PERFORMANCE INVESTIGATION ON DT$\mathcal{C}$WT BASED ON THE COMMON FACTOR TECHNIQUE

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ABSTRACT

This paper investigates the performance on a class of DT$\mathcal{C}$WTs based on the common factor technique. The common factor technique was first proposed in [6] by Selesnick, where two corresponding scaling lowpass filters are required to satisfy the half-sample delay condition, and then was modified to improve the analyticity [9] and frequency selectivity [8]. The analyticity of complex wavelet can be improved by minimizing the phase error in the approximation band of allpass filters with given flatness condition. The frequency selectivity of scaling lowpass filters can be improved by locating a specified number of zeros at $z = -1$ and applying the Remez exchange algorithm to minimize the magnitude response in the stopband. How to determine the approximation band and stopband will influence the performance of DT$\mathcal{C}$WT. Therefore, this paper is dedicated to how to choose the approximation band and stopband properly.

Index Terms— DT$\mathcal{C}$WT, common factor technique, analyticity, frequency selectivity, FIR filter.

1. INTRODUCTION

The dual tree complex wavelet transform (DT$\mathcal{C}$WT) has been proposed as an important signal and image processing tool for a variety of applications [2] $\sim$ [7]. The corresponding scaling lowpass filters are required to satisfy the half-sample delay condition if two wavelet bases are a pair of Hilbert transform [6]. Selesnick had proposed a simple common factor method in [6], where allpass filters had been used to construct the corresponding scaling lowpass filters. Once the allpass filter suffice the half-sample delay condition, the design becomes how to satisfy the condition of orthonormality and the regularity of wavelets. Therefore, this approach is simple and effective. In [6], Selesnick had used the maximally flat allpass filters, but the maximally flat allpass filters have a larger phase error as ω increases, which influences the analyticity of complex wavelet. In [9], the allpass filter with a specified degree of flatness and equiripple phase response in the approximation band had been used to improve the analyticity. In [6], Selesnick had given a class of FIR orthonormal solutions, where the scaling lowpass filters have as many zeros at $z = -1$ as possible, resulting in the maximally flat magnitude responses. However, the maximally flat filters have a poor frequency selectivity. It is known that the frequency selectivity is regraded as an important property for many applications of signal and image processing. Thus, the frequency selectivity has been improved by locating the number of zeros at z = 1 and applying the Remez exchange algorithm in the stopband [8].

In this paper, we investigate the performance of DT$\mathcal{C}$WTs based on the common factor technique. We first improve the analyticity of complex wavelet by designing the allpass filters with the specified degree of flatness and equiripple phase response in the approximation band. Next, to improve the frequency selectivity of the scaling lowpass filters, we specify the number of vanishing moments and apply the Remez exchange algorithm in the stopband to minimize the magnitude response. Finally, we investigate the relationship between the maximum error and the cutoff frequency, from which we can obtain the best performance of DT$\mathcal{C}$WT by choosing the approximation band and stopband properly.

2. DUAL TREE COMPLEX WAVELET TRANSFORM

Generally, a complex wavelet $\psi_c(t)$ is consisted of two real wavelet bases, denoted by $\psi_1(t)$ and $\psi_2(t)$, respectively. It is known that if two wavelet functions are a pair of Hilbert transform, the complex wavelet $\psi_c(t) = \psi_1(t) + j\psi_2(t)$ is analytic, i.e., the spectrum is one-sided:

$$\Psi_c(\omega) = \Psi_1(\omega) + j\Psi_2(\omega) = \begin{cases} 2\Psi_1(\omega) & \omega > 0 \\ 0 & \omega < 0 \end{cases}. \quad (1)$$

where $\Psi_i(\omega)$ is the Fourier transform of $\psi_i(t)$.

It has been proved in [6] that two wavelet functions are a Hilbert transform pair, if and only if the corresponding scaling lowpass filters $H_1(z)$, $H_2(z)$ satisfy

$$H_2(e^{j\omega}) = H_1(e^{j\omega})e^{-j\frac{\omega}{2}} \quad |\omega| < \pi. \quad (2)$$

Eq.(2) is the so called half-sample delay condition, which is the necessary and sufficient condition for two wavelet bases to be a Hilbert transform pair. However, the ideal Hilbert transform pair cannot be achieved. Eq.(2) can only be approximated with real filters. Therefore, to evaluate the analyticity, we use the p-norm of the spectrum $\Psi_c(\omega)$ to define an objective measure of quality as

$$E_p = \frac{||\Psi_c(\omega)||_{p,[-\infty,0]}}{||\Psi_c(\omega)||_{p,[0,\infty]}} \quad (3)$$

where

$$||\Psi_c(\omega)||_{p,[\Omega]} = \left(\int_\Omega |\Psi_c(\omega)|^p d\omega\right)^{\frac{1}{p}}. \quad (4)$$

If $p = \infty$, $E_\infty = \lim_{p\to\infty} E_p$ is the peak error in the negative frequency domain. If $p = 2$, E_2 is the square root of the negative frequency energy. In this paper, we will use E_∞ and E_2 to evaluate the analyticity of the complex wavelet.

