MINIMAX DESIGN OF SPARSE FIR DIGITAL FILTERS

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ABSTRACT

In this paper, we present a novel algorithm to design sparse FIR digital filters in the minimax sense. To tackle the nonconvexity of the design problem, an efficient iterative procedure is developed to find a potential sparsity pattern. In each iteration, a subproblem in a simpler form is constructed. Instead of directly resolving these nonconvex subproblems, we resort to their respective dual problems. It can be proved that under a weak condition, globally optimal solutions of these subproblems can be attained by solving their dual problems. In this case, the overall iterative procedure can converge to a locally optimal solution of the original design problem. The real minimax design can then be achieved by refining the FIR filter obtained by the iterative procedure. The design procedure described above can be repeated for several times to further improve the sparsity of design results. The output of the previous stage can be used as the initial point of the subsequent design. Simulation results demonstrate the effectiveness of our proposed algorithm.

Index Terms— Finite impulse response (FIR) digital filter, sparse filter design, minimax.

1. INTRODUCTION

Traditional digital filter design algorithms mainly focus on the development of efficient and reliable numerical design methods, and seldom take into account the implementation efficiency during the design stage. In this paper, we consider the sparse FIR digital filter design problems in the minimax sense. The resulting FIR filters have a considerable number of coefficients equal to zero, such that the multipliers corresponding to zero-valued coefficients are no longer required. In general, the l_0-norm of filter coefficient vector is adopted to measure the sparsity of an FIR filter. However, it is known that the l_0-norm is highly nonconvex.

Lately, inspired by the advance of sparse and redundant representation of signals [1]-[2], some techniques have been developed to tackle the nonconvexity of such nonconvex design problems. In [3], the orthogonal matching pursuit (OMP) is employed to design sparse linear-phase FIR filters. The OMP algorithm is a greedy one, which successively adds one additional index to a set of active indices which indicate zero-valued coefficients, such that the residual approximation error is minimized. However, the OMP algorithm can only handle sparse filter design problems with linear equality constraints or a single quadratic constraint, which restricts its capability in resolving design problems in a more general form. Two heuristic approaches are proposed in [4] to design sparse linear-phase FIR filters by using linear programming (LP) as building blocks. The first one successively thin filter coefficients based on two selection rules, i.e., smallest-coefficient rule and minimum-increase rule. As its name implies, the first rule chooses one index at which the corresponding coefficient has the smallest magnitude, while the second one chooses an index at which nullifying the corresponding coefficient leads to the minimum increase of approximation error. The second design algorithm shares a similar idea with the basis pursuit (BP) algorithm [5]. By replacing l_0-norm by its l_1-norm counterpart, the original nonconvex design problem is relaxed to a convex optimization problem. Another design algorithm based on l_1-norm is presented in [6]. The l_1-norm of coefficients is mixed with the minimax approximation error as a regularization term in the objective function of the design problem.

The rest of paper is organized as follows. The proposed design algorithm is developed in Section 2. Two numerical examples are presented in Section 3 to demonstrate the effectiveness of the proposed algorithm. Conclusions are finally drawn in Section 4.

2. PROPOSED DESIGN ALGORITHM

2.1. Problem Formulation

Let $D(\omega)$ be an ideal frequency response and Ω_I represent the union of frequency bands of interest. The frequency response of an Nth-order FIR digital filter is defined by

$$H(e^{j\omega}) = \mathbf{v}^T(\omega)\mathbf{h} \quad (1)$$

This work was supported in part by the Fundamental Research Funds for the Central Universities of China under grants 2011B11214 and 2011B11114, the National Nature Science Foundation of China under grants 61101158 and 60905060, the Scientific Research Foundation for the Excellent Middle-Aged and Youth Scientists of Shandong Province of China under grant BS2010DX025, and the National Basic Research Program of China (973 Program) under grant 2010CB327900.

