty}[\boldsymbol{U}]_{k,k} = 0$. Therefore, all the elements of $\lim_{N\to\infty}\boldsymbol{U}$ are equal to zero except for its first diagonal element given by (21).

In order to further simplify (20), only the elements of the first column of $(\breve{\boldsymbol{\Psi}}^T\breve{\boldsymbol{\Psi}})^{-1}\breve{\boldsymbol{\Psi}}^T$ are required. We have shown in the previous section that these elements are all equal to L/W^2. Therefore, (20) can be simplified to

$$\lim_{N\to\infty}\mathcal{C}_{\hat{\boldsymbol{p}}}=\left(\frac{W}{2\pi}\right)^2\sigma^4\left(2\pi\frac{W}{L}\right)^2\left(\frac{L}{W^2}\right)^2\mathbf{1}_{\boldsymbol{LL}}-\sigma^4\mathbf{1}_{LL}=0.$$

In other words, the variance of the estimator tends to zero as the number of samples goes to infinity, which proves the consistency of the estimator. □

In our derivations, we considered the signal to be noise-free. For the case of independent additive white Gaussian noise, the correlogram for undersampled data is an unbiased and consistent estimator of the sum of the signal and noise.

5. NUMERICAL EXAMPLES

In this section, we investigate the behavior of the estimator for finite-length signals based on the obtained analytical results. The estimation variance depends on the number of the sampling channels q, the number of the spectral segments L, and the length of the signal N_x. Here, the power of the signal is set to $\sigma^2 = 4$ and the Nyquist sampling rate is considered to be $W = 1000$ Hz. The time offsets c_i $(1 \le i \le q)$ are found randomly for each (L, q)-pair and kept unchanged for different signal lengths. Fig. 1 depicts the variance of the estimator $[\mathcal{C}_{\hat{\boldsymbol{p}}}]_{1,1}$ versus the length of the Nyquist samples N_x for different values of (L, q)-pairs. The average sampling rate is also given by qW/L. From the curves corresponding to the $(51, 12)$, $(101, 25)$, and $(201, 50)$-pairs, it can be seen that the performance of the estimator degrades by increasing the number of spectral segments L, i.e., by increasing frequency resolution. The average sampling rate is kept almost the same in this scenario. Consider next the case when the frequency resolution L is the same but the average sampling rate is different. It can be seen from the curves corresponding to the $(101, 20)$ and $(101, 25)$-pairs that the estimation variance is lower when the average sampling rate is higher.

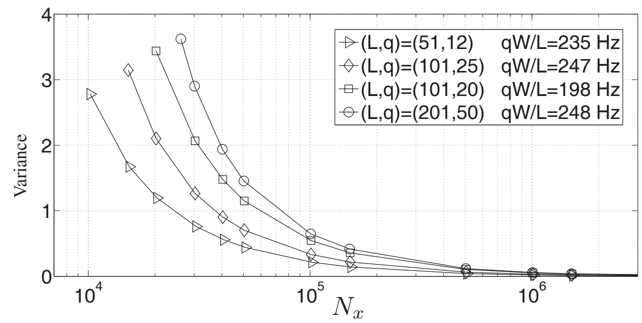

Fig. 1. Variance versus Nyquist signal length.

6. CONCLUSION

The correlogram estimation method based on undersampled data has been analyzed in this paper. The bias of the estimator has been computed and it has been shown that the estimator is unbiased for any signal length. The variance of the method has been also derived and it has been proved that the variance tends to zero asymptotically. Therefore, this method is a consistent estimator. The behavior of the estimator for finite-length signals has been also investigated, and it has been illustrated that there is a tradeoff between the accuracy of the estimator and the frequency resolution. It has been shown that at a fixed average sampling rate, the performance of the estimator degrades for the estimation with higher frequency resolution. Furthermore, for a given frequency resolution, the performance improves by increasing the average sampling rate. Finally, it has been shown that the variance tends to zero as the length of the signal tends to infinity.

7. REFERENCES

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