3. THE COMMON FACTOR TECHNIQUE

In [6], Selesnick had proposed the common factor technique, where the scaling lowpass filters $H_1(z)$ and $H_2(z)$ have the following form

$$\begin{cases} H_1(z) &= F(z)D(z) \\ H_2(z) &= F(z)z^{-L}D(z^{-1}) \end{cases}. \tag{5}$$

It can be easily verified that

$$H_2(z) = H_1(z)z^{-L}\frac{D(z^{-1})}{D(z)} = H_1(z)A(z) \tag{6}$$

where $A(z)$ is an allpass filter defined as

$$A(z) = z^{-L}\frac{D(z^{-1})}{D(z)} = z^{-L}\frac{1+\sum_{n=1}^{L} d(n)z^{n}}{1+\sum_{n=1}^{L} d(n)z^{-n}}, \tag{7}$$

where L is the degree of $A(z)$ and $d(n)$ are real filter coefficients. Therefore, it is clear that the half sample condition in Eq.(2) is achieved if $A(e^{j\omega}) \approx e^{-j\frac{\omega}{2}}(-\pi < \omega < \pi)$. Then two wavelet bases form a Hilbert transform pair.

Since the Hilbert transform pair is non-ideal, we define the error function $E(\omega)$ between two scaling lowpass filters as

$$E(\omega) = H_2(e^{j\omega}) - H_1(e^{j\omega})e^{-j\frac{\omega}{2}}. \tag{8}$$

According to Eq.(6), we have $E(\omega) = H_1(e^{j\omega})[A(e^{j\omega}) - e^{-j\frac{\omega}{2}}]$, thus the magnitude response of $E(\omega)$ is

$$|E(\omega)| = 2|H_1(e^{j\omega})||\sin\frac{\theta(\omega)+\frac{\omega}{2}}{2}|, \tag{9}$$

where $\theta(\omega)$ is the phase response of $A(z)$. It is clear that $|E(\omega)|$ depends on the magnitude response $|H_1(e^{j\omega})|$ and the phase error of $A(z)$. Since $H_1(z)$ is a lowpass filter, it is necessary to minimize the phase error both in the passband and transition band of scaling lowpass filter to improve the analyticity of complex wavelet.

4. DESIGN OF DT$\mathcal{C}$WT WITH IMPROVED PERFORMANCE

4.1. Design of allpass filters with given flatness condition

In the following, we discuss how to improve the analyticity of complex wavelet. In [6], the maximally flat allpass filters had been used where $\omega = 0$ is chosen as the point of approximation and the phase error becomes large as ω increases. Now, we consider $A(z)$ has the given degree of flatness at $\omega = 0$. It is required that the derivatives of $\theta(\omega)$ are equal to

$$\left.\frac{\partial^{2r+1}\theta(\omega)}{\partial\omega^{2r+1}}\right|_{\omega=0} = \begin{cases} -\frac{1}{2} & (r=0) \\ 0 & (r=1,2,\cdots,J-1) \end{cases} \tag{10}$$

where J $(0 \le J \le L)$ controls the degree of flatness.

Next, we want to minimize the phase error $\theta_e(\omega)$

$$\min\left\{\max_{0\le\omega\le\omega_c}\theta_e(\omega)\right\} = \min\left\{\max_{0\le\omega\le\omega_c}(\theta(\omega)+\frac{1}{2}\omega)\right\}, \tag{11}$$

where the desired phase response is $-\frac{1}{2}\omega$ and ω_c is the cutoff frequency of the approximation band. We can apply the Remez exchange algorithm to obtain an equiripple phase response in the approximation band $[0, \omega_c]$by using the remaining degree of freedom, as shown in [9].