where $\mathbf{v}(\omega) = [1 \quad e^{-j\omega} \quad \dots \quad e^{-jN\omega}]^T$ and $\mathbf{h} = [h_0 \quad \dots \quad h_N]^T$. Then, the sparse FIR filter design problem in the minimax sense can be expressed by

$$\min \quad \|\mathbf{h}\|_0 \tag{2}$$

$$\text{s.t.} \quad W(\omega_k)\left|[H(e^{j\omega_k}) - D(\omega_k)]\right| = \|\mathbf{F}_k\mathbf{h} - \mathbf{d}_k\|_2 \leq \delta(\omega_k), k = 1,2,\dots,K, \tag{2.a}$$

where $W(\omega)$ represents a given weighting function, ω_ks are frequency points sampled over Ω_I, and

$$\mathbf{F}_k = W(\omega_k)\begin{bmatrix}\mathbf{v}_r^T(\omega_k)\\ \mathbf{v}_i^T(\omega_k)\end{bmatrix}, \tag{3}$$

$$\mathbf{d}_k = W(\omega_k)[D_r(\omega_k) \quad D_i(\omega_k)]^T. \tag{4}$$

In (3), $\mathbf{v}_r(\omega)$ and $\mathbf{v}_i(\omega)$ are real and imaginary parts of $\mathbf{v}(\omega)$, respectively. Similar notations are used in (4). If a sparsity pattern is given, a real minimax design can be attained by solving the following problem

$$\min \quad t \tag{5}$$

$$\text{s.t.} \quad \|\mathbf{F}_k\mathbf{h} - \mathbf{d}_k\|_2 \leq \delta(\omega_k) + t, \; k = 1,\dots,K, \tag{5.a}$$

$$h_n = 0, \; n \in \mathcal{Z}, \tag{5.b}$$

where $\mathcal{Z}$ is a subset of indices of zero-valued coefficients.

2.2. Design Strategy

The proposed design algorithm is inspired by the iterative shrinkage/thresholding algorithms (see [7], [8], and references therein). The proposed design algorithm mainly consists of two steps. We first employ an iterative procedure, which is to be developed in the next subsection, to find a potential sparsity pattern. Then, a real minimax design can be attained by solving (5) with the sparsity pattern obtained in the first step. Since (5) is a convex optimization problem, the core of the entire design algorithm is to determine a potential sparsity pattern. To achieve a better result, we can successively run the design procedure described above for several times. The first stage starts from a given initial point, which can be obtained by solving (5) with an empty $\mathcal{Z}$, while the following ones start from the output of the previous stage. This procedure continues until the sparsity of the design result cannot be further improved.

2.3. Iterative Procedure

The iterative procedure starts from an given $\mathbf{h}^{(0)}$, which can be obtained by solving (5) with an empty $\mathcal{Z}$. Let the solution obtained in the lth iteration be $\mathbf{h}^{(l)}$. In the $(l+1)$th iteration, we first construct a subproblem with the same objective function as that in (2), whereas constraints are modified as

$$\|\mathbf{F}_k\mathbf{h} - \mathbf{d}_k\|_2^2 + s_k\left(\mathbf{h}, \mathbf{h}^{(l)}\right) \leq \delta^2(\omega_k) \tag{6}$$

where $s_k\left(\mathbf{h}, \mathbf{h}^{(l)}\right)$ is defined by

$$s_k\left(\mathbf{h}, \mathbf{h}^{(l)}\right) = c_k\left\|\mathbf{h} - \mathbf{h}^{(l)}\right\|_2^2 - \left\|\mathbf{F}_k\mathbf{h} - \mathbf{F}_k\mathbf{h}^{(l)}\right\|_2^2 = \left(\mathbf{h} - \mathbf{h}^{(l)}\right)^T(c_k\mathbf{I} - \mathbf{F}_k^T\mathbf{F}_k)\left(\mathbf{h} - \mathbf{h}^{(l)}\right). \tag{7}$$

In (7), $\mathbf{I}$ denotes an identity matrix and c_k is chosen so that $s_k\left(\mathbf{h}, \mathbf{h}^{(l)}\right)$ is convex for all k s, which implies $c_k \geq \lambda_{\max}(\mathbf{F}_k^T\mathbf{F}_k)$ where $\lambda_{\max}(\cdot)$ denotes the maximal eigenvalue of a symmetric matrix. In our designs, c_k is always set to $\max_j \lambda_{\max}(\mathbf{F}_j^T\mathbf{F}_j)$. After some manipulations, (6) can cast as

$$\left\|\mathbf{h} - \mathbf{b}_k^{(l)}\right\|_2^2 \leq u_k^{(l)} \tag{8}$$

where

$$\mathbf{b}_k^{(l)} = \mathbf{F}_k^T\mathbf{v}_k^{(l)} + \mathbf{h}^{(l)}, \tag{9}$$

$$u_k^{(l)} = \frac{1}{c_k}\delta^2(\omega_k) - \mathbf{v}_k^{(l)T}(c_k\mathbf{I} - \mathbf{F}_k\mathbf{F}_k^T)\mathbf{v}_k^{(l)}, \tag{10}$$

$$\mathbf{v}_k^{(l)} = \frac{1}{c_k}\left(\mathbf{d}_k - \mathbf{F}_k\mathbf{h}^{(l)}\right). \tag{11}$$

Then, replacing (2.a) by (8), we have

$$\min \quad \|\mathbf{h}\|_0 \tag{12}$$

$$\text{s.t.} \quad \left\|\mathbf{h} - \mathbf{b}_k^{(l)}\right\|_2^2 \leq u_k^{(l)}, \; k = 1,2,\dots,K. \tag{12.a}$$

Although (12) is much simpler than (2), it is still nonconvex. To tackle (12), we shall resort to its dual problem. Note that any feasible solution to (12) is also feasible to (2) as $s_k\left(\mathbf{h}, \mathbf{h}^{(l)}\right)$ is always nonnegative. Hence, the iterative procedure can continue until the sparsity of obtained filter cannot be further improved or the design result obtained by the dual problem is infeasible to (12).

The Lagrangian function of (12) is given by

$$\begin{aligned} f(\mathbf{h}, \boldsymbol{\lambda}) &= \|h\|_0 + \sum_{k=1}^{K}\lambda_k\left(\left\|\mathbf{h} - \mathbf{b}_k^{(l)}\right\|_2^2 - u_k^{(l)}\right) \\ &= \sum_{n=0}^{N}\left[|h_n|_0 + \sum_{k=1}^{K}\lambda_k\left(h_n - b_{k,n}^{(l)}\right)^2\right] - \sum_{k=1}^{K}\lambda_k u_k^{(l)} \end{aligned} \tag{13}$$

where $\boldsymbol{\lambda} = [\lambda_1 \quad \dots \quad \lambda_K]^T$ is a Lagrangian multiplier vector with nonnegative entries and $b_{k,n}^{(l)}$ is the nth component of $\mathbf{b}_k^{(l)}$, and $|x|_0$ is equal to 1 if x is nonzero or 0 otherwise. Since $f(\mathbf{h}, \boldsymbol{\lambda})$ can be decomposed to a set of functions of h_ns independently, the minimizer of $f(\mathbf{h}, \boldsymbol{\lambda})$ can be obtained by minimizing each function with respect to h_n. In this way, we attain the dual problem of (12)

$$\max \quad g(\boldsymbol{\lambda}, \mathbf{z}) = \boldsymbol{\lambda}^T\mathbf{p}^{(l)} - \mathbf{1}^T\mathbf{z} \tag{14}$$

$$\text{s.t.} \quad \frac{\left(\boldsymbol{\lambda}^T\mathbf{b}_n^{(l)}\right)^2}{\mathbf{1}^T\boldsymbol{\lambda}} \leq 1 + z_n, \; n = 0,1,\dots,N \tag{14.a}$$