4.2. Design of scaling lowpass filters with improved frequency selectivity

It is well-known that frequency selectivity is a useful property for many applications of signal and image processing. In [8], a design method had been proposed to specify the number of zeros at $z = -1$ in advance, and to use the remaining degree of freedom to get the best possible frequency selectivity. To obtain $F(z)$ in Eq.(5) with K vanishing moments, $F(z)$ is chosen as

$$F(z) = Q(z)(1+z^{-1})^K, \tag{12}$$

thus Eq.(5) becomes

$$\begin{cases} H_1(z) =Q(z)(1+z^{-1})^K D(z) \\ H_2(z) =Q(z)(1+z^{-1})^K z^{-L}D(z^{-1}) \end{cases}. \tag{13}$$

Then we have the product filter $P(z)$ as

$$\begin{aligned} P(z) &= H_1(z)H_1(z^{-1}) = H_2(z)H_2(z^{-1}) \\ &= Q(z)Q(z^{-1})(z+2+z^{-1})^K D(z)D(z^{-1}) \end{aligned}. \tag{14}$$

Defining

$$R(z) =Q(z)Q(z^{-1}) = \sum_{n=-R}^{R} r(n)z^{-n}, \tag{15}$$

$$S(z) =(z+2+z^{-1})^K D(z)D(z^{-1}) = \sum_{n=-L-K}^{L+K} s(n)z^{-n}, \tag{16}$$

where $r(n) = r(-n)$ for $1 \le n \le R$, $s(n) = s(-n)$ for $1 \le n \le L+K$. Note that $P(z)$ must be a halfband filter, i.e., $P(z) - P(-z) = 2$. This is equivalent to

$$\sum_{k=I_{min}}^{I_{max}} s(2n-k)r(k) = \begin{cases} 1 & (n=0) \\ 0 & (n\ne 0) \end{cases}, \tag{17}$$

where $I_{\min} = \max\{-R, 2n-L-K\}$ and $I_{\max} = \min\{R, 2n+L+K\}$. Therefore, the degree of $H_i(z)$ is $M = N+L+K$ and M is an odd number. In Eq.(17), there exist $(M+1)/2$ equations with respect to $N+1$ unknown coefficients $r(n)$. The only solution can be obtained if $(M+1)/2 = N+1$. Given N and L, the maximal K is $K_{max} = N-L+1$, resulting in the maximally flat scaling lowpass filters [6].

Next, we consider the case of $K < K_{max}$. By applying the Remez exchange algorithm in the stopband $[\omega_s, \pi]$, we suppose that ω_i ($\omega_s = \omega_0 < \omega_1 < \cdots < \omega_{2m} < \pi$) are a set of extremal frequencies and formulate $P(e^{j\omega})$ as

$$P(e^{j\omega_i}) = R(e^{j\omega_i})S(e^{j\omega_i}) = (1+(-1)^i)\delta, \tag{18}$$

where $\delta > 0$ is an error and $K_{max} - K = 2m$. Note that we force $P(e^{j\omega_i}) \ge 0$ to ensure the spectral factorization of $R(z)$. Therefore, Eq.(18) is equivalent to

$$r(0) + 2\sum_{n=1}^{N} r(n)\cos(n\omega_i) - \frac{1+(-1)^i}{S(e^{j\omega_i})}\delta = 0, \tag{19}$$

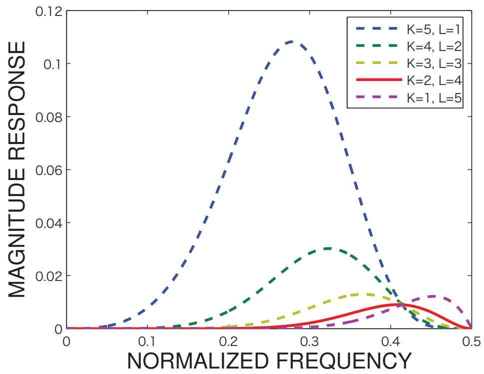

Fig. 1. Magnitude responses of $E(\omega)$.

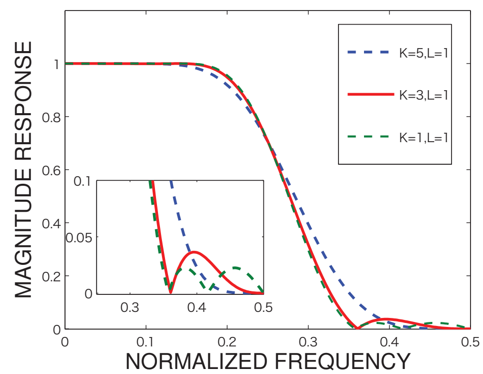

Fig. 2. Magnitude responses of scaling lowpass filters $H_i(z)$.

for $i = 0, 1, \cdots, 2m$. Eqs.(17) and (19) have totally $(M+1)/2 + 2m+1 = N+2$ equations with respect to $(N+1)$ filter coefficients $r(n)$. Therefore, a set of filter coefficients $r(n)$ can be obtained by solving a system of linear equations. Furthermore, we make use of an iteration procedure to obtain the equiripple magnitude responses of $P(z)$. Finally, we can obtain $Q(z)$ from $R(z)$ by using a spectral factorization approach.