$$\boldsymbol{\lambda} \geq \mathbf{0}, \quad \mathbf{z} \geq \mathbf{0} \tag{14.b}$$

where $\mathbf{1}$ denotes a vector with all the entries equal to 1, z_ns are a set of auxiliary variables introduced to simplify the dual problem formulation, each entry of $\mathbf{p}^{(l)}$ is computed by $p_k^{(l)} = \sum_{n=0}^{N}\left(b_{k,n}^{(l)}\right)^2 - u_k^{(l)}$, and $\mathbf{b}_n^{(l)} = \left[b_{1,n}^{(l)} \quad \dots \quad b_{K,n}^{(l)}\right]^T$. It is clear that (14) is convex and can be efficiently solved.

Let the optimal solution of (14) be $(\boldsymbol{\lambda}^*, \mathbf{z}^*)$. Then the primal solution $\mathbf{h}^{(l+1)}$ can be recovered by

$$h_n^{(l+1)} = \begin{cases} 0, & \text{if } \left(\boldsymbol{\lambda}^{*T}\mathbf{b}_n^{(l)}\right)^2 \leq \mathbf{1}^T\boldsymbol{\lambda}^* \text{ or } z_n^* = 0, \\ \bar{b}_n^{(l)}, & \text{if } \left(\boldsymbol{\lambda}^{*T}\mathbf{b}_n^{(l)}\right)^2 > \mathbf{1}^T\boldsymbol{\lambda}^* \text{ or } z_n^* > 0, \end{cases} \tag{15}$$

where $\bar{b}_n^{(l)} = \frac{\boldsymbol{\lambda}^{*T}\mathbf{b}_n^{(l)}}{\mathbf{1}^T\boldsymbol{\lambda}^*}$. Let $\mathcal{Z}^{(l)}$ represent the subset of indices of zero-valued coefficients computed by (15). Based on the weak duality property, the objective value of (14) only provides a lower bound of that of (12). However, the following proposition indicates that by solving (14) and applying (15), we could achieve an optimal solution of (12).

Proposition 1: If

$$\frac{\left(\boldsymbol{\lambda}^{*T}\mathbf{b}_n^{(l)}\right)^2}{\mathbf{1}^T\boldsymbol{\lambda}^*} \neq 1, \quad \forall n \in \mathcal{Z}^{(l)}, \tag{16}$$

(15) is an optimal solution of (12).

Proof: In order to prove the Proposition 1, we have to demonstrate that if (16) is satisfied, (15) is a feasible solution to (12), and the duality gap between (12) and (14) is zero. In the proof, we shall drop all the superscripts $^{(l)}$ for ease of notation.

The Lagrangian function of (14) can be written by

$$L(\boldsymbol{\lambda}, \mathbf{z}, \mathbf{r}, \mathbf{s}, \mathbf{t}) = (\mathbf{1} - \mathbf{r} - \mathbf{t})^T\mathbf{z} - \boldsymbol{\lambda}^T(\mathbf{p} + \mathbf{s}) + \sum_{n=0}^{N} r_n\left[\frac{(\mathbf{b}_n^T\boldsymbol{\lambda})^2}{\mathbf{1}^T\boldsymbol{\lambda}} - 1\right] \tag{17}$$

where $\mathbf{r} = [r_0 \ \cdots \ r_N]^T$, $\mathbf{s} = [s_1 \ \cdots \ s_K]^T$, and $\mathbf{z} = [z_0 \ \cdots \ z_N]^T$ are the Lagrangian multiplier vectors associated, respectively, with (15.a), (15.b), and (15.c). Let the optimal solution of the dual problem of (14) be $(\mathbf{r}^*, \mathbf{s}^*, \mathbf{t}^*)$, which can attained by minimizing $L(\boldsymbol{\lambda}^*, \mathbf{z}^*, \mathbf{r}, \mathbf{s}, \mathbf{t})$. Then, $(\boldsymbol{\lambda}^*, \mathbf{z}^*)$ and $(\mathbf{r}^*, \mathbf{s}^*, \mathbf{t}^*)$ should satisfy the Karush-Kuhn-Tucker (KKT) optimality conditions [9]