5. PERFORMANCE INVESTIGATION

In this section, we present several examples to investigate the performance on DT$\mathcal{C}$WT based on the common factor technique. First, we have constructed the maximally flat scaling lowpass filters with $M = 11$, $R = 5$. We can choose different K and L, where $K + L = 6$. The magnitude responses of $E(\omega)$ are shown in Fig.1. It is clear that when $K = 2$, the magnitude responses of $E(\omega)$ becomes the minimum, while it is maximum when $K = 5$. Then, we consider the scaling lowpass filters $H_i(z)$ with the improved frequency selectivity. To obtain $H_i(z)$ with one equiripple in the stopband, we set $K = 3$, $L = 1$, so $R = 7$, and the remaining degree of freedom is $2m = 2$. The stopband is set at $\omega_s = 0.70\pi$. The resulting magnitude response of $H_i(z)$ is shown in solid line in Fig.2. For the comparison, the maximally scaling lowpass filter ($R = 5$) and the scaling lowpass filter with two equiripples in the stopband ($R = 9$) are also plotted in Fig.2. It is seen in Fig.2 that the magnitude responses of $H_i(z)$ with the improved frequency selectivity are more sharp than the filter with $R = 5$. We then consider the scaling lowpass filters with both analyticity and frequency selectivity, where $K = 2$, $L = 2$, $J = \{0, 1\}$ and $\omega_c = 0.56\pi$, $\omega_s = 0.82\pi$. Fig.3 shows the magnitude responses of $E(\omega)$. It is obvious that the maximally flat filter has the maximum error while our proposed filters $H_i(z)$ can decrease the $E(\omega)$ efficiently. Next, we have designed allpass filters with $L = 2$, $J = 1$, $\omega_c = \{0.35\pi, 0.55\pi, 0.75\pi\}$, and constructed the scaling lowpass filters $H_i(z)$ with $K = 2$, $R = 7$, $\omega_s = 0.80\pi$. The magnitude responses of $E(\omega)$ are shown in Fig.4. It is clear that the minimum $E(\omega)$ can be obtained when $\omega_c = 0.55\pi$. To the oppositie, if ω_c is too small or too big, the maximum error of $E(\omega)$ would increase, resulting in the poor analyticity of complex wavelet. That is to say, how to determine ω_c will influence the maximum error of $E(\omega)$, that is, the analyticity of complex wavelets. Fig.5 shows the relationship between E_∞, E_2 and ω_c, where the optimal ω_c is $\omega_c^{opt} = 0.59\pi$. Moreover, the spectrum $\Psi_i(\omega)$ of the resulting wavelet functions $\psi_i(t)$ is shown in solid line in Fig.6. The spectrum $\Psi_i(\omega)$ composed by the maximally scaling lowpass filter is also shown in dash line in Fig.6. The shape of $\Psi_i(\omega)$ are the almost same. Furthermore, the spectrum $\Psi_c(\omega)$ are given in Fig.7 where the complex wavelet with the improved analyticity and frequency selectivity indicates a better analyticity than the maximum case. Finally, we choose $J = 1$, $\{K, L\} = \{4, 2\}$, $\{3, 3\}$,$\{2, 4\}$ and set $\omega_c = [0.35\pi \sim 0.95\pi]$ and $\omega_s = [0.55\pi \sim 0.95\pi]$ to find the optimal ω_c. Fig.8 shows the relationship between ω_c^{opt} and ω_s, where ω_c^{opt} are almost constant when $\omega_s \geq 0.70\pi$.

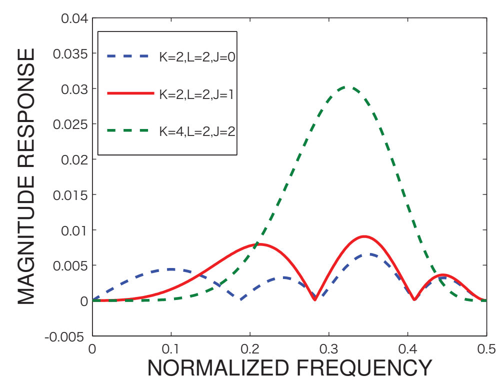

Fig. 3. Magnitude responses of $E(\omega)$.