$$\frac{(\mathbf{b}_n^T\boldsymbol{\lambda}^*)^2}{\mathbf{1}^T\boldsymbol{\lambda}^*} \leq 1 + z_n^*, \quad n = 0,1,\dots,N, \tag{18}$$

$$\boldsymbol{\lambda}^* \geq \mathbf{0}, \quad \mathbf{z}^* \geq \mathbf{0}, \tag{19}$$

$$\mathbf{r}^* \geq \mathbf{0}, \quad \mathbf{s}^* \geq \mathbf{0}, \quad \mathbf{t}^* \geq \mathbf{0}, \tag{20}$$

$$r_n^*\left[\frac{(\mathbf{b}_n^T\boldsymbol{\lambda}^*)^2}{\mathbf{1}^T\boldsymbol{\lambda}^*} - 1 - z_n^*\right] = 0, \quad n = 0,1,\dots,N, \tag{21}$$

$$s_k^*\lambda_k^* = 0, \quad k = 1,2,\dots,K, \tag{22}$$

$$t_n^*z_n^* = 0, \quad n = 0,1,\dots,N, \tag{23}$$

$$\nabla_\lambda L(\boldsymbol{\lambda}^*, \mathbf{z}^*, \mathbf{r}^*, \mathbf{s}^*, \mathbf{t}^*) = -(\mathbf{p} + \mathbf{s}^*) + \sum_{n=0}^{N} r_n\left(2\bar{b}_n\mathbf{b}_n - \bar{b}_n^2\mathbf{1}\right) = \mathbf{0}, \tag{24}$$

$$\nabla_z L(\boldsymbol{\lambda}^*, \mathbf{z}^*, \mathbf{r}^*, \mathbf{s}^*, \mathbf{t}^*) = \mathbf{1} - \mathbf{r}^* - \mathbf{t}^* = \mathbf{0}. \tag{25}$$

Due to (23) and (25), we have

$$t_n^* = 0, \quad r_n^* = 1, \quad \forall n \notin \mathcal{Z}. \tag{26}$$

For $n \in \mathcal{Z}$, we have to distinguish two situations. First, if (18) is inactive, we obtain by (21) and (25)

$$t_n^* = 1, \quad r_n^* = 0, \quad \forall n \in \mathcal{Z}_+ \tag{27}$$

where $\mathcal{Z}_+ = \{n | (\mathbf{b}_n^T\boldsymbol{\lambda}^*)^2 < \mathbf{1}^T\boldsymbol{\lambda}^*, \ n = 0,\dots,N\}$. Second, if (18) is active for some $n \in \mathcal{Z}$, the corresponding r_n^* and t_n^* are uncertain. Let $\mathcal{Z}_0$ be the subset of such indices. Then, using (26) and (27), (24) can be rewritten by

$$\begin{aligned} s_k^* &= -\left(\sum_{n=0}^{N} b_{k,n}^2 - u_k\right) + \sum_{n=0}^{N} r_n^*\left(2\bar{b}_n b_{k,n} - \bar{b}_n^2\right) \\ &= u_k - \sum_{n=0}^{N} b_{k,n}^2 + \sum_{n\notin\mathcal{Z}}\left(2\bar{b}_n b_{k,n} - \bar{b}_n^2\right) \\ &\quad + \sum_{n\in\mathcal{Z}_0} r_n^*\left(2\bar{b}_n b_{k,n} - \bar{b}_n^2\right). \end{aligned} \tag{28}$$

Table I

SPECIFICATIONS OF EXAMPLE 1

Passband region	$[0, 0.0436\pi]$
Stopband region	$[0.0872\pi, \pi]$
Filter order N	80 and 90
Passband magnitude	Within ± 0.5dB of unity
Stopband magnitude	Below −20, −25, −30, −35, and −40dB