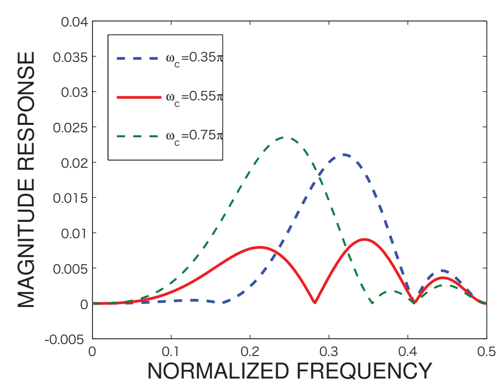

Fig. 4. Magnitude responses of $E(\omega)$.

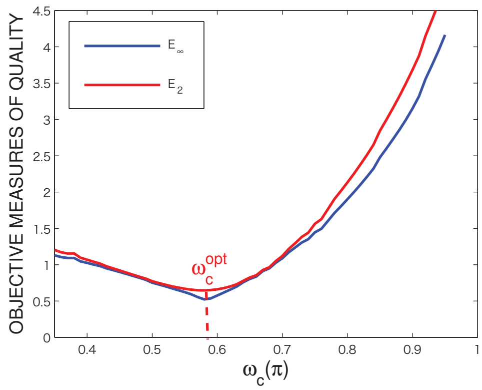

Fig. 5. Relationship between E_∞, E_2 and ω_c.

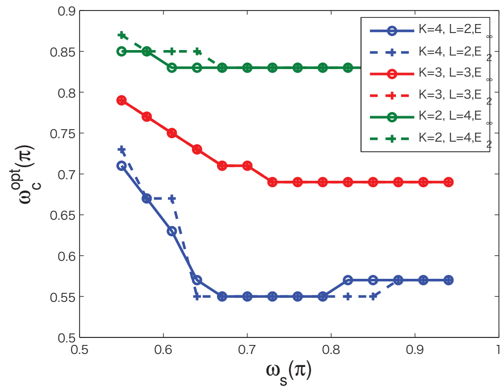

Fig. 8. Relationship between ω_c^{opt} and ω_s.

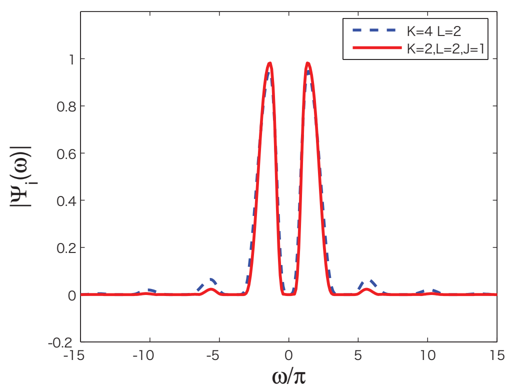

Fig. 6. Magnitude responses of $\Psi_i(\omega)$.

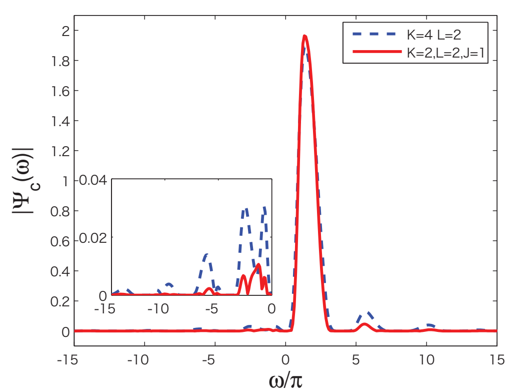

Fig. 7. Magnitude responses of $\Psi_c(\omega)$.

6. CONCLUSION

In this paper, we have investigated the performance on DTCWT based on the common factor technique. To improve the analyticity of complex wavelet, we first have described the design method of allpass filters with the specified degree of flatness and equiripple phase response in the approximation band. Next, we have specified the number of vanishing moments and applied the Remez exchange algorithm in the stopband to improve the frequency selectivity of the scaling lowpass filters. The investigation shows that a proper approximation band can reduce the maximum error and improve the analyticity of complex wavelet efficiently. In addition, it is shown that the optimal approximation band is almost unchanged when the stopband is not too wide.

7. REFERENCES

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