Table II

DESIGN RESULTS OF EXAMPLE 1

Stopband magnitude level (dB)	Proposed		Minimum 1-norm [4]	
	$N = 80$	$N = 90$	$N = 80$	$N = 90$
–20	48	52	33	49
–25	40	52	42	52
–30	34	44	34	44
–35	26	32	20	30
–40	16	24	8	18

It can be observed from (28) that in order to guarantee the feasibility of (15) to (12), we should have

$$\begin{aligned} & u_k - \sum_{n=0}^{N} b_{k,n}^2 + \sum_{n\notin\mathcal{Z}}\left(2\bar{b}_n b_{k,n} - \bar{b}_n^2\right) \\ &= u_k - \sum_{n\in\mathcal{Z}} b_{k,n}^2 - \sum_{n\notin\mathcal{Z}}\left(\bar{b}_n - b_{k,n}\right)^2 \\ &= s_k^* - \sum_{n\in\mathcal{Z}_0} r_n^*\left(2\bar{b}_n b_{k,n} - \bar{b}_n^2\right) \\ &\geq 0, \qquad k = 1,2,\dots,K. \end{aligned} \tag{29}$$

If $\mathcal{Z}_0$ is empty or, equivalently, (16) is satisfied, the summation term involving r_n^* can be removed from (29). In view of (20), we then conclude that (15) is feasible to (12).

Multiplying $\boldsymbol{\lambda}^*$ on both sides of (24) yields

$$\begin{aligned} & -(\mathbf{p} + \mathbf{s}^*)^T\boldsymbol{\lambda}^* + \left[\sum_{n=0}^{N} r_n\left(2\bar{b}_n\mathbf{b}_n - \bar{b}_n^2\mathbf{1}\right)\right]^T\boldsymbol{\lambda}^* \\ &= -\mathbf{p}^T\boldsymbol{\lambda}^* + \sum_{n=0}^{N} r_n^*\left(2\bar{b}_n\mathbf{b}_n^T\boldsymbol{\lambda}^* - \bar{b}_n^2\mathbf{1}^T\boldsymbol{\lambda}^*\right) \\ &= -\mathbf{p}^T\boldsymbol{\lambda}^* + \sum_{n=0}^{N} r_n^*\frac{(\mathbf{b}_n^T\boldsymbol{\lambda}^*)^2}{\mathbf{1}^T\boldsymbol{\lambda}^*} = 0 \end{aligned} \tag{30}$$

where we use (22) and the definition of $\bar{b}_n$ to obtain the first and second equalities, respectively. Then, by successively applying (21), (25), and (23) on (30), we further have

$$\begin{aligned} & -\mathbf{p}^T\boldsymbol{\lambda}^* + \sum_{n=0}^{N} r_n^*\frac{(\mathbf{b}_n^T\boldsymbol{\lambda}^*)^2}{\mathbf{1}^T\boldsymbol{\lambda}^*} \\ &= -\mathbf{p}^T\boldsymbol{\lambda}^* + \mathbf{1}^T\mathbf{r}^* + \mathbf{z}^{*T}\mathbf{r}^* \\ &= -\mathbf{p}^T\boldsymbol{\lambda}^* + \mathbf{1}^T\mathbf{r}^* + \mathbf{z}^{*T}(\mathbf{1} - \mathbf{t}^*) \\ &= \mathbf{1}^T\mathbf{z}^* - \mathbf{p}^T\boldsymbol{\lambda}^* + \mathbf{1}^T\mathbf{r}^* \\ &= 0. \end{aligned} \tag{31}$$

Taking $g(\boldsymbol{\lambda}^*, \mathbf{z}^*)$, (26), and (27) into (31) further yields

$$\begin{aligned} & \mathbf{1}^T\mathbf{z}^* - \mathbf{p}^T\boldsymbol{\lambda}^* + \mathbf{1}^T\mathbf{r}^* \\ &= -g(\boldsymbol{\lambda}^*, \mathbf{z}^*) + \sum_{n\notin\mathcal{Z}} 1 + \sum_{n\in\mathcal{Z}_0} r_n^* \\ &= 0. \end{aligned} \tag{32}$$

It can be seen that if (16) is satisfied (or $\mathcal{Z}_0$ is empty), we finally have

$$g(\boldsymbol{\lambda}^*, \mathbf{z}^*) = \sum_{n \notin Z} 1, \tag{33}$$

which means that the duality gap between (12) and (14) is 0. ■

It should be mentioned that (16) is just a sufficient condition, which implies that even if (16) is violated in some iterations, the obtained solutions could be still feasible to (12). In addition, due to the existence numerical errors, in practical designs we employ a soft condition to replace (16)

$$\left| \frac{\left(\mathbf{b}_n^{(l)T} \boldsymbol{\lambda}^*\right)^2}{\mathbf{1}^T \boldsymbol{\lambda}^*} - 1 \right| \geq \varepsilon, \ \ \forall n \in \mathcal{Z}^{(l)}, \tag{34}$$

where ε is a small parameter specified by designers.

4. NUMERICAL EXAMPLES

In this section, two numerical examples are presented to demonstrate the effectiveness of the proposed design algorithm. The multi-stage design strategy is employed to implement all the designs. In our designs, ε used in (34) is set to 10^{-6}, and the weighting function $W(\omega)$ is always chosen equal to 1 over Ω_I and 0 otherwise. In all the designs, 401 frequency points are uniformly sampled over the normalized frequency band $[0, 1]$. The dual problem (14) is solved by SeDuMi [10].

In the first example, a lowpass linear-phase FIR filter is designed by the proposed algorithm. The specifications are given in Table I. For each N, a set of designs with different stopband magnitude levels are implemented. The design results in terms of the number of zero-valued coefficients are summarized in Table II. For comparison, we also employ the minimum 1-norm algorithm proposed in [4] to design the sparse FIR filters under the same set of specifications, and the design results are also reported in Table II. It can be observed that the proposed algorithm can achieve better results in 6 designs of total 10 designs, whereas the minimum 1-norm algorithm can overperform our proposed algorithm only in one design.

The second example is to design another lowpass FIR filter. The detailed specifications are illustrated in Table III. For a fair comparison, we first implement a nonsparse FIR filter with a specified order, and the upper bound of approximation error of the obtained filer is adopted as $\delta(\omega)$ used in (2.a). Then, we implement a sparse filter using the proposed algorithm. The design results in terms of the number of nonzero-valued coefficients are summarized in Table IV.

Table III
SPECIFICATIONS OF EXAMPLE 2

Passband region	$[0, 0.55\pi]$
Stopband region	$[0.6\pi, \pi]$
Filter order N for proposed algorithm	64
Filter order N for nonsparse filter design	40, 42, 44, 46, 48, 50, 52, 54, 56, and 58

Table IV
DESIGN RESULTS OF EXAMPLE 2

$\delta(\omega)$ ($\times 10^{-2}$)	Number of nonzero-valued coefficients		$\delta(\omega)$ ($\times 10^{-2}$)	Number of nonzero-valued coefficients	
	Proposed	Nonsparse		Proposed	Nonsparse
5.573	32	41	3.571	44	51
5.557	35	43	3.454	42	53
4.804	36	45	2.884	45	55
4.484	38	47	2.848	44	57
4.096	37	49	2.498	48	59

5. CONCLUSIONS

In this paper, a novel algorithm is proposed for sparse FIR filter designs in the minimax sense. The core of the design algorithm is the dual problem (14), which can be reliably solved and lead to the optimal solutions of the nonconvex primal subproblems (12) under a weak optimality condition (16). Simulation results show that the proposed design algorithm can improve the sparsity of designed FIR filters at a cost of slightly increasing filter orders. Furthermore, compared with some heuristic design algorithms, which are based on the l_1-norm optimization, the proposed algorithm can attain sparser designs.